# THE KING'S SCHOOL

2006
Higher School Certificate
**Trial Examination**

# lathematics Extension 1

**eral Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

**Total marks – 84**

- Attempt Questions 1-7
- All questions are of equal value
- Start a new booklet for each Question
- Put your Student Number and the Question Number on the front of each booklet

**Total marks – 84**
**Attempt Questions 1-7**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

---

Marks

**Question 1** **(12 marks)** Use a SEPARATE writing booklet.

(a) Differentiate $y = \log_e(\cos x)$ expressing your answer in simplest form. **2**

(b) Evaluate:

(i) $\displaystyle\int_2^3 \frac{2x}{\sqrt{x^2-4}}\,dx$ using the substitution $u = x^2 - 4$. **3**

(ii) $\displaystyle\int_{\pi}^{\frac{4\pi}{3}} \sin x \cos x\, dx$ **3**

(c) Solve the following inequaltiy for $x$, graphing the solution on a number line

$$\frac{1}{x+2} < 3$$ **2**

(d) Determine the acute angle between the straight lines whose equations are

$$x - y + 1 = 0 \text{ and } 2y = x + 1$$ **2**

**End of Question 1**

**Question 2** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $y = 2\cos^{-1}\frac{x}{3}$.

(i) Sketch the graph of this function clearly showing the domain and range. **2**

(ii) Find the angle, $\theta$, that the tangent to the curve $y = 2\cos^{-1}\frac{x}{3}$ at $x = 0$ makes with the positive direction of the $x$-axis. **3**

(b) Find the volume of the solid of revolution formed when the curve $y = \sin x$ is rotated around the $x$-axis between the lines $x = 0$ and $x = \frac{\pi}{4}$. **3**

(c) Find all values of $\theta$ for which $2\sin\theta - \sqrt{2} = 0$. **2**

(d) Find the point P which divides the interval joint A(-3, 5) and B(7,10) externally in the ratio 2 : 7. **2**

**End of Question 2**

**Question 3** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A spherical balloon is expanding so that its volume $V$ mm$^3$ increases at a constant rate of 72 mm$^3$ per second.

What is the rate of increase of its surface area $A$ mm$^2$, when the radius is 12 mm? **4**

(b) Factorise $x^3 - 3x^2 - 10x + 24$, given that $x = 2$ is a zero, and hence solve $x^3 + 24 > 3x^2 + 10x$. **4**

(c) Solve $3\sin x + 2\cos x = 2$, for $0 \leq x \leq 360°$, to the nearest minute. **4**

**End of Question 3**

**Question 4** (12 marks) Use a SEPARATE writing booklet. **Marks**

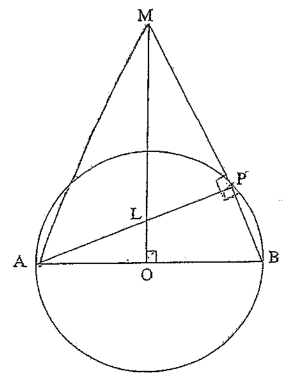
(a) Find the term independent of $x$ in the expansion of $\left(x + \frac{1}{x^2}\right)^9$. **3**

(b) Consider the expansion of $(1 + 2x)^n$.

(i) Write down an expression for the coefficient of the term in $x^4$. **2**

(ii) The ratio of the coefficient of the term in $x^4$ to that of the term in $x^6$ is 5 : 8. Find $n$. **3**

(b)




O is the centre of the circle, MPB is a straight line and OLM is perpendicular to AOB as shown. Prove that:

(i) A, O, P, M are concyclic, and **2**

(ii) $\angle$ OPA = $\angle$ OMB. **2**

**End of Question 4**

**Question 5** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) $TA$ and $TB$ are two tangents drawn to a circle from an external point $T$. $A$ and $B$ are the point of contact of the tangent with the circle.

(i) Draw a neat diagram clearly showing this information.

(ii) Prove that $TA = TB$. 2

(b) Prove by Mathematical Induction that $7^n - 1$ is divisible by 6 for all positive integers of $n$. 4

(c) The elevation of hill at a place $A$ due east of it is $39^\circ$, at a place $B$ due south of $A$, the elevation is $27^\circ$.

If the distance from $A$ to $B$ is 500m, find the height of the hill, to the nearest metre. 4

(d) Show that $\frac{d}{dx}(\log_e 2x) = \frac{d}{dx}(\log_e x)$.

~~Does this mean that $\log_e 2x = \log_e x$?~~

Give reasons for your answers. 2

**Question 6** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Consider the parabola $y = x^2$.

Find the equation of the tangent to the parabola at the point $P(t, t^2)$. 2

(ii) Show that the line passing through the focus of the parabola and perpendicular to the tangent at $P$ had equation $x = \frac{t}{2}(1 - 4y)$. 2

(iii) Find the locus of $Q\ (X, Y)$, the point of intersection of the tangent and the line through the focus perpendicular to the tangent. 2

(b) ~~A particle A is projected horizontally~~ at 50 $m/s$ from the top of a tower 100m high. At the same instant, another particle $B$ is projected from the bottom of the tower, in the same vertical plane at 100 $m/s$ with elevation $60^\circ$.

Prove that the particles will collide and find where they do so. (Use $g = 10ms^{-2}$.) 6

**End of Question 6**

**Question 7** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve for $0^\circ \le \theta \le 360^\circ$, $\sin x = 3\cos(x + 65^\circ)$. 3

(b) Prove that the area of a $\Delta ABC$ is $\frac{a^2 \sin B \sin C}{2 \sin A}$. 2

(c) Given that $y = \frac{x^2 + \lambda}{x + 2}$ and $x$ is real, find:

(i) the set of value(s) of $\lambda$ for which $y$ can take all but one real value. 1

(ii) If when $\lambda = 5$, by sketching $y = \frac{x^2 + 5}{x + 2}$, find the range of the function. 3

(d) Find the values of $m$ for which the line $y = mx$ touches the curve $y = \frac{2x^2 + 1}{2(x + 2)}$. 3

**End of Examination**

TKS THSC MATK1 2006

1(a) $y = \log_e(\cos x)$

$y' = \dfrac{-\sin x}{\cos x}$

$= -\tan x$

(b)(i) $\displaystyle\int_2^3 \frac{2x\,dx}{\sqrt{x^2-4}}$ $\quad u = x^2 - 4$

$\dfrac{du}{dx} = 2x$

$x = 2,\ u = 0$

$x = 3,\ u = 5$

$I = \displaystyle\int_0^5 \frac{2x}{\sqrt{u}} \times \frac{du}{2x}$

$= \displaystyle\int_0^5 u^{-\frac{1}{2}}\,du$

$= \left[2u^{\frac{1}{2}}\right]_0^5$

$= 2\sqrt{5}.$

(ii) $\displaystyle\int_\pi^{\frac{4\pi}{3}} \sin x \cos x\,dx$

$= \displaystyle\int_\pi^{\frac{4\pi}{3}} \tfrac{1}{2}\sin 2x\,dx$

$= \left[-\tfrac{1}{4}\cos 2x\right]_\pi^{\frac{4\pi}{3}}$

$= -\tfrac{1}{4}\cos\tfrac{8\pi}{3} + \tfrac{1}{4}\cos 2\pi$

$= -\tfrac{1}{4}\cos\tfrac{2\pi}{3} + \tfrac{1}{4}$

$= \tfrac{1}{8} + \tfrac{1}{4}$

$= \tfrac{3}{8}.$

(c) $\dfrac{1}{x+2} < 3 \qquad x \neq -2$

C.P. $x = -2$

$1 = 3(x+2)$

$x = -1\tfrac{2}{3}$

$-2 \qquad -1\tfrac{2}{3}$

test $x = 0 \quad \tfrac{1}{2} < 3$ true

$\therefore\ x < -2$ or $x > -1\tfrac{2}{3}$.

(d) $l_1: m_1 = 1$

$l_2: m_2 = \tfrac{1}{2}$

$\tan\theta = \left|\dfrac{1 - \frac{1}{2}}{1 + 1\times\frac{1}{2}}\right|$

$= \left|\dfrac{\frac{1}{2}}{\frac{3}{2}}\right|$

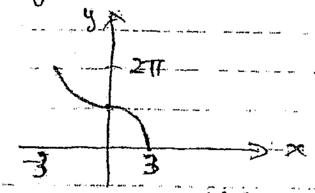
$= \tfrac{1}{3}$

$\therefore\ \theta = 18°26'$

2(a) $y = \cos^{-1}x \quad D: -1 \le x \le 1$

$R: 0 \le y \le \pi$

(i) $y = 2\cos^{-1}\frac{x}{3}$




(ii) $y = 2\cos^{-1}\frac{x}{3}$

$y' = 2 \times \dfrac{-1}{\sqrt{1 - \frac{x^2}{9}}} \times \dfrac{1}{3}$

$= \dfrac{-2}{\sqrt{9 - x^2}}$

at $x = 0 \quad y' = -\frac{2}{3}$

$\therefore\ \tan\theta = -\frac{2}{3}$

$\theta = 146°19'$

(b) $V = \pi\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

$= \dfrac{\pi}{2}\displaystyle\int_0^{\frac{\pi}{4}} (1 - \cos 2x)\,dx$

$= \dfrac{\pi}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \dfrac{\pi}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - (0)\right]$

$= \dfrac{\pi}{2} \times \left(\tfrac{\pi}{4} - \tfrac{1}{2}\right)$

$= \dfrac{\pi(\pi - 2)}{8}\ u^3$

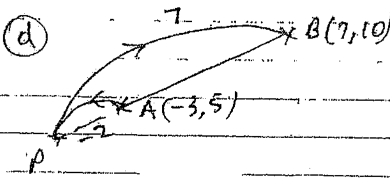
(c) $2\sin\theta - \sqrt{2} = 0$

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$= \dfrac{1}{\sqrt{2}}$

$\theta = n\pi + (-1)^n \dfrac{\pi}{4}$ where $n$ is any integer.

(d)




$x = \dfrac{-14-21}{7-2} \qquad y = \dfrac{-20+35}{7-2}$

$= -\dfrac{35}{5} \qquad = \dfrac{15}{5}$

$= -7 \qquad = 3$

$P(-7, 3)$

Q3 (a) $\frac{dV}{dt} = 72$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4\pi \times 12^2} \times 72$

$= \frac{1}{8\pi}$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 8\pi \times 12 \times \frac{1}{8\pi}$

$= 12 \text{ mm}^2/\text{sec.}$

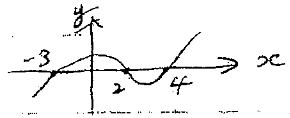
(b)

$$\begin{array}{r} x^2 - x - 12 \\ x-2 \enclose{longdiv}{x^3 - 3x^2 - 10x + 24} \\ x^3 - 2x^2 \\ -x^2 - 10x \\ -x^2 + 2x \\ -12x + 24 \\ -12x + 24 \end{array}$$

$x^3 - 3x^2 - 10x + 24 = (x-2)(x-4)(x+3)$

$x^3 + 24 > 3x^2 + 10x$

$x^3 - 3x^2 - 10x + 24 > 0$




$-3 < x < 2$ or $x > 4$

(c) $3\sin x + 2\cos x = 2$

$3\sin x + 2\cos x$

$\equiv R\sin(x+\alpha)$

$\equiv R[\sin x.\cos\alpha + \cos x \sin\alpha]$

$R = \sqrt{3^2+4^2} = \sqrt{13}$

$\cos\alpha = \frac{3}{\sqrt{13}}$

$\alpha = 33°41'$

$\therefore \sqrt{13}\sin(x+33°41') = 2$

$\sin(x+33°41') = \frac{2}{\sqrt{13}}$

$\therefore x + 33°41' = 33°41', 146°18'$ or $393°41'$

$\therefore x = 0°, 112°38'$ or $360°$

or

$3\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) = 2$

$6t + 2 - 2t^2 = 2 + 2t^2$

$2t^2 - 3t = 0$

$t(2t-3) = 0$

$t = 0$ or $\frac{3}{2}$

$\tan\frac{x}{2} = 0$ or $\frac{3}{2}$

$\frac{x}{2} = 0°, 180°$ or $56°19'$

$x = 0°, 360°$ or $112°38'$

Q4) (a)

General Term $= {}^9C_r\, x^r . \left(\frac{1}{x^2}\right)^{9-r}$

$= {}^9C_r \frac{x^r}{x^{18-2r}}$

$= {}^9C_r . x^{3r-18}$

$\therefore 3r - 18 = 0$

$r = 6$

Term is ${}^9C_6 = 84.$

(b) (i) $(2x+1)^n$

General term $= {}^nC_r (2x)^r . 1^{n-r}$

$= {}^nC_r (2x)^r$

Term in $x^4$ $r = 4$

$\therefore$ coeff $= {}^nC_4 . 2^4$

(ii) Term in $x^6$ $r = 6$

coeff $= {}^nC_4 . 2^6$

$\therefore \frac{{}^nC_4 . 2^4}{{}^nC_6 . 2^6} = \frac{5}{8}$

$\frac{\frac{n!}{4!(n-4)!}}{\frac{n!}{6!(n-6)!} . 2^2} = \frac{5}{8}$

$\frac{\cancel{n!}}{4!(n-4)!} \times \frac{6!(n-6)!}{\cancel{n!}} = \frac{4\times 5}{8}$

$\frac{6\times 5}{(n-4)(n-5)} = \frac{5}{2}$

$12 = n^2 - 9n + 20$

$n^2 - 9n + 8 = 0$

$(n-1)(n-8) = 0$

$n \neq 1 \therefore n = 8$

(c) (i) $\angle AOM = 90°$ (data)

$\angle APB = 90°$ (angle in semicircle)

$\angle APM = 90°$ (st. line)

$\therefore A, O, M, P$ as

$\angle AOM = \angle APM$

(angles in same seg on circle)

(ii)

$\angle OAP = \angle OMP$

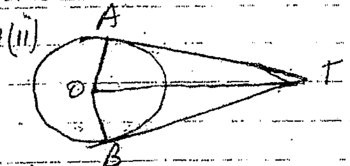
(angles in same seg)

$\angle OAP = \angle OPA$

(base angles of isos $\Delta$)

$\therefore \angle OPA = \angle OMP$

Q5 (a) (ii)




In $\triangle$'s $OAT$ & $OBT$.

$OA = OB$ (radii of circle)

$OT$ is common

$\angle OAT = \angle OBT = 90^\circ$

(tang & radius make $90^\circ$)

$\therefore$ $\triangle$'s congruent (RHS)

$\therefore$ $TA = TB$

(corresponding sides of congruent triangles)

(b) when $n=1$ $7^1 - 1$ is divisible by 6

assume true for $n = k$

i.e. $\frac{7^k - 1}{6} = M$ where $M$ is +ve integer

$\therefore 7^k = 6M + 1$

prove true for $n = k+1$

i.e. $\frac{7^{k+1} - 1}{6} = M_1$ (a +ve integer)

$LHS = \frac{7 \times 7^k - 1}{6}$

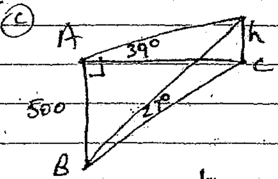
$= \frac{7 \times (6M+1) - 1}{6}$

$= \frac{42M + 7 - 1}{6}$

$= (7M + 1)$ (a positive integer)

$\therefore$ if true for $n = k$, it is true for $x = k+1$
since it is true for $n = 1$, it is true for $n = 2$
and so on for any positive integer

Q5 (c)




$\tan 39^\circ = \frac{h}{AC}$

$AC = \frac{h}{\tan 39^\circ}$

$\tan 27^\circ = \frac{h}{BC}$

$BC = \frac{h}{\tan 27^\circ}$

$BC^2 = 500^2 + AC^2$

$\frac{h^2}{\tan^2 27^\circ} = 500^2 + \frac{h^2}{\tan^2 39^\circ}$

$h^2\left(\frac{1}{\tan^2 27^\circ} - \frac{1}{\tan^2 39^\circ}\right) = 500^2$

$h = \frac{500}{\sqrt{\frac{1}{\tan^2 27^\circ} - \frac{1}{\tan^2 39^\circ}}}$

$= 327.8\ldots$

$= 328\,m$ (to nearest metre)

(d) $\frac{d}{dx}(\log_e 2x)$

$= \frac{2}{2x}$

$= \frac{1}{x}$

$\frac{d}{dx}(\log_e x)$

$= \frac{1}{x}$

$\therefore \frac{d}{dx}(\log_e 2x) = \frac{d}{dx}(\log_e x)$

No – as they differ by a constant

**Q6 (a)(i)** $y = x^2$

$\frac{dy}{dx} = 2x$

at $x = t$

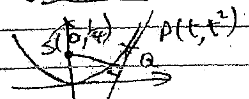
$\frac{dy}{dx} = 2t$

eqn of tang at $P$ is

$y - t^2 = 2t(x-t)$

$y = 2tx - t^2$ (2)

**(ii)**




$m_{SQ} = -\frac{1}{2t}$

eqn of SQ is

$y - \frac{1}{4} = -\frac{1}{2t}(x-0)$

$-2ty + \frac{1}{2}t = x$

$x = \frac{t}{2}(1-4y)$ — (1)

**(iii)** solving (1) & (2)

$y = \cancel{2}t \times \frac{t}{\cancel{2}}(1-4y) - t^2$

$y = t^2 - 4t^2y - t^2$

$y(1+4t^2) = 0$

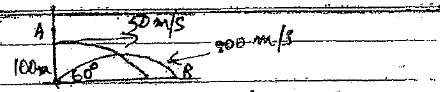
$y = 0$

$x = \frac{t}{2}$

$Q\left(\frac{t}{2}, 0\right)$

∴ locus of Q is the x axis.

**(b)**




$A: t=0, x=0, y=100, \dot{x}=50, \dot{y}=0$

$B: t=0, x=0, y=0, \dot{x}=100\cos 60° = 50$

$\dot{y} = 100\sin 60° = 50\sqrt{3}$

$A$: $\ddot{x}=0$ $\quad \ddot{y} = -10$

$\dot{x} = c_1$ $\quad \dot{y} = -10t + c_3$

$\dot{x} = 50$ $\quad c_3 = 0$

$x = 50t + c_2$ $\quad y = -5t^2 + c_4$

$c_2 = 0$ $\quad c_4 = 100$

$x = 50t$ $\quad y = -5t^2 + 100$.

$B$: $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = c_1$ $\quad \dot{y} = -10t + c_3$

$c_1 = 50$ $\quad c_3 = 50\sqrt{3}$.

$\dot{x} = 50$ $\quad \dot{y} = -10t + 50\sqrt{3}$

$x = 50t + c_2$ $\quad y = -5t^2 + 50\sqrt{3}t + c_4$

$c_2 = 0$ $\quad c_4 = 0$

$x = 50t$ $\quad y = -5t^2 + 50\sqrt{3}\,t$

to collide x & y must be equal at the same time.

$y = -5t^2 + 100 = -5t^2 + 50\sqrt{3}t$

$t = \frac{2}{\sqrt{3}}$ sec

if $t = \frac{2}{\sqrt{3}}$ $\quad x = 50 \times \frac{2}{\sqrt{3}} = \frac{100}{\sqrt{3}}$

$y = -5 \times \left(\frac{2}{\sqrt{3}}\right)^2 + 100$

$= 93\frac{1}{3}$

∴ particles collide after $\frac{2}{\sqrt{3}}$ secs at $\left(\frac{100}{\sqrt{3}}, 93\frac{1}{3}\right)$

**Q7 (a)** $\sin x = 3\cos(x + 65°)$

$\sin x = 3\cos x . \cos 65° - 3\sin x . \sin 65°$

$\sin x(1 + 3\sin 65°) = 3\cos x . \cos 65°$

$\tan x = \frac{3\cos 65°}{1 + 3\sin 65°}$ $\quad (\cos x \neq 0, x \neq 90°)$

$\therefore x = 18°50', 198°50'$

**(b)** $A = \frac{1}{2}ab\sin C$

$\frac{b}{\sin B} = \frac{a}{\sin A}$

$b = \frac{a\sin B}{\sin A}$

$\therefore$ Area $= \frac{1}{2} a \times \frac{a\sin B}{\sin A}$

$= \frac{a^2 \sin B}{2\sin A}$

**(c) (i)** $\lambda = -4$

(d)(ii) $y = \frac{x^2+5}{x+2} \quad (x \neq -2)$

$y \neq 0$ as $x^2+5 \neq 0$

$x = 0, \; y = \frac{5}{2}$

$y' = \frac{(x+2)2x - (x^2+5)\cdot 1}{(x+2)^2}$

$= \frac{2x^2+4x-x^2-5}{(x+2)^2}$

$= \frac{x^2+4x-5}{(x+2)^2}$

for max or min $y' = 0$

$x^2+4x-5 = 0$

$(x+5)(x-1) = 0$

$x = -5$ or $1$

$x = -5$ LHS $x = -6$ $y' > 0$

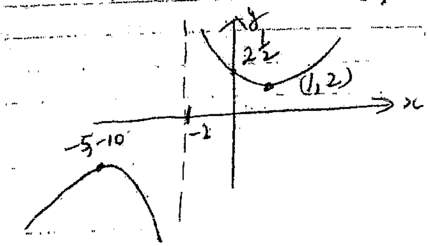
RHS $x = -4$ $y' < 0$

$\therefore$ rel max at $(-5, -10)$

$x = 1$ LHS $x = 0$ $y' < 0$

RHS $x = 2$ $y' > 0$

$\therefore$ rel min at $(1, 2)$




$\therefore \; y \geq 2$ or $y \leq -10$.

(d) $mx = \frac{2x^2+1}{2x+4}$

$2mx^2+4mx = 2x^2+1$

$x^2(2m-2)+4mx-1 = 0$

equal roots

$16m^2 - 4(2m-2)\times -1 = 0$

$16m^2+8m-8 = 0$

$2m^2+m-1 = 0$

$(2m-1)(m+1) = 0$

$\therefore \; m = \frac{1}{2}$ or $-1$.