Test 1: Extension Algebra, Trigonometry, Coordinate Geometry and Graphing

Total 40 marks (Suggested time 45 minutes)

Directions to students

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- The marks for each question are indicated at the start of the question.

QUESTION 1. (15 marks) — Marks

(a) Solve the following inequalities: **9**

(i) $\dfrac{2}{|x-1|} < 1$

(ii) $(x+3)(x-1)(2x-1) > 0$

(iii) $\dfrac{1}{2x-1} \le 2$.

(b) Solve the system of simultaneous equations: **6**

$$\left.\begin{aligned} a+2b-c&=-2\\ 2a-b-2c&=6\\ a+b-3c&=6 \end{aligned}\right\}$$

QUESTION 2. (12 marks) — Marks

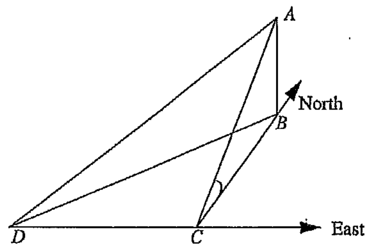
(a) A and B are the points (–5, 12) and (4, 9) respectively. Find the co-ordinates of P which divides AB externally in the ratio 5:2. **2**

(b) (i) Show that the acute angle θ between the straight lines $y = x+2$ and $y = mx+c$ is given by $\tan\theta = \left|\dfrac{m-1}{m+1}\right|$ **5**

(ii) Write down a similar result for the angle γ between the straight lines $y = 3x-1$ and $y = mx+c$.

(iii) Hence find the gradient(s) of the lines bisecting the angles between the straight lines $y = x+2$ and $y = 3x-1$.

(c) **5**



The angle of elevation of the top A of a building from a point C due south of it is 25°. At a second point D, which is 160 metres due west of C, the angle of elevation of the top of the building is 20°. Point B is the bottom of the vertical building and on the same horizontal plane as D and C.

(i) Copy and complete the diagram adding all the given information.

(ii) Find the height AB of the building to the nearest metre.

QUESTION 3. (13 marks)

Consider the function $y = \dfrac{12x}{(x-3)^2}$.

(a) What is the domain of the function? **1**

(b) Show that the graph of this function passes through the origin. **1**

(c) Determine if the function is odd or even or neither. Justify your answer. **2**

(d) Show that a minimum turning point occurs at (–3, –1). **4**

(e) What happens to the value of y as x approaches positive infinity? **1**

(f) What happens to the value of y as x approaches negative infinity? **1**

(g) Sketch the curve, showing important features including asymptote(s). **2**

(h) From the graph, determine the values of x for which the function is increasing. **1**

Test 1: Extension Algebra, Trigonometry, Coordinate Geometry and Graphing

Suggested Solutions

QUESTION 1. (15 marks)

(a) (i) $\frac{2}{|x-1|} < 1$

$2 < |x-1|$

$|x-1| > 2$

$x-1>2,\ x-1<-2$

$x>3,\ x<-1.$

Or, this could be done by taking reciprocals of both sides.

$\frac{|x-1|}{2} > 1$

$|x-1| > 2$

$x-1>2,\ x-1<-2$

$\therefore x>3,\ x<-1.$ 3

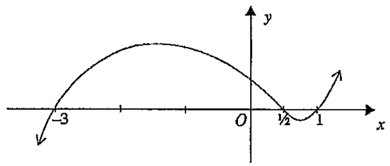
Notes: $|x-1| \neq 0 \therefore x \neq 1$

$|x-1| > 0$ so multiply by sides by a positive without reversing inequality.

Note: Reverse the inequality sign when taking reciprocals, where both sides have the same sign.

(ii) $(x+3)(x-1)(2x-1) > 0$

Solved by drawing a sketch of the corresponding cubic function $y=(x+3)(x-1)(2x-1)$.



Read the solution set from the graph for $y>0$.

$\therefore -3 < x < \frac{1}{2},\ x>1.$ 2

(iii) $\frac{1}{2x-1} \le 2,\ \left(x \neq \frac{1}{2}\right)$

Note: Denominator $(2x-1) \neq 0$

Multiply both sides by $(2x-1)^2$

$1(2x-1) \le 2(2x-1)^2$

$(2x-1) - 2(2x-1)^2 \le 0$

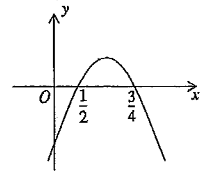
$(2x-1)[1-2(2x-1)] \le 0$

$(2x-1)(3-4x) \le 0$

Note: The sign of the inequality remains the same if both sides of the inequality are multiplied by a positive number (such as a perfect square).

Note: When using this method always take out a common factor first.

Note: To solve the quadratic inequality, sketch the graph of the corresponding quadratic function and determine the values of x for which $y \le 0$. Observe here that the x^2 term is negative.



$\therefore x < \frac{1}{2},\ x > \frac{3}{4}$ 4

Note: $x \neq \frac{1}{2}$.

ALTERNATIVE SOLUTION:

Consider $2x-1>0$ i.e. $x > \frac{1}{2}$

Note: In this solution, we multiply both sides by $(2x-1)$. We must consider the two separate cases: $2x-1>0$ and $2x-1<0$. (Denominator $2x-1 \neq 0$)

$\frac{1}{2x-1} \le 2$

$1 \le 2(2x-1)$

$1 \le 4x-2$

$4x \ge 3$

$x \ge \frac{3}{4}$

which lies completely in the domain $x > \frac{1}{2}$ (A)

Consider $2x-1<0$ i.e. $x<\frac{1}{2}$

$\frac{1}{2x-1} \le 2$

$1 \ge 2(2x-1)$

$1 \ge 4x-2$

$4x \le 3$

$x \le \frac{3}{4}$

Note: Inequality sign is reversed because we are multiplying both sides by a negative number.

But $x<\frac{1}{2}\ \therefore x<\frac{1}{2}$ (B)

Combining (A) and (B), the solution is $x<\frac{1}{2},\ x \ge \frac{3}{4}$.

Note: Alternatively, could graph $y=\frac{1}{2x-1}$ and $y=2$, solve the two equations simultaneously, and check the graph for behaviour either side of the points of intersection.

(b) $a + 2b - c = -2$ (1)

$2a - b - 2c = 6$ (2)

$a + b - 3c = 6$ (3)

Eliminating b from (2) and (3)

(2) + (3): $3a - 5c = 12$ (4)

Eliminating b from (1) and (2)

(2) × 2: $4a - 2b - 4c = 12$ (2a)

(1) + (2a): $5a - 5c = 10$

$a - c = 2$ (5)

Solving (4) and (5) by substitution.

From (5) $a = c + 2$ (5a)

Substitute (5a) into (4): $3(c + 2) - 5c = 12$

$3c + 6 - 5c = 12$

$-2c = 12 - 6$

$-2c = 6$

$c = -3$

Substitute $c = -3$ into (5a): $a = -3 + 2$

$a = -1$

Substitute $a = -1$ and $c = -3$ into (3):

$(-1) + b - 3(-3) = 6$

$-1 + b + 9 = 6$

$b = 6 - 8$

$b = -2$

$\therefore a = -1,\ b = -2,\ c = -3$ **6**

Note: The aim is to form two equations with same two unknowns. Eliminate a, b or c from two pairs of equations. In this solution, b is eliminated, forming two equations in a and c.

Note: Solving two equations with two unknowns.

QUESTION 2. (12 marks)

(a) $A(-5, 12)$ and $B(4, 9)$

Ratio: 5 : 2; external.

$$P:\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$m = 5,\ n = -2$

$x_1 = -5,\ y_1 = 12,\ x_2 = 4,\ y_2 = 9$

$$P:\left(\frac{5(4) + (-2)(-5)}{5-2}, \frac{5(9) + (-2)(12)}{5-2}\right)$$

$$P:\left(\frac{20 + 10}{3}, \frac{45 - 24}{3}\right)$$

$P:(10, 7)$ **2**

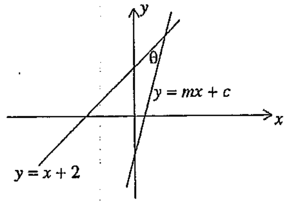
Note: This formula is often presented with different pronumerals for the ratio.

Note: The ratio is negative for external division. Common practice is to place the negative sign on the smaller ratio number.

Note: This result can also be obtained by a sketch and using similar triangles.

(b) (i) Line $y = x + 2$ has gradient 1.

Line $y = mx + c$ has gradient m.



$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\therefore \tan\theta = \left|\frac{m-1}{m+1}\right|$$ **1**

Note: This is the formula for finding the acute angle θ between two straight lines whose gradients are m_1 and m_2.

(ii) $y = 3x - 1$ and $y = mx + c$

$$\tan\gamma = \left|\frac{m-3}{1+3m}\right|$$ **1**

Note: For the line $y = 3x - 1$ the gradient is 3.

(iii) Let the line $y = mx + c$ be the bisector of the two lines $y = x + 2$ and $y = x - 3$.

Then $\theta = \gamma$ so $\tan\theta = \tan\gamma$.

$$\left|\frac{m-1}{1+m}\right| = \left|\frac{m-3}{1+3m}\right|$$

$|(m-1)(1+3m)| = |(1+m)(m-3)|$

$|3m^2 - 2m - 1| = |m^2 - 2m - 3|$

$3m^2 - 2m - 1 = -(m^2 - 2m - 3)$

or $3m^2 - 2m - 1 = m^2 - 2m - 3$

i.e. $4m^2 - 4m - 4 = 0$ or $2m^2 + 2 = 0$

$m^2 - m - 1 = 0$, $2m^2 + 2 = 0$ has no solution

$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore m = \frac{1 \pm \sqrt{5}}{2}$$ **3**

$\therefore$ the gradients are $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$.

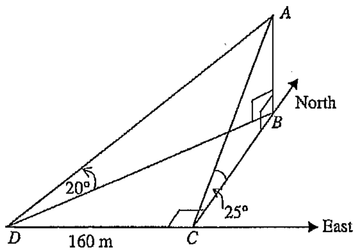
Note: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$|ab| = |a||b|$

Note: Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Note: Both solutions are relevant. One gradient applies to the bisector of the acute angle while the other applies to the bisector of the obtuse angle.

(c) (i)



1

(ii) In triangle ABC, $\frac{AB}{BC} = \tan 25°$

$$BC = \frac{AB}{\tan 25°}$$

$BC = AB \tan 65°$ ....(1)

Note: $\frac{1}{\tan 25°} = \cot 25° = \tan 65°$

In triangle ABD, $\frac{AB}{BD} = \tan 20°$

$$BD = \frac{AB}{\tan 20°}$$

$BD = AB \tan 70°$ ....(2)

Note: $\frac{1}{\tan 20°} = \cot 20° = \tan 70°$

In triangle BCD,
$BD^2 = BC^2 + CD^2$ ....(3)

Note: Right angle at C. Theorem of Pythagoras

Substitute (1) and (2) into (3):

$$AB^2 \tan^2 70° = AB^2 \tan^2 65° + 160^2$$

$$AB^2(\tan^2 70° - \tan^2 65°) = 160^2$$

$$AB^2 = \frac{160^2}{\tan^2 70° - \tan^2 65°}$$

$$AB = 93.159987...$$

$\therefore$ the building is 93 m high (nearest metre). 4

QUESTION 3. (13 marks)

$$y = \frac{12x}{(x-3)^2}$$
$$= \frac{12x}{x^2 - 6x + 9}$$

(a) Domain: all x except $x = 3$. 1

Note: Denominator cannot be zero.

(b) When $x = 0$, $y = \frac{0}{(-3)^2}$

$$= 0$$

So (0, 0) is on the curve.
Hence the graph passes through the origin. 1

Solutions to Mathematics Extension 1 Diagnostic Topic Test
Test 1: Extension Algebra, Trigonometry, Coordinate Geometry and Graphing

(c) $f(-x) = \frac{12(-x)}{(-x-3)^2}$

$$= \frac{-12x}{(x+3)^2}$$

$f(-x) \neq f(x)$ $\therefore f(x)$ is not even.
$f(-x) \neq -f(x)$ $\therefore f(x)$ is not odd.
$\therefore f(x)$ is neither even nor odd. 2

(d) $\frac{dy}{dx} = 12\left[\frac{(x-3)^2 \times 1 - x \times 2(x-3)^1 \times 1}{(x-3)^4}\right]$

$$= 12\left[\frac{(x-3)(x-3-2x)}{(x-3)^4}\right]$$

Note: Common factor of $(x-3)$ in numerator.

$$= 12\left[\frac{-(x+3)}{(x-3)^3}\right]$$

Note: Cancel $(x-3)$.

When $\frac{dy}{dx} = 0, x = -3$

When $x = -3$, $y = \frac{12(-3)}{36} = -1$

$\therefore$ stationary point at $(-3, -1)$

When $x = -3 - \varepsilon$, $\frac{dy}{dx} = 12\left[\frac{-(-3-\varepsilon+3)}{(-3-\varepsilon-3)^3}\right] = \frac{(-)(-)}{(-)} < 0$

When $x = -3 + \varepsilon$, $\frac{dy}{dx} = 12\left[\frac{-(-3+\varepsilon+3)}{(-3+\varepsilon-3)^3}\right] = \frac{(-)(+)}{(-)} > 0$

Note:

x	$-3-\varepsilon$	-3	$-3+\varepsilon$
$\frac{dy}{dx}$	\	—	/

$\therefore$ relative minimum turning point at $(-3, -1)$. 4

Note: The outcome can also be tested using the second derivative.

(e) $y = \frac{12x}{x^2 - 6x + 9}$

$$= \frac{\frac{12}{x}}{1 - \frac{6}{x} + \frac{9}{x^2}}$$

Note: Divide numerator and denominator by the highest power of x, i.e. x^2

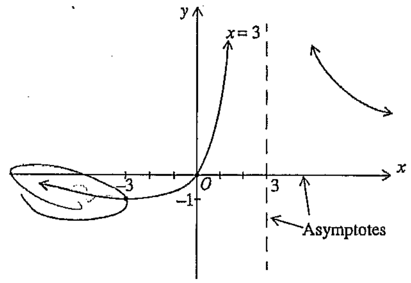
As $x \to \infty$, $y \to 0$ from the positive side. 1

Note: $y = \frac{0^+}{1 - 0 + 0} = 0$: from above because both the numerator and denominator are positive.

(f) As $x \to -\infty$, $y \to 0$ from the negative side. 1

Note: $y = \frac{0^-}{1 - 0 + 0} = 0$: from below because the numerator is negative and the denominator is positive.

(g)



2

(h) The function is increasing for $-3 < x < 3$. 1