Name: ............................................ Maths Class: ...................

# SYDNEY TECHNICAL HIGH SCHOOL

Year 11

# Mathematics Extension 2

**HSC Course**

**Assessment 1**

**November, 2015**

*Time allowed: 70 minutes*

***General Instructions:***

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Reference Formulae is provided. You are not to write in this booklet as it must be returned at the end of the examination.

| | | |
|---|---|---|
| Section 1 | Multiple Choice<br>Questions 1-5<br>5 Marks | |
| Section II | Questions 6-9<br>50 Marks | |

## SECTION II

**Use the answer booklet provided, and start each new question on a new page.**

***Allow about 63 minutes for this section.***

**QUESTION 6: (13 Marks)**

**Marks**

(a) For the complex number $6-8i$, find **4**

(i) $|z|$ (ii) $\bar{z}$ (iii) $z\bar{z}$ (iv) arg z (to the nearest minute)

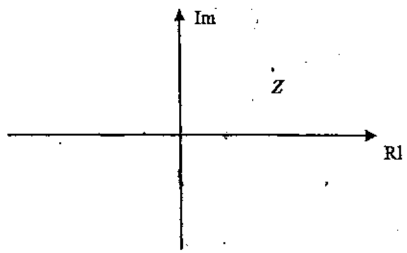
(b) Write $\frac{2-3i}{3+2i}$ in the form $a+ib$ **2**

(c) Express $\sqrt{5+12i}$ in the form $a+ib$ **3**

(d) (i) Express $z=1+i\sqrt{3}$ in mod-arg form **3**

(ii) Hence find the value of $(1+i\sqrt{3})^9$

(e) Given that Z, representing the complex number, z, lies on the Argand Diagram, as shown below, redraw the diagram and plot the point representing $i^3z$ **1**



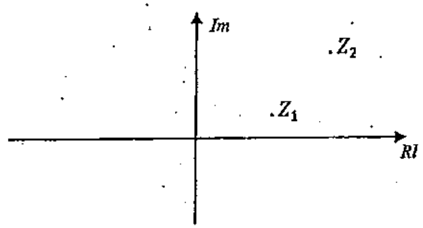
## QUESTION 7: (13 Marks) (Start a new page)

Marks

(a) Given that $Im(z^2) = 2$, plot the locus of the point Z (z) on the Argand Diagram. **2**

(b) If $x = 1$ is a double root of the equation $x^4 + ax^3 + bx^2 - 5x + 1 = 0$, Find the values of $a$ and $b$. **2**

(c) The points $Z_1$ and $Z_2$ representing the complex numbers $z_1$ and $z_2$ respectively, are shown on the Argand Diagram below. **2**




P (z) is a point which moves so that $\arg(z - z_1) = \arg(z - z_2)$. Sketch the locus of the point P.

(d) Sketch the locus given by $\arg(z - 1 - i) = \frac{\pi}{4}$ **1**

(e) Suppose that the point Z representing the complex number z, lies on the unit circle, and that

$$0 \leq \arg z \leq \frac{\pi}{4}$$

(i) Sketch the locus of Z **1**

(ii) Prove that $2\arg(z+1) = \arg z$ (give all reasons) **2**

(f) For the polynomial $4x^3 + 8x^2 + x - 3 = 0$ one root is the sum of the other 2 roots. **3**

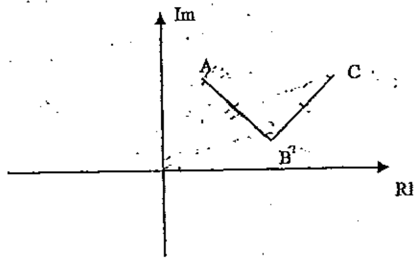
Find the values of the 3 roots.

## QUESTION 8: (13 Marks) (Start a new page)

Marks

(a) In the diagram below, A, B and C represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively

$\Delta$ ABC is isosceles and right-angled at B.




(i) Show that $(z_1 - z_2)^2 = -(z_3 - z_2)^2$ **2**

(ii) Suppose D is the point which makes ABCD a square. **1**

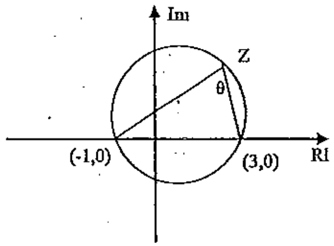
Find D in terms of $z_1$, $z_2$ and $z_3$

(b) Sketch the locus of the point Z representing the complex number z, if **3**

$$z\bar{z} + 2(z + \bar{z}) \leq 0$$

*QUESTION 8 continues over......)*

(c) Z moves on a circle which passes through the points (-1, 0) and (3, 0) as shown below. **3**




It is given that $arg\left[\frac{z-3}{z+1}\right] = \frac{\pi}{3}$.

Find the value of θ and the y-value of the centre of the circle.

(d) P represents the complex number z, where $|z-2| = 1$

If O is the origin, find:

(i) the minimum distance OP? **1**

(ii) the maximum distance OP? **1**

(iii) the largest value of arg z ? **2**

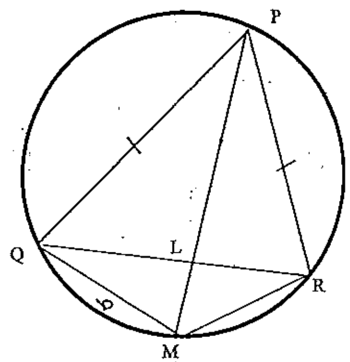
## QUESTION 9: 11 Marks (Start a New Page)

(a) (i) For $k \geq 1$, prove that $\frac{1}{k} - \frac{1}{k+1} > \frac{1}{(k+1)^2}$ **2**

(ii) Prove, by the process of mathematical induction, that for $n \geq 1$, **4**

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots \ldots\ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

(b)




PQR is an isosceles triangle with PQ = PR.

L is the point of intersection of the diagonals of the cyclic quadrilateral PRMQ.

(i) Prove that $\Delta QLM$ is similar to $\Delta PLR$ **2**

(ii) Show that $QM \times LR = LM \times PR$ **1**

(iii) It can be further proved that $\Delta QLP$ is similar to $\Delta MLR$ which leads to the statement that $MR \times QL = QP \times LM$ **2**
*(You do not have to prove this. It may be assumed for the next section.)*

Prove that:

$$\frac{1}{QM} + \frac{1}{MR} = \frac{QR}{LM \times PR}$$

## SOLUTIONS

### Questions 1 to 5

1/ C  2/ A  3/ D  4/ A  5/ C

### Question 6:

(a) (i) 10  (ii) $6+8i$  (iii) 100

(iv) $\tan\theta = -8/6$

$\theta = -53^\circ 8'$ (or $306^\circ 52'$)

(b) $\frac{2-3i}{3+2i} \times \frac{3-2i}{3-2i} = -i$

(c) $x^2 - y^2 + 2ixy = 5 + 12i$

$x^2 - y^2 = 5$ and $xy = 6$

$y = 6/x$

$\therefore x^2 - (6/x)^2 = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$x = \pm 3$ or $x = \pm 2i$

if $x = 3$, $y = 2$

$\sqrt{5+12i} = \pm(3+2i)$

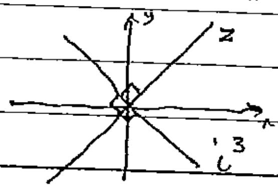
(d) (i) $r = 2$, $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

$\therefore z = 2\,\text{cis}\,\pi/3$

(ii) $z^9 = 2^9\,\text{cis}\,3\pi$

$= -512$ or $512\,\text{cis}\,\pi$

(e)



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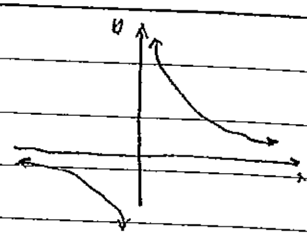
### Question 7

(a) $\text{Im}(z^2) = 2$

$\therefore \text{Im}(x^2 - y^2 + 2ixy) = 2$

$xy = 1$



(b) $P(x) = x^4 + ax^3 + bx^2 - 5x + 1$

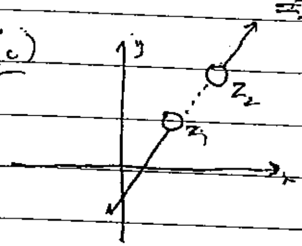
$P'(x) = 4x^3 + 3ax^2 + 2bx - 5$

Now $P(1) = 0 \Rightarrow a + b = 3$

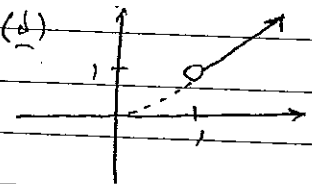
$P'(1) = 0 \Rightarrow 3a + 2b = 1$

$\Rightarrow a = -5 \quad b = 8$

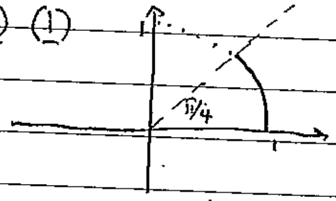
(c)



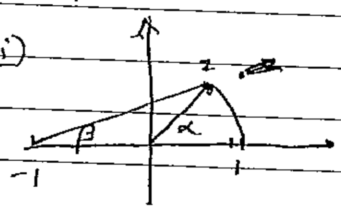

(d)




(e) (i)




(ii)




Let $\arg z = \alpha$ and $\arg(z+1) = \beta$

The triangle formed is isosceles (because z is on unit circle)

$\therefore \alpha = 2\beta$ (exterior angle theorem)

(f) $\gamma = \alpha + \beta$

Sum of Roots $= 2(\alpha+\beta) = -2$

$\therefore \alpha + \beta = -1$ (1)

Product: $\alpha\beta(\alpha+\beta) = 3/4$

$\therefore \alpha\beta = -3/4$ (2)

Solving (1) and (2) gives

$\alpha = 1/2$ or $\alpha = -3/2$

$\therefore$ Roots are $1/2, -3/2, -1$

QUESTION 8:

(a) (i) $z_1 - z_2 = i(z_3 - z_2)$  (ii) $z_1 + z_3 - z_2$

$(z_1 - z_2)^2 = i^2(z_3 - z_2)^2$

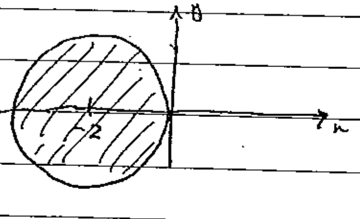
$= -(z_3 - z_2)^2$

(b) Let $z = x + iy$

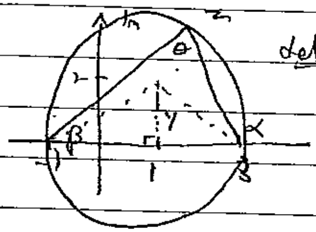
$\bar{z} = x - iy$  $z\bar{z} + 2(z + \bar{z}) \le 0$

$\Rightarrow x^2 + y^2 + 4x \le 0$

$\therefore (x+2)^2 + y^2 \le 1$




(c)




Let $\arg(z - 3) = \alpha$

and $\arg(z + 1) = \beta$

$\therefore \arg(z-3) - \arg(z+1) = \theta$

$\therefore \arg\left(\frac{z-3}{z+1}\right) = \theta$

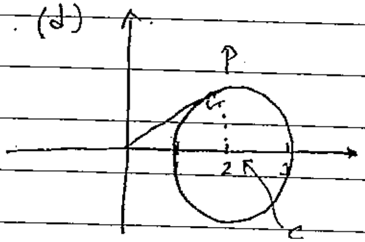
$\therefore \theta = \frac{\pi}{3}$

(ii) $y/2 = \frac{1}{\tan \pi/3}$

because the y-line bisects the angle AND that angle is $2\theta$ (angle at the centre is twice that on the circumference)

$\therefore y = 2/\sqrt{3}$

(d)




(i) MIN OP = 1

(ii) MAX OP = 3

(iii) Largest value of arg z is when OP is a tangent. This means that $OP \perp PC$. $\therefore PC = 1$ and OC is 2

If $\arg z = \theta$

$\sin\theta = 1/2 \Rightarrow \theta = \pi/6$.





QUESTION 9:

(a) (i) $\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)}$

$= \frac{1}{k^2+k}$

$> \frac{1}{(k+1)^2}$ because $k^2 + 2k + 1 > k^2 + k$

(ii) For $n=1$ LHS = 1, RHS = 1

~~∴ LHS = RHS ∴ trivial (proven)~~

For $n=2$ LHS = $5/4$ RHS = $2 - 1/2$

$= 3/2$

∴ LHS < RHS

∴ the formula is true for $n = 1, 2$

Assume the formula is true for $n = k$

$\therefore 1 + 1/4 + 1/9 + \ldots + 1/k^2 \le 2 - 1/k$

For $n = k+1$

$1 + 1/4 + 1/9 + \ldots + 1/k^2 + \frac{1}{(k+1)^2} \le 2 - 1/k + \frac{1}{(k+1)^2}$

$= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)$

$< 2 - \frac{1}{k+1}$

from the result in part (i)

Which is of the same form as for $n = k$

∴ If it is true for $n = k$, it is true for $n = k+1$

But the formula is true for $n = 2$.

∴ " " " " " $n = 3$

and so on.

∴ true for all $n$.

QUESTION 9 (b)

(i) In $\triangle QLM$ and $\triangle PLR$

$\angle MQL = \angle LPR$ (angles at the circumference standing on the arc MR)

$\angle QLM = \angle PLR$ (vertically opposite angles)

$\therefore \triangle QLM \parallel\!\mid \triangle PLR$ (equiangular)

(ii) $\dfrac{QM}{LM} = \dfrac{PR}{LR}$ (corresponding sides in similar triangles are in ratio)

$\therefore QM \times LR = LM \times PR$ (1)

(iii) Similarly $MR \times QL = QP \times LM$ (2)

From (1) $QM = \dfrac{LM \times PR}{LR}$

From (2) $MR = \dfrac{QP \times LM}{QL}$

$\therefore \dfrac{1}{QM} + \dfrac{1}{MR} = \dfrac{LR}{LM.PR} + \dfrac{QL}{LM.PQ}$

and since $PQ = PR$.

$\therefore \dfrac{1}{QM} + \dfrac{1}{MR} = \dfrac{LR + QL}{LM.PR}$

$= \dfrac{QR}{LM.PR}$