CRANBROOK SCHOOL

YEAR 12 EXT1 MATHEMATICS – TEST

5th June, 2006 Circle teacher: CJL JJA SKB

– **Trigonometric functions (non-calculus and calculus)** **Time: 50mins**
- **Inverse functions (non-calculus and calculus)**

All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Begin each question on a new page.

1. **(12 marks) (Begin a new page)** **JJA**

(a) Simplify the following: $2\cos\left(\frac{\pi}{4}+\phi\right)\cos\left(\frac{\pi}{4}-\phi\right)$ 3

(b) Find the general solution for: $\sqrt{2}\sin x = 1$ 2

(c) Find, in exact form, the solution of: $\sin\phi + \sqrt{2}\cos\phi = -\sqrt{3}$ for $0 \le \phi \le 2\pi$ 3

(d) Evaluate $\lim_{x\to 0}\left[\frac{\sin 3x}{x} + \frac{6x}{\tan 3x}\right]$ 2

(e) Sketch for $0 \le x \le 2\pi$ $y = 3\sin\left(x - \frac{\pi}{2}\right)$ 2

2. **(12 marks) (Begin a new page)** **CJL**

(a) Find $\frac{d}{dx}\left[\cos^4 x\right]$ 1

(b) (i) Differentiate $e^{3x}(\cos x - 3\sin x)$ 2

(ii) Hence, or otherwise, find $\int e^{3x}\sin x\,dx$ 1

(c) Find the area bounded by the curve $y = \cos^2 3x$, the x and y axes and the line $x = \frac{\pi}{18}$ in exact form. 3

(d) Find the equation of the normal to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$ 3

(e) Find the exact value of $\int_{0.5}^{1} \sin \pi x\, dx$ 2

3. **(12 marks) (Begin a new page)** **SKB**

(a) Find the inverse function $f^{-1}(x)$ if $f(x) = 1 + \sin x$, $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

Hence sketch $y = f^{-1}(x)$ stating its domain. 3

(b) Find, showing all necessary working, the exact value of $\cos(2\tan^{-1}\frac{-1}{3})$. 3

(c) **(i)** Sketch $y = \sin^{-1}(\cos 3x), 0 \le x \le \pi$. 2

(ii) For what values of x is $\frac{dy}{dx}$ undefined, $0 \le x \le \pi$? 1

(d) The area bounded by the curve $y = \frac{1}{\sqrt{4+5x^2}}$, the x and y axes and the line $x = 1$ is rotated about the x-axis. Find the volume generated in exact form. 3

1) (a) $2\cos\left(\frac{\pi}{4}+\phi\right)\cos\left(\frac{\pi}{4}-\phi\right)$

$= 2\left[\cos\frac{\pi}{4}\cos\phi - \sin\frac{\pi}{4}\sin\phi\right] \times \left[\cos\frac{\pi}{4}\cos\phi + \sin\frac{\pi}{4}\sin\phi\right]$

$= 2\left[\left(\cos\frac{\pi}{4}\cos\phi\right)^2 - \left(\sin\frac{\pi}{4}\sin\phi\right)^2\right]$

$= 2\left[\frac{1}{2}\left(\cos^2\phi - \sin^2\phi\right)\right]$

$= \cos 2\phi$

(b) $\sqrt{2}\sin x = 1$

$\therefore \sin x = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\therefore x = n\pi + (-1)^n\frac{\pi}{4}$, n is any integer.

(c) $\sin\phi + \sqrt{2}\cos\phi = -\sqrt{3}$

$\therefore \sqrt{3}\left(\frac{1}{\sqrt{3}}\sin\phi + \frac{\sqrt{2}}{\sqrt{3}}\cos\phi\right) = -\sqrt{3}$

$\therefore \sin(\phi+\alpha) = -1$

where $\cos\alpha = \frac{1}{\sqrt{3}}$, $\sin\alpha = \frac{\sqrt{2}}{\sqrt{3}}$

$\therefore \tan\alpha = \sqrt{2}$

$\therefore \alpha = \tan^{-1}(\sqrt{2})$, α is acute.

$\therefore \sin(\phi + \tan^{-1}(\sqrt{2})) = -1$

$\therefore \phi + \tan^{-1}(\sqrt{2}) = -\frac{\pi}{2}$

$\therefore \phi = -\frac{\pi}{2} - \tan^{-1}(\sqrt{2}) + 2\pi$

$\therefore \phi = \frac{3\pi}{2} - \tan^{-1}(\sqrt{2})$, for $0 \le \phi \le 2\pi$.

(d) $\lim_{x\to 0}\left[\frac{\sin 3x}{x} + \frac{6x}{\tan 3x}\right]$

$= \lim_{x\to 0}\left[\frac{\sin 3x}{3x}\cdot\frac{3}{1} + \frac{3x}{\tan 3x}\cdot\frac{2}{1}\right]$

$= 1\cdot 3 + 1\cdot 2$

$= 5$

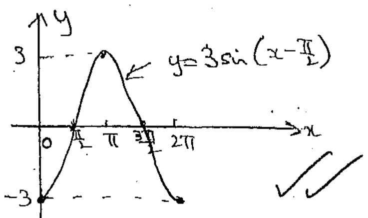
(e) $y = 3\sin\left(x - \frac{\pi}{2}\right)$

$= 3\sin\left[-\left(\frac{\pi}{2} - x\right)\right]$

$= -3\sin\left(\frac{\pi}{2} - x\right)$

$= -3\cos x$

$\therefore$ Period $= 2\pi$
sub-interval width $= \frac{\pi}{2}$



2 (a) $\frac{d}{dx}\left[\cos^4 x\right] = 4\cos^3 x \cdot -\sin x$
$= -4\sin x\cos^3 x$

(b) (i) Let $y = e^{3x}(\cos x - 3\sin x)$

$\therefore \frac{dy}{dx} = e^{3x}(-\sin x - 3\cos x) + (\cos x - 3\sin x)\cdot 3e^{3x}$

$= e^{3x}(-\sin x - 3\cos x + 3\cos x - 9\sin x)$

$= e^{3x}(-10\sin x)$

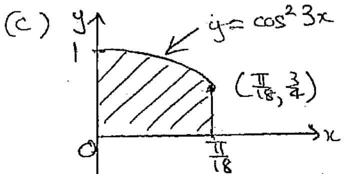
$= -10e^{3x}\sin x$

(ii) $I = \int e^{3x}\sin x\, dx$

$= -\frac{1}{10}\int -10e^{3x}\sin x\, dx$

$= -\frac{1}{10}e^{3x}(\cos x - 3\sin x) + c$

(c)



$\therefore \text{Area} = \int_0^{\frac{\pi}{18}} \cos^2 3x\, dx$

$\cos 2x = 2\cos^2 x - 1$
$\therefore \cos^2 x = \frac{1}{2}[1 + \cos 2x]$
$\therefore \cos^2 3x = \frac{1}{2}[1 + \cos 6x]$

$= \frac{1}{2}\int_0^{\frac{\pi}{18}}(1 + \cos 6x)\, dx$

$= \frac{1}{2}\left[x + \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{18}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{18} + \frac{\sin\frac{\pi}{3}}{6}\right) - 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{18} + \frac{\sqrt{3}}{12}\right]$

$= \frac{1}{2}\left[\frac{2\pi + 3\sqrt{3}}{36}\right]$

$= \frac{2\pi + 3\sqrt{3}}{72}$ units^2

1) $\because y = \tan x$ $\therefore \frac{dy}{dx} = \sec^2 x$

at $\left(\frac{\pi}{4}, 1\right)$ $\frac{dy}{dx} = \sec^2\frac{\pi}{4} = 2 = m_{tang}$.

$\therefore m_{norm} = -\frac{1}{2}$

$\therefore$ Eqn of req'd normal is: $y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$

$\therefore 2y - 2 = -x + \frac{\pi}{4}$

$\therefore x + 2y - 2 - \frac{\pi}{4} = 0$

(e) $I = \int_{0.5}^{1} \sin\pi x\, dx$

$= \left[\frac{-\cos\pi x}{\pi}\right]_{0.5}^{1}$

$= -\frac{1}{\pi}\left[\cos\pi x\right]_{0.5}^{1}$

$= -\frac{1}{\pi}\left[\cos\pi - \cos\frac{\pi}{2}\right]$

$= -\frac{1}{\pi}[-1 - 0]$

$= \frac{1}{\pi}$

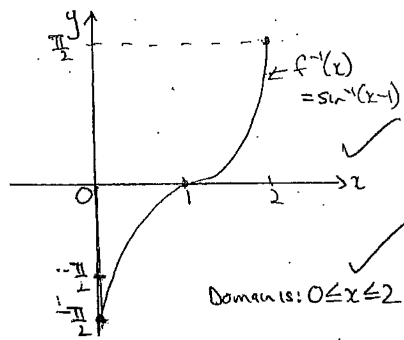
3 (a) Let $y = 1 + \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\therefore$ range is: $0 \le y \le 2$.

For inverse function interchange x for y

$\therefore x = 1 + \sin y$ $\therefore y = \sin^{-1}(x-1)$

$\therefore f^{-1}(x) = \sin^{-1}(x-1)$.



Domain is: $0 \le x \le 2$

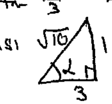
(b) Let $E = \cos\left(2\tan^{-1}\frac{-1}{3}\right)$

$= \cos\left(-2\tan^{-1}\frac{1}{3}\right)$

$= \cos\left(2\tan^{-1}\frac{1}{3}\right)$

Let $\alpha = \tan^{-1}\frac{1}{3}$ $\therefore \tan\alpha = \frac{1}{3}$

Basic Triad is:



$\therefore E = \cos 2\alpha = 2\cos^2\alpha - 1$

$\therefore E = 2\left(\frac{[illegible]}{\sqrt{10}}\right) - 1 = \frac{[illegible]}{5}$

(c) (i) $y = \sin^{-1}(\cos 3x)$, $0 \le x \le \pi$

$\therefore y' = \frac{1}{\sqrt{1 - \cos^2 3x}}\cdot -3\sin 3x$

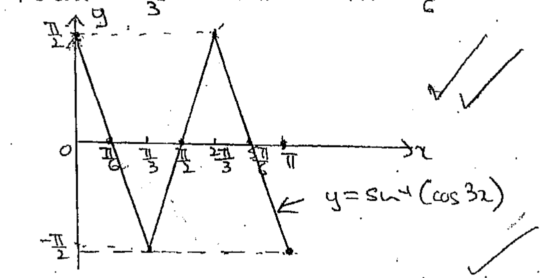
$= \frac{-3\sin 3x}{\sqrt{\sin^2 3x}}$

$= \frac{-3\sin 3x}{|\sin 3x|}$

$= -3$ if $|\sin 3x| = \sin 3x$

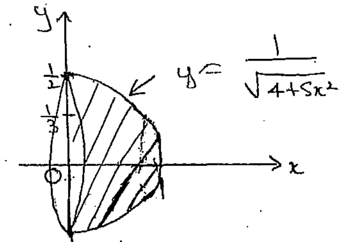
or 3 if $|\sin 3x| = -\sin 3x$

Period $= \frac{2\pi}{3}$ $\therefore$ sub-interval width $= \frac{\pi}{6}$



(ii) $\frac{dy}{dx}$ is undefined when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$.

(d)



$\text{Volume} = \pi\int_0^1 \left(\frac{1}{\sqrt{4+5x^2}}\right)^2 dx$

$= \pi\int_0^1 \frac{dx}{4+5x^2}$

$= \pi\int_0^1 \frac{dx}{5\left(\frac{4}{5}+x^2\right)}$

$= \frac{\pi}{5}\int_0^1 \frac{dx}{\left(\frac{2}{\sqrt{5}}\right)^2 + x^2}$

$= \frac{\pi}{5}\cdot\frac{1}{2/\sqrt{5}}\left[\tan^{-1}\left(\frac{\sqrt{5}x}{2}\right)\right]_0^1$

$= \frac{\sqrt{5}\pi}{10}\left[\tan^{-1}\left(\frac{\sqrt{5}}{2}\right) - 0\right]$

$= \frac{\sqrt{5}\pi}{10}\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$ units^3