Year 12 Extension 1 Mathematics

Calculus with Trigonometry, Inverse Functions

Term 2, 2010 | Week 5 Tue, 18 May *Time Allowed*: 50 mins *Marks*: 45

Show all working to gain maximum marks

Marks will be deducted for poorly presented or illegible work

Question 1 (15 marks) **Start a new booklet** **Marked by HRK**

a) Prove using t results, where $t=\tan\frac{\theta}{2}$ this result:

$$\frac{\cos\theta-\sin\theta+1}{\cos\theta+\sin\theta-1}=\cot\frac{1}{2}\theta$$ 2

b) Find the general solution of $\sin x+\sqrt{3}\cos x=1$ 3

c) Find $\frac{d}{dx}\left(\frac{1}{\sqrt{\sin^3 4x}}\right)$ 2

d) Find $\frac{d}{dx}\left(\cot^2\left(\frac{\pi}{2}-2x\right)\right)$ 2

e) Find the exact value of $\cos\frac{7\pi}{12}$ 2

f) Evaluate $\int_0^2\sqrt{4-x^2}\,dx$, using the substitution $x=2\sin\theta$. 4

Question 2 (15 marks) **Start a new booklet** **Marked by DWH**

a) Evaluate $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$, leaving your answer in exact form. 2

b) Evaluate $\sin^{-1}\left(2\sin\frac{\pi}{6}\right)$ 2

c) Using an appropriate compound-angle formula, evaluate

$$\cos\left(\sin^{-1}\left(\frac{8}{17}\right)+\tan^{-1}\left(\frac{4}{3}\right)\right)$$ 3

d) Consider $y=2\sin^{-1}x$

(i) State the domain and range of this function. 2

(ii) Sketch the curve. 3

e) Evaluate the domain and range of the function $y=2\sin^{-1}\left(\frac{x^2-10}{6}\right)$ 3

Question 3 (15 marks) **Start a new booklet** **Marked by RPN**

a) Differentiate the following functions with respect to x:

(i) $f(x)=\cos^{-1}2x$ 2

(ii) $f(x)=\tan^{-1}(e^x)$ 2

b) Evaluate $\int_0^{\frac{1}{4}}\frac{dx}{\sqrt{1-4x^2}}$, leaving your answer in exact form. 3

c) Consider the tangent to the curve $y=x^2\tan^{-1}x$ where $x=1$.

(i) Find $\frac{d}{dx}\left(x^2\tan^{-1}x\right)$ 2

(ii) Find the gradient of the tangent where $x=1$. 1

(iii) Find the y-intercept of this tangent. 2

d) Find $\int\frac{x+1}{x^2+1}dx$ 3

(First express the integral with two separate fractions.)

1(a) ② $\checkmark = 1$ mark

LHS

$$= \frac{\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + \frac{1+t^2}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - \frac{1+t^2}{1+t^2}} \times \frac{1+t^2}{1+t^2} \quad \checkmark$$

$$= \frac{1 \cancel{-t^2} - 2t + 1 \cancel{+t^2}}{\cancel{1} - t^2 + 2t \cancel{-1} - t^2}$$

$$= \frac{2-2t}{2t-2t^2}$$

$$= \frac{2(\cancel{1-t})}{2t(\cancel{1-t})}$$

$$= \frac{1}{t} \quad \checkmark$$

$$= \cot\frac{\theta}{2}$$

$= \text{RHS}$

Knowledge of t results was good BUT more care needed with fractions + algebra!

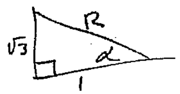
b) ③ 1ST RESTATE LHS

$1\sin x + \sqrt{3}\cos x = R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

EQUATING COEFFICIENTS OF $\sin x$, $\cos x$

$1 = R\cos\alpha \qquad \sqrt{3} = R\sin\alpha$

$\therefore \cos\alpha = \frac{1}{R} \qquad \sin\alpha = \frac{\sqrt{3}}{R}$



$R = \sqrt{3+1} = 2$

$\sin\alpha = \frac{\sqrt{3}}{2} \quad \therefore \alpha = \frac{\pi}{3}$

(OR use $\cos\alpha$ or $\tan\alpha$)

THEN $\sin x + \sqrt{3}\cos x = 1$ ✓

can be written as

$$2\sin\left(x + \frac{\pi}{3}\right) = 1 \quad \checkmark$$

ie $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ ✓

GEN SOLUTION

$$\left(x + \frac{\pi}{3}\right) = n\pi + (-1)^n\frac{\pi}{6}$$

NOTE LAST STEP IS TO SUBTRACT THE $\frac{\pi}{3}$

$$x = n\pi + (-1)^n\frac{\pi}{6} - \frac{\pi}{3}$$

(c) ② $y = \dfrac{1}{\sqrt{\sin^3 4x}}$

$$= (\sin 4x)^{-\frac{3}{2}} \quad \checkmark$$

$$y' = -\frac{3}{\cancel{2}}(\sin 4x)^{-\frac{5}{2}}\cos 4x \times \cancel{4}^{2}$$

$$= \frac{-6\cos 4x}{(\sin 4x)^{5/2}} \quad \text{OR} \quad \frac{-6\cos 4x}{\sqrt{\sin^5 4x}} \quad \checkmark$$

Time Time Time !!! ☹ ....IS PRECIOUS especially in 3U(EXT) TASKS

USE INDEX LAWS ☺

PLEEEESE learn from this!

(d) ② $\dfrac{d}{dx}\cot^2\left(\frac{\pi}{2} - 2x\right)$

Recall $\cot(90° - x) = \tan x$ CO-RATIOS

$$= \frac{d}{dx}\tan^2 2x \quad \checkmark$$

and simple chain rule

$$= \frac{d}{dx}(\tan 2x)^2$$

$$= 2(\tan 2x)^1 \times \sec^2 2x \times 2 \quad \checkmark$$

$$= 4\tan 2x\sec^2 2x$$

(Don't regret all those time wasting lines(pages) just learn from this too! ☺

e) ② $\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$ ✓

$\frac{3\pi}{12} + \frac{4\pi}{12}$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}} \quad \text{OR} \quad \frac{\sqrt{2}-\sqrt{6}}{4} \quad \checkmark$$

f) ④ $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

$x=0$: $0 = 2\sin\theta$, $\theta = 0$
$x=2$: $2 = 2\sin\theta$, $\theta = \frac{\pi}{2}$ ✓

$x = 2\sin\theta$
$\frac{dx}{d\theta} = 2\cos\theta$
$dx = 2\cos\theta\,d\theta$ ✓

$$= \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\;2\cos\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \quad \checkmark$$

$$= \cancel{4}^{2}\int_0^{\frac{\pi}{2}} \tfrac{1}{\cancel{2}}(1+\cos 2\theta)\,d\theta$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \quad \checkmark$$

$$= \pi$$

② (a) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

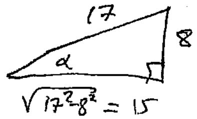
(note range of $\sin^{-1}$ fn is restricted to $\left(-\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}\right)$ and students lost 1 mark for answers in the incorrect domain)

(b) $\sin^{-1}\left(2\sin\frac{\pi}{6}\right) = \sin^{-1}\left(2\times\frac{1}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2}$.

(careful with range of $\sin^{-1}x$ again)

(c) $\cos\left(\sin^{-1}\left(\frac{8}{17}\right) + \tan^{-1}\left(\frac{4}{3}\right)\right)$

let $\alpha = \sin^{-1}\left(\frac{8}{17}\right)$



$= \cos(\alpha+\beta)$

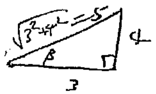
$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{15}{17}\times\frac{3}{5} - \frac{8}{17}\times\frac{4}{5}$

$= \frac{45-32}{85} = \frac{13}{85}$

$\sqrt{17^2-8^2} = 15$

let $\beta = \tan^{-1}\left(\frac{4}{3}\right)$



(d) $y = 2\sin^{-1}(x)$

(i) domain: $-1 \le x \le 1$

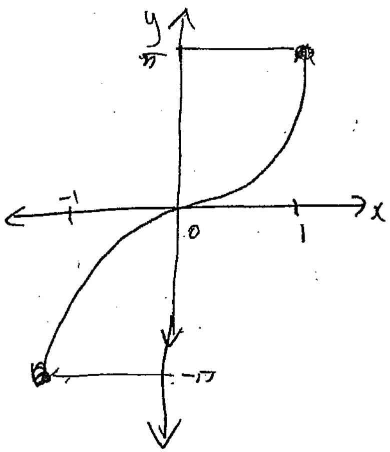
range: $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$-\pi \le 2\sin^{-1}x \le \pi$

$\therefore -\pi \le y \le \pi$

(fairly well done this qn)

(ii)



(these are 3 very easy marks!)

(e) $y = 2\sin^{-1}\left(\frac{x^2-10}{6}\right)$

$-\pi \le y \le \pi$ (1 easy mark.

$-1 \le \frac{x^2-10}{6} \le 1$

$-6 \le x^2-10 \le 6$

$4 \le x^2 \le 16$ (1 mark for correct inequality)

$2 \le x \le 4$

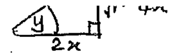
OR

$-4 \le x \le -2$ (the tricky third mark!

3. (a) (i) $f(x) = y = \cos^{-1}(2x)$ ...

⑮

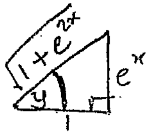
$-\sin y \frac{dy}{dx} = 2 \quad \therefore \frac{dy}{dx} = \frac{-2}{\sin y}$



Hence $f'(x) = \frac{-2}{\sqrt{1-4x^2}}$ [2]

(ii) $f(x) = y = \tan^{-1}(e^x) \quad \therefore \tan y = e^x$

$\sec^2 y \frac{dy}{dx} = e^x$ so $\frac{dy}{dx} = \cos^2 y (e^x)$



Hence $f'(x) = \frac{e^x}{1+e^{2x}}$ [2]

Many forgot to multiply by e^x

(b) $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} = \int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-(2x)^2}}$ let $u = 2x$, $du = 2dx$

many didn't do this!

$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\left[\sin^{-1}u\right]_0^{\frac{1}{2}} = \frac{1}{2}\left(\frac{\pi}{6}-0\right) = \frac{\pi}{12}$ [3]

(c) (i) $y = x^2\tan^{-1}(x) \quad \frac{dy}{dx} = 2x\tan^{-1}(x) + x^2\left(\frac{1}{1+x^2}\right)$ [2]

(ii) at $x=1$, $\frac{dy}{dx} = 2.\frac{\pi}{4} + \frac{1}{2} = \frac{1}{2}(\pi+1)$ [1]

(iii) $y - y_1 = m(x-x_1)$ and $y_1 = 1^2.\frac{\pi}{4} = \frac{\pi}{4}$

$y - \frac{\pi}{4} = \frac{1}{2}(\pi+1)(x-1)$ is eqn. of tangent.

at $x=0$, $y = -\frac{1}{2}(\pi+1) + \frac{\pi}{4} = -\frac{\pi}{4} - \frac{1}{2}$ $\therefore$ y intercept is $\left(0, -\frac{(2+\pi)}{4}\right)$ [2]

You should write a y-intercept as coordinates!

(d) $\int \frac{x+1}{x^2+1}dx = \frac{1}{2}\int\frac{2x}{x^2+1}dx + \int\frac{1}{x^2+1}dx = \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C$ [3]

Very few got this right!

ln| | Not ln( )