Year 12 Extension 1 Mathematics: *Trig & Inverse Fns & Inverse Trig* Term 2, 20

Week 5 . HSC TASK #2

Time Allowed: 50 minutes | Total Marks: 40 Name: _______

1. **Use a new booklet** **Marked by GHW** **16 marks**

(a) Multiple Choice: Write down the correct alternative on your booklet. 5

(i) If $f(x) = \frac{x+4}{7}$, the inverse $f^{-1}(x)$ of $f(x)$ is:

A. $f^{-1}(x) = \frac{4}{x+7}$ B. $f^{-1}(x) = \frac{4x+7}{4}$ C. $f^{-1}(x) = 4x+7$ D. $f^{-1}(x) = 7x-4$

(ii) $\cos^{-1}(-x) =$

A. $\cos^{-1}(x)$ B. $\sin^{-1}(x)$ C. $-\cos^{-1}(x)$ D. $\pi - \cos^{-1}(x)$

(iii) If $y = \sin^{-1}(\sqrt{1-x^2})$ then $\frac{dy}{dx} =$

A. $-x\sqrt{1-x^2}$ B. $\frac{-x}{\sqrt{1-x^2}}$ C. $\frac{2x}{\sqrt{1-x^2}}$ D. $\frac{-1}{\sqrt{1-x^2}}$

(iv) The evaluation of $\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$ is

A. $\frac{\pi}{6}$ B. $\frac{\pi}{4}$ C. $\frac{2\pi}{3}$ D. $\frac{3\pi}{2}$

(v) The derivative of $y = x\ln(\tan^{-1}x)$ is

A. $\frac{x}{1+x^2} + \ln(\tan^{-1}x)$ B. $\frac{x(1+x^2)}{\tan^{-1}x} + \ln(\tan^{-1}x)$

C. $\frac{x}{(1+x^2)\tan^{-1}x} + \ln(\tan^{-1}x)$ D. $\frac{x(1+x^2)}{\tan^{-1}x} + x\ln(\tan^{-1}x)$

(b) (i) Write down the result for $\sin 2x$ in terms of $\sin x$ and $\cos x$. 1

(ii) Hence show that $\sin(\frac{\pi}{4} - x)\cos(\frac{\pi}{4} - x) = \frac{1}{2}\cos 2x$. 2

(c) Solve: $\sin\theta - \sqrt{3}\cos\theta = 1$, for $0 \le \theta \le 2\pi$ 3

(d) Find the general solutions of: $\sin^2\theta - \frac{1}{2}\sin\theta = 0$. 3

(e) Evaluate $\int_0^{\frac{\pi}{6}} 3\sin^2 x\, dx$. 2

2. **Use a new booklet** **Marked by RPN** **12marks**

(a) If $f(x) = \log_e(x-1)$

(i) Find $f^{-1}(x)$

(ii) Show that $f(x)$ and $f^{-1}(x)$ are mutually inverse functions,

i.e. $f(f^{-1}(x)) = f^{-1}(f(x))$ 3

(b) If the domain of $y = x^2 - 4x$ is restricted to a monotonic increasing curve:

(i) sketch $y = f(x)$ 2

(ii) find the inverse function $y = f^{-1}(x)$ 2

(iii) state the domain and range of the inverse function 1

(c) Evaluate: (i) $\cos^{-1}(-\frac{1}{\sqrt{2}})$ 1

(ii) $\tan(\sin^{-1}(\frac{\sqrt{3}}{2}))$ 1

(d) Find the exact value of $\sin(2\tan^{-1}\frac{12}{13})$ 2

3. **Use a new booklet** **Marked by HRK** **8marks**

(a) Sketch $y = 4\sin^{-1}(\frac{x}{3})$ 2

(b) Find $\frac{d}{dx}(\cos^{-1}8x)$ 1

(c) Evaluate: $\int_{-3}^{0} \frac{dx}{9+x^2}$ 2

(d) Differentiate $y = \sin^{-1}(5x^2)$ with respect to x. 1

(e) Find $\int \frac{dx}{\sqrt{49-4x^2}}$. 2

Start here.

1 (a) (i) let $y = \frac{x+4}{7}$

for inverse fn interchange x for y:

$\therefore x = \frac{y+4}{7}$

$\therefore y = 7x-4$

$\therefore f^{-1}(x) = 7x-4 \quad \therefore D$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x) \quad \therefore D$

(iii) $y = \sin^{-1}(\sqrt{1-x^2})$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$

$= \frac{-x}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-(1-x^2)}}$

$= \frac{-x}{\sqrt{1-x^2} \cdot x}$

$= \frac{-1}{\sqrt{1-x^2}} \quad \therefore D$

(iv) $\int_0^{2\sqrt{3}} \frac{dx}{4+x^2} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^{2\sqrt{3}}$

$= \frac{1}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}0\right]$

$= \frac{1}{2} \cdot \frac{\pi}{3}$

$= \frac{\pi}{6} \quad \therefore A$

(v) $y = x \ln(\tan^{-1}x)$

$\therefore \frac{dy}{dx} = x \cdot \frac{1}{\tan^{-1}x} \cdot \frac{1}{1+x^2} + \ln(\tan^{-1}x) \cdot 1$

$= \frac{x}{(1+x^2)\tan^{-1}x} + \ln(\tan^{-1}x)$

$\therefore C$

(b) (i) $\sin 2x = 2\sin x \cos x$

(ii) TO SHOW: $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right) = \frac{1}{2}\cos 2x$

SOLN: LHS $= \sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

$= \frac{1}{2}\sin\left(2\left(\frac{\pi}{4}-x\right)\right)$

$= \frac{1}{2}\sin\left(\frac{\pi}{2}-2x\right)$

$= \frac{1}{2}\cos 2x$

$=$ RHS

(c) $\sin\theta - \sqrt{3}\cos\theta = 1, \quad 0 \le \theta \le 2\pi$

$\therefore 2\left(\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right) = 1$

$\therefore \sin(\theta - \alpha) = \frac{1}{2}$

where $\cos\alpha = \frac{1}{2}$, $\sin\alpha = \frac{\sqrt{3}}{2}$ $\therefore \tan\alpha = \sqrt{3}$

$\therefore \alpha = \frac{\pi}{3}$

$\therefore \sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore \theta - \frac{\pi}{3} = \frac{\pi}{6}$ (require 1st, 2nd quads)

$\therefore \theta = \frac{\pi}{6} + \frac{\pi}{3}$ or $\pi - \frac{\pi}{6} + \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{2}$ or $\frac{7\pi}{6}$

(d) $\sin^2\theta - \frac{1}{2}\sin\theta = 0$

$\therefore \sin\theta\left(\sin\theta - \frac{1}{2}\right) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\therefore \sin\theta = \sin 0$ or $\sin\theta = \sin\frac{\pi}{6}$

$\therefore \theta = \pi n + (-1)^n \cdot 0$ or $\pi n + (-1)^n \frac{\pi}{6}$

$\therefore \theta = \pi n$ or $\pi n + (-1)^n \frac{\pi}{6}$ where $n \in J$ (set of integers)

(e) $I = \int_0^{\frac{\pi}{6}} 3\sin^2 x \, dx$

$= 3\int_0^{\frac{\pi}{6}} \frac{1}{2}\left[1-\cos 2x\right] dx$

$= \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{6}}$

$= \frac{3}{2}\left[\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) - 0\right]$

$= \frac{3}{2}\left(\frac{2\pi - 3\sqrt{3}}{12}\right)$

$= \frac{1}{8}(2\pi - 3\sqrt{3})$

2 (a) $f(x) = \log_e(x-1)$

(i) Let $y = \log_e(x-1)$

For inverse fn interchange x for y:

$\therefore x = \log_e(y-1)$

$\therefore e^x = y-1 \quad \therefore y = e^x + 1$

$\therefore f^{-1}(x) = e^x + 1$

(ii) Now $f(f^{-1}(x)) = f(e^x+1)$

$= \log_e(e^x + 1 - 1)$

$= \log_e(e^x)$

$= x$

Also $f^{-1}(f(x)) = f^{-1}(\log_e(x-1))$

$= e^{\log_e(x-1)} + 1$

$e^{\log_e x} = x \quad = x - 1 + 1$

$= x$

$\Rightarrow f(f^{-1}(x)) = f^{-1}(f(x))$.

$\therefore f(x)$ and $f^{-1}(x)$ are mutually inverse functions.

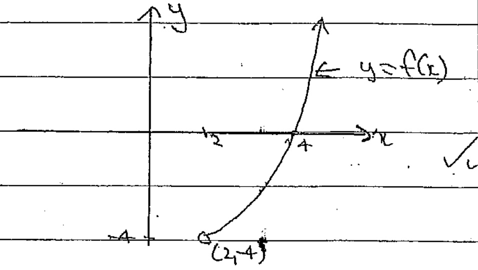
(b) $y = x^2 - 4x \quad = x(x-4)$

$\therefore \frac{dy}{dx} = 2x - 4$

For monotonic increasing $\frac{dy}{dx} > 0$

$\therefore 2x - 4 > 0 \quad \therefore x > 2$ and $y > -4$

(i)



(ii) Let $y = x^2 - 4x$

For inverse fn interchange x for y

$\therefore x = y^2 - 4y$

$\therefore x = (y-2)^2 - 4$

$\therefore y = \cancel{\pm}\sqrt{x+4} + 2$

$\therefore f^{-1}(x) = \sqrt{x+4} + 2$

(iii) Domain of $f^{-1}(x)$ is: $x > -4$

Range of $f^{-1}(x)$ is: $y > 2$.

(c) (i) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

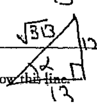
$= \pi - \frac{\pi}{4} \quad = \frac{3\pi}{4}$

(ii) $\tan\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \tan\frac{\pi}{3} \quad = \sqrt{3}$

(d) Let $E = \sin\left(2\tan^{-1}\frac{12}{13}\right)$

Let $\alpha = \tan^{-1}\frac{12}{13} \quad \therefore \tan\alpha = \frac{12}{13}$

$\therefore$ Basic Triad is:



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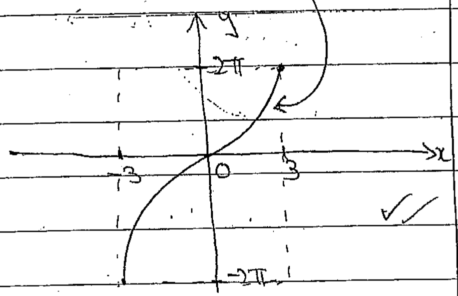
Start here. $\therefore E = \sin 2\alpha$

$= 2\sin\alpha\cos\alpha$

$= 2\,\frac{12}{\sqrt{313}}\cdot\frac{13}{\sqrt{313}}$

$= \frac{312}{313}$

3. (a) $y = 4\sin^{-1}\left(\frac{x}{3}\right)$



(b) $\frac{d}{dx}\left(\cos^{-1} 8x\right)$

$= \frac{-1}{\sqrt{1-(8x)^2}}\cdot 8$

$= \frac{-8}{\sqrt{1-64x^2}}$

(c) $I = \int_{-3}^{0}\frac{dx}{9+x^2}$

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{0}$

$= \frac{1}{3}\left[\tan^{-1}0 - \tan^{-1}(-1)\right]$

$= \frac{\pi}{12}$

(d) $y = \sin^{-1}(5x^2)$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(5x^2)^2}}\cdot 10x$

$= \frac{10x}{\sqrt{1-25x^4}}$

(e) $I = \int \frac{dx}{\sqrt{49-4x^2}}$

$= \int \frac{dx}{\sqrt{4\left(\frac{49}{4}-x^2\right)}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{7}{2}\right)^2-x^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{x}{7/2}\right)+c$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right)+c$