Topic 3. Complex numbers.
Level 3.
Problem COM3_01.

If $z_1 = 2-3i$ and $z_2 = 1+4i$, evaluate (a) z_1+z_2; (b) z_1-z_2; (c) z_1z_2; (d) z_1^2; (e) $\frac{1}{z_2}$; (f) $z_2 \div z_1$; (g) $z_1^2 - z_2^2$; (h) $z_1^3 - z_2^3$.

Answer: (a) $3+i$; (b) $1-7i$; (c) $14+5i$; (d) $-5-12i$; (e) $\frac{1}{17}-\frac{4}{17}i$; (f) $-\frac{10}{13}+\frac{11}{13}i$; (g) $10-20i$; (h) $1+43i$.

Solution: (a) $z_1+z_2 = 3+i$

(b) $z_1 - z_2 = 1-7i$

(c) $z_1z_2 = 2-12i^2-3i+8i = 14+5i$

(d) $z_1^2 = 4-12i+9i^2 = -5-12i$

(e) $\frac{1}{z_2} = \frac{1}{1+4i} = \frac{1-4i}{(1+4i)(1-4i)} = \frac{1-4i}{1+16} = \frac{1}{17}-\frac{4}{17}i$

(f) $\frac{z_2}{z_1} = \frac{1+4i}{2-3i} = \frac{(1+4i)(2+3i)}{(2-3i)(2+3i)} = \frac{(2-12)+(8+3)i}{4+9} = -\frac{10}{13}+\frac{11}{13}i$

(g) $z_1^2 - z_2^2 = (z_1-z_2)(z_1+z_2) = (1-7i)(3+i) = (3+7)+(-21+1)i = 10-20i$

(h) $z_1^3 - z_2^3 = (z_1-z_2)(z_1^2+z_1z_2+z_2^2) =$

$= (1-7i)((-5-12i)+(14+5i)+(1+8i+16i^2)) =$

$= (1-7i)(-6+i) = (-6+7)+(42+1)i = 1+43i.$

Problem COM3_02.

$z \in \mathbf{C}$ such that $\frac{z}{z-i}$ is real. Show that z is imaginary.

Solution: $\frac{z}{z-i}$ is real $\Rightarrow \frac{z-i+i}{z-i} = 1+\frac{i}{z-i} = 1+\frac{i\cdot i}{i(z-i)} = 1-\frac{1}{iz+1}$ is real.

$\therefore \frac{1}{iz+1}$ is real $\Rightarrow \frac{-i\bar{z}+1}{(iz+1)(-i\bar{z}+1)}$ is real. Hence $i\bar{z}$ is real $\Rightarrow i(i\bar{z})$ is imaginary. Thus $\bar{z}$ is imaginary $\Rightarrow z$ is imaginary.

Problem COM3_03.

Complex numbers $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1+2i}$ where a and b are real, are such that $z_1 + z_2 = 1$. Find a and b.

Answer: $a=4$, $b=-5$.

Solution: $z_1 = \frac{a}{1+i} = \frac{a(1-i)}{(1+i)(1-i)} = \frac{a-ia}{1+1} = \frac{a}{2}-\frac{a}{2}i$,

$z_2 = \frac{b}{1+2i} = \frac{b(1-2i)}{(1+2i)(1-2i)} = \frac{b-2ib}{1+4} = \frac{b}{5}-\frac{2b}{5}i$.

Hence $z_1+z_2 = \left(\frac{a}{2}+\frac{b}{5}\right) - i\left(\frac{a}{2}+\frac{2b}{5}\right)$. But $z_1+z_2=1$ and a, b are real. Equating real and imaginary parts:

$\frac{a}{2}+\frac{b}{5}=1$ and $\frac{a}{2}+\frac{2b}{5}=0$. Therefore $a=4$, $b=-5$.

Problem COM3_04.

The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z}+2iz = 12+6i$. Find the possible values of z.

Answer: $3-i$, $3+3i$.

Solution: Let $z = x+iy$, where x, y are real. Then $x^2+y^2+2ix-2y = 12+6i$. Equating real and imaginary parts, $x^2+y^2-2y=12$ and $2ix=6i$. Hence $x=3$ and $y^2-2y-3=0$. Therefore the possible values of z are $3-i$ and $3+3i$.

Problem COM3_05.

Solve $x^2+2x+2=0$.

Answer: $-1\pm i$.

Solution: $\Delta = -4 = 4i^2 \Rightarrow x = \frac{-2\pm 2i}{2} = -1\pm i$.

Problem COM3_06.

Solve $x^2-4x+(1-4i)=0$.

Answer: $x=-i$ or $x=4+i$.

Solution: Find Δ: $\Delta = 4^2-4(1-4i) = 12+16i$.

Find square roots of Δ: Let $(a+ib)^2 = 12+16i, a, b \in \mathbf{R}$. Then $(a^2-b^2)+(2ab)i = 12+16i$.

Equating real and imaginary parts, $a^2-b^2=12$ and $2ab=16$.

$a^2-\frac{64}{a^2}=12 \Rightarrow a^4-12a^2-64=0$, $(a^2-16)(a^2+4)=0$, a real.

$\therefore a=4, b=2$ or $a=-4, b=-2$. Hence Δ has square roots $4+2i$, $-4-2i$. Use the quadratic formula: $x^2-4x+(1-4i)=0$ has the solutions $x=\frac{4\pm(4+2i)}{2}$,

$\therefore x=-i$ or $x=4+i$.

Problem COM3_07.

Solve $x^2+(2-i)x-2i=0$.

Answer: $i, -2$.

Solution: Find Δ: $\Delta = (2-i)^2 + 8i = 3+4i$. Find square roots of Δ: Let $(a+ib)^2 = 3+4i$, $a, b \in \mathbf{R}$. Then $a^2 - b^2 = 3$ and $2ab = 4$.
$\therefore a^4 - a^2b^2 = 3a^2$ and $a^2b^2 = 4$. Thus $a^4 - 3a^2 - 4 = 0 \Rightarrow (a^2-4)(a^2+1) = 0$, a real, $\therefore a = 2, b = 1$ or $a = -2, b = -1$. Hence Δ has the square roots $2+i, -2-i$.
Use the quadratic formula: $x^2 + (2-i)x - 2i = 0$
has solutions $x = \dfrac{-(2-i) \pm (2+i)}{2}$ $\therefore x = i$ or $x = -2$.

Problem COM3_08.
$1+i$ is a root of the equation $x^2 + (a+2i)x + (5+ib) = 0$, where a and b are real. Find the values of a and b.

Answer: $a = -3$, $b = -1$.

Solution: Substituting $x = 1+i$, $(1+i)^2 + (a+2i)(1+i) + (5+ib) = 0$,
$\therefore (1-1) + 2i + (a-2) + i(a+2) + 5 + ib = 0$,
$\therefore (a+3) + i(a+b+4) = 0$, $a, b \in \mathbf{R}$.
Equating real and imaginary parts: $a+3=0$ and $a+b+4=0$.
Therefore $a = -3$, $b = -1$.

Problem COM3_09.
$1-2i$ is one root of the equation $x^2 + (1+i)x + k = 0$. Find the other root and the value of k.

Answer: $x = -2+i$, $k = 5i$.

Solution: Let z be the other root of the equation $x^2 + (1+i)x + k = 0$. Then $z + (1-2i) = -(1+i)$ and $z \cdot (1-2i) = k$. Therefore $z = -(1+i) - (1-2i) = -2+i$ and $k = (-2+i)(1-2i) = (-2+2) + i(4+1) = 5i$. Hence $k = 5i$ and equation $x^2 + (1+i)x + k = 0$ has roots $x = 1-2i$ and $x = -2+i$.

Problem COM3_10.
a and b are real numbers such that the sum of the squares of the roots of the equation $x^2 + (a+ib)x + 3i = 0$ is 8. Find all possible pairs of values a, b.

Answer: $a = 3, b = 1$ or $a = -3, b = -1$.

Solution: Let z_1, z_2 are the roots of the equation $x^2 + (a+ib)x + 3i = 0$. Then $z_1^2 + (a+ib)z_1 + 3i = 0$ and $z_2^2 + (a+ib)z_2 + 3i = 0$. But $z_1^2 + z_2^2 = 8$. Hence $8 + (a+ib)(z_1+z_2) + 6i = 0$. But $z_1 + z_2 = -(a+ib)$. Therefore $8 - (a+ib)^2 + 6i = 0$,
$\therefore (a+ib)^2 = 8+6i$, $a, b \in \mathbf{R}$. Thus $(a^2-b^2) + (2ab)i = 8+6i$. Equating real and imaginary parts, $a^2 - b^2 = 8$ and $2ab = 6$. $a^2 - \dfrac{9}{a^2} = 8 \Rightarrow a^4 - 8a^2 - 9 = 0$. $(a^2-9)(a^2+1) = 0$, a real. $\therefore a = 3, b = 1$ or $a = -3, b = -1$.

Problem COM3_11.
Express $(6+5i)(7+2i)$ in the form $a+ib$, where a and b are real, and write down $(6-5i)(7-2i)$ in a similar form. Hence find the prime factors of $32^2 + 47^2$.

Answer: $32+47i$, $32-47i$, $32^2 + 47^2 = 61 \cdot 53$.

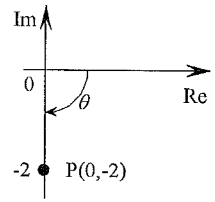
Solution: $z = (6+5i)(7+2i) = (42-10) + i(12+35) = 32+47i$. Since $(6-5i)(7-2i) = \bar{z}$, then $(6-5i)(7-2i) = 32-47i$. It is clear $|z|^2 = |6+5i|^2|7+2i|^2$. Therefore we obtain $32^2 + 47^2 = (6^2+5^2)(7^2+2^2) = 61 \cdot 53$.

Problem COM3_12.
Find the modulus and principle argument of (a) $-2i$; (b) $-1+\sqrt{3}i$; (c) $i(i+1)$.

Answer: (a) $2, -\dfrac{\pi}{2}$; (b) $2, \dfrac{2\pi}{3}$; (c) $\sqrt{2}, \dfrac{3\pi}{4}$.

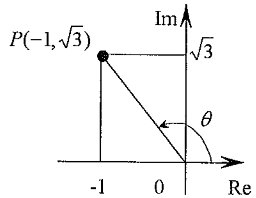
Solution: In each case $P(a,b)$ represents the complex number $z = a+ib$ and θ is the principal argument of z.

(a)



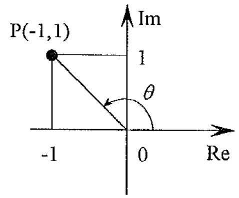
$z = -2i$
$|z| = 2$
$\arg z = -\dfrac{\pi}{2}$

(b)



$z = -1+\sqrt{3}i$
$|z| = \sqrt{1+3} = 2$
$\theta = \pi - \dfrac{\pi}{3} \Rightarrow \arg z = \dfrac{2\pi}{3}$

(c)



$z = i(i+1) = -1+i$
$|z| = \sqrt{1+1} = \sqrt{2}$
$\theta = \pi - \frac{\pi}{4} \Rightarrow \arg z = \dfrac{3\pi}{4}$

Problem COM3_13.

If $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$, find $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$.

Answer: $\sqrt{2}$; $\frac{7\pi}{12}$.

Solution: $|z_1| = 2$ and $\arg z_1 = \frac{\pi}{3}$, $|z_2| = \sqrt{2}$ and $\arg z_2 = -\frac{\pi}{4}$.

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{2}{\sqrt{2}} = \sqrt{2}$, $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}$.

Problem COM3_14.

Use the properties of modulus and argument of a complex number to deduce that (a) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$; (b) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{(\overline{z})}$; (c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\overline{z_1})}{(\overline{z_2})}$.

Solution: Let $r_1 = |z_1|$, $r_2 = |z_2|$ and $\theta_1 = \arg z_1$, $\theta_2 = \arg z_2$. Then
$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $\overline{z_1} = r_1(\cos(-\theta_1) + i\sin(-\theta_1))$,
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $\overline{z_2} = r_2(\cos(-\theta_2) + i\sin(-\theta_2))$.

(a) $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \Rightarrow \overline{z_1 z_2} = r_1 r_2(\cos(-(\theta_1+\theta_2)) + i\sin(-(\theta_1+\theta_2)))$.
But $\overline{z_1} \cdot \overline{z_2} = r_1 r_2(\cos((-\theta_1)+(-\theta_2)) + i\sin((-\theta_1)+(-\theta_2)))$. Therefore $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

(b) Let $r = |z|$ and $\arg z = \theta$. Then $z = r(\cos\theta + i\sin\theta)$, $\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$ and $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$. Thus $\overline{\left(\frac{1}{z}\right)} = \frac{1}{r}(\cos\theta + i\sin\theta)$ and $\frac{1}{(\overline{z})} = \frac{1}{r}(\cos\theta + i\sin\theta)$. Hence $\overline{\left(\frac{1}{z}\right)} = \frac{1}{(\overline{z})}$.

(c) $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = \frac{r_1}{r_2}(\cos(-(\theta_1-\theta_2)) + i\sin(-(\theta_1-\theta_2)))$. But $\frac{(\overline{z_1})}{(\overline{z_2})} = \frac{r_1}{r_2}(\cos(-\theta_1+\theta_2) + i\sin(-\theta_1+\theta_2))$. Therefore, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\overline{z_1})}{(\overline{z_2})}$.

Problem COM3_15.

If $z = 1+i$, find $\left|z^{10}\right|$ and $\arg\left(z^{10}\right)$.

Answer: 32; $\frac{\pi}{2}$.

Solution: $z = 1+i = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$,

$\therefore |z| = \sqrt{2}$ and $\arg z = \frac{\pi}{4}$. Then $\left|z^{10}\right| = |z|^{10} = \left(\sqrt{2}\right)^{10} = 32$, $\arg\left(z^{10}\right) = 10\arg z = 10\cdot\frac{\pi}{4} = \frac{5\pi}{2}$. But $\frac{5\pi}{2} > \pi$. The principal argument of z^{10} is $\frac{5\pi}{2} - 2\pi = \frac{\pi}{2}$.

Problem COM3_16.

$z = 1+\sqrt{3}i$. Find the smallest positive integer n for which z^n is real and evaluate z^n for this value of n. Show that there is no integral value of n for which z^n is imaginary.

Answer: 3, $z^3 = -8$.

Solution: $z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \Rightarrow |z| = 2, \arg z = \frac{\pi}{3}$. If z^n is real, then $\arg\left(z^n\right) = k\pi$, k is integral. But $\arg\left(z^n\right) = n\arg z$. Therefore $n\cdot\frac{\pi}{3} = k\pi$, $k = 0, \pm1, \pm2, \ldots$,
$\therefore n = 3k, k = 0, \pm1, \pm2, \ldots$

Hence the smallest positive integer n such that z^n is real is 3.
$\left|z^3\right| = 2^3 = 8$ and $\arg\left(z^3\right) = \pi$, $\therefore z^3 = -8$.

If z^n is imaginary, then $\arg\left(z^n\right) = \frac{\pi}{2} + k\pi$, k is integral. But $\arg\left(z^n\right) = n\arg z$. Therefore $n\cdot\frac{\pi}{3} = \frac{\pi}{2} + k\pi$, $k = 0, \pm1, \pm2, \ldots$, $\therefore n = \frac{3}{2} + 3k, k = 0, \pm1, \pm2, \ldots$

Hence there is no integral value of n for which z^n is imaginary.

Problem COM3_17.

Find the modulus of $\frac{7-i}{3-4i}$. Evaluate $\tan\left\{\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)\right\}$. Hence find the principal argument of $\frac{7-i}{3-4i}$ in terms of π.

Answer: $\sqrt{2}$, 1, $\frac{\pi}{4}$.

Solution: Let $z_1 = 7-i$, $z_2 = 3-4i$, and $z = \frac{7-i}{3-4i}$. Then $|z_1| = \sqrt{49+1} = 5\sqrt{2}$ and $\arg z_1 = -\tan^{-1}\left(\frac{1}{7}\right)$, $|z_2| = \sqrt{9+16} = 5$ and $\arg z_2 = -\tan^{-1}\left(\frac{4}{3}\right)$, $|z| = \frac{|z_1|}{|z_2|} = \sqrt{2}$ and $\arg z = \arg z_1 - \arg z_2 = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)$. Use a well-known formula:

$\tan\left\{\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)\right\} = \frac{\tan\left(\tan^{-1}\frac{4}{3}\right) - \tan\left(\tan^{-1}\frac{1}{7}\right)}{1 + \tan\left(\tan^{-1}\frac{4}{3}\right)\cdot\tan\left(\tan^{-1}\frac{1}{7}\right)} = \frac{\frac{4}{3} - \frac{1}{7}}{1 + \frac{4}{3}\cdot\frac{1}{7}} = 1$. Hence $\tan\arg z = 1$. But

$\frac{4}{3} > \frac{1}{7}$. Therefore $\arg z = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) \in \left(0, \frac{\pi}{2}\right)$. Thus principal value of argument z is $\frac{\pi}{4}$.

$\therefore$ Modulus of $\frac{7-i}{3-4i}$ is $\sqrt{2}$, $\tan\left\{\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)\right\} = 1$, principal argument of $\frac{7-i}{3-4i}$ is $\frac{\pi}{4}$.

Problem COM3_18.

If $|z| = r$ and $\arg z = \theta$, show that $\frac{z}{z^2 + r^2}$ is real and give its value.

Answer: $\frac{1}{2r\cos\theta}$.

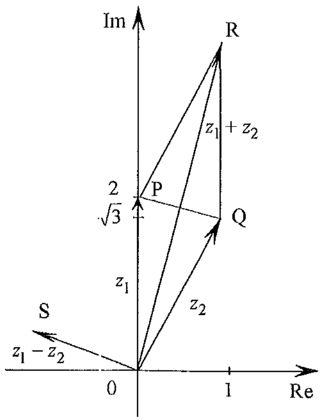
Solution: Noting $r^2 = z\bar{z}$, $\frac{z}{z^2+r^2} = \frac{z}{z^2+z\bar{z}} = \frac{1}{z+\bar{z}} = \frac{1}{2\operatorname{Re} z}$. Hence $\frac{z}{z^2+r^2}$ is real. Since $\operatorname{Re} z = r\cos\theta$, $\frac{z}{z^2+r^2} = \frac{1}{2r\cos\theta}$.

Problem COM3_19.

Find the modulus and argument of each of the complex numbers $z_1 = 2i$ and $z_2 = 1+\sqrt{3}i$. Mark on an Argand diagram the points P, Q, R and S representing z_1, z_2, z_1+z_2 and $z_1 - z_2$ respectively. Deduce the exact values of $\arg(z_1+z_2)$ and $\arg(z_1-z_2)$.

Answer: $\arg(z_1-z_2) = \frac{11\pi}{12}$, $\arg(z_1+z_2) = \frac{5\pi}{12}$.

Solution:



$z_1 = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, $\therefore$ $|z_1| = 2$ and $\arg z_1 = \frac{\pi}{2}$.

$z_2 = 1+\sqrt{3}i = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $\therefore$ $|z_2| = 2$ and $\arg z_2 = \frac{\pi}{3}$.

$OP = |z_1|$, $OQ = |z_2|$. But $|z_1| = |z_2|$. Hence $OP = OQ$ and $OPRQ$ is a rhombus. Therefore $\angle POR = \angle QOR$. Thus $\arg(z_1+z_2) = \frac{1}{2}(\arg z_1 + \arg z_2) = \frac{5\pi}{12}$. Since diagonals OR and QP of the rhombus $OPRQ$ meet at right angle, $\arg(z_1 - z_2) = \arg(z_1+z_2) + \frac{\pi}{2} = \frac{11\pi}{12}$.

$\therefore \arg(z_1+z_2) = \frac{5\pi}{12}$, $\arg(z_1-z_2) = \frac{11\pi}{12}$.

Problem COM3_20.

(a) Solve the equations $x^2+x+1=0$ and $x^2-\sqrt{3}x+1=0$. Plot on an Argand diagram the points A and B representing the solutions of the first equation, and C and D representing the solutions of the second, choosing A and C to lie above the real axis.

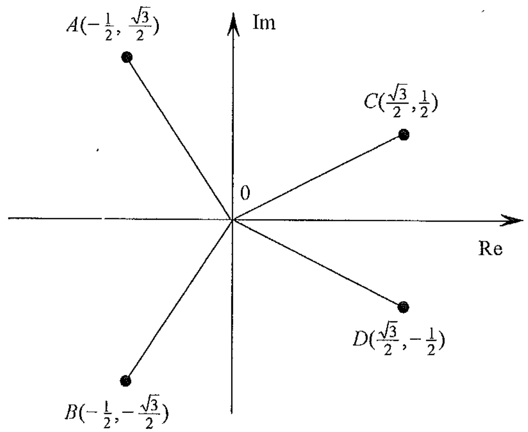
(b) Find the angles $\angle AOB$, $\angle COD$, $\angle COA$ and $\angle ACB$.

Answer: (a) $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, $\frac{\sqrt{3}}{2} \pm \frac{1}{2}i$; (b) $\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}$.

Solution: (a) Using the quadratic formula:

$$x^2+x+1=0 \Rightarrow \Delta = -3 \Rightarrow x = \frac{-1\pm\sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i,$$

$$x^2-\sqrt{3}x+1=0 \Rightarrow \Delta = -1 \Rightarrow x = \frac{\sqrt{3}\pm i}{2} = \frac{\sqrt{3}}{2} \pm \frac{1}{2}i.$$



(b) Let $x_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $x_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ are the solutions of the first equation, and $x_3 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $x_4 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ are the solutions of the second equation.

Then $\arg x_1 = \frac{2\pi}{3}$, $\arg x_2 = -\frac{2\pi}{3}$, $|x_1| = |x_2| = 1$, $\arg x_3 = \frac{\pi}{6}$, $\arg x_4 = -\frac{\pi}{6}$, $|x_3| = |x_4| = 1$.

Hence $\angle AOB = 2\pi - (\arg x_1 - \arg x_2) = \frac{2\pi}{3}$, $\angle COD = \arg x_3 - \arg x_4 = \frac{\pi}{3}$,

$\angle COA = \arg x_1 - \arg x_3 = \frac{\pi}{2}$. $\angle ACB = \angle ACO + \angle BCO$.

But $\angle ACO = \frac{1}{2}(\pi - \angle AOC)$, since $AO = OC = 1$, and

$\angle BCO = \frac{1}{2}(\pi - \angle BOC)$, since $BO = OC = 1$.

Therefore $\angle ACB = \pi - \frac{1}{2}(\angle AOC + \angle BOC) = \pi - \frac{1}{2}(2\pi - \angle AOB) = \frac{1}{2}\angle AOB = \frac{\pi}{3}$.

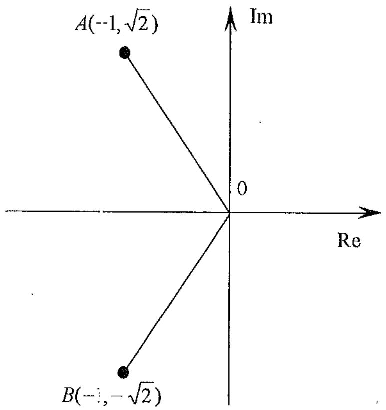
$\therefore \angle AOB = \frac{2\pi}{3}$, $\angle COD = \frac{\pi}{3}$, $\angle COA = \frac{\pi}{2}$, $\angle ACB = \frac{\pi}{3}$.

Problem COM3_21.

Let H and K be the points representing the roots of $x^2 + 2px + q = 0$, where p and q are real and $p^2 < q$. Find the algebraic relation satisfied by p and q when $\angle HOK$ is a right angle.

Answer: $q - 2p^2 = 0$.

Solution:



Using the quadratic formula:

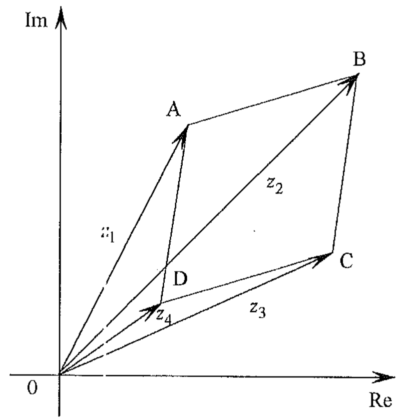
$x^2 + 2px + q = 0 \Rightarrow \Delta = 4p^2 - 4q \Rightarrow x = \frac{-2p \pm i2\sqrt{q-p^2}}{2} = -p \pm \sqrt{q-p^2}\,i$, since $p^2 < q$.

Let $x_3 = -p + i\sqrt{q-p^2}$ and $x_4 = -p - i\sqrt{q-p^2}$. Since $\angle HOK = 2\arg x_3$, if $p < 0$, or $\angle HOK = 2\pi - 2\arg x_3$, if $p > 0$, $\angle HOK = \frac{\pi}{2} \Rightarrow \arg x_3 = \frac{\pi}{4}$ when $p < 0$ or $\arg x_3 = \frac{3\pi}{4}$ when $p > 0$. In each case $\frac{\sqrt{q-p^2}}{|p|}$ must be equal to 1. Hence $\angle HOK$ is a right angle when $q - 2p^2 = 0$.

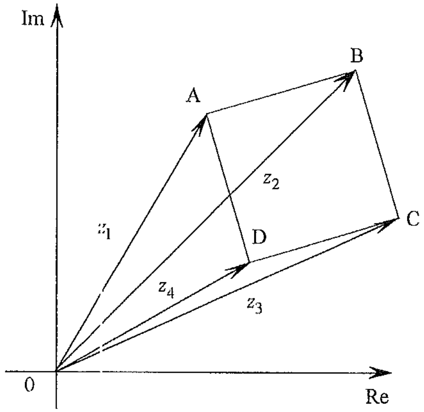
Problem COM3_22.

On an Argand diagram the points A, B, C and D represent z_1, z_2, z_3 and z_4 respectively. Show that if $z_1 - z_2 + z_3 - z_4 = 0$, then ABCD is a parallelogram, and if also $z_1 + iz_2 - z_3 - iz_4 = 0$, then ABCD is a square.

Solution:



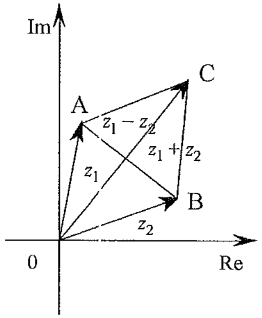
If $z_1 - z_2 + z_3 - z_4 = 0$, then $z_1 - z_2 = z_4 - z_3$. But $\overrightarrow{BA}$ represents $z_1 - z_2$, $\overrightarrow{CD}$ represents $z_4 - z_3$. Therefore $\overrightarrow{BA}$ and $\overrightarrow{CD}$ are parallel. On the other hand, $z_1 - z_4 = z_2 - z_3$. But $\overrightarrow{DA}$ represents $z_1 - z_4$, $\overrightarrow{CB}$ represents $z_2 - z_3$. Hence $\overrightarrow{DA}$ and $\overrightarrow{CB}$ are parallel. So we proved that $ABCD$ is a parallelogram.



If $z_1 + iz_2 - z_3 - iz_4 = 0$, then $z_1 - z_3 = i(z_4 - z_2)$. Hence the diagonals CA and BD of the parallelogram $ABCD$ meet at right angle and $CA = BD$. Therefore $ABCD$ is a square.

Problem COM3_23.

Use the vector representation of z_1 and z_2 on an Argand diagram to show that, if $0 < \arg z_2 < \arg z_1 < \frac{\pi}{2}$, and $\arg(z_1 - z_2) - \arg(z_1 + z_2) = \frac{\pi}{2}$, then $|z_1| = |z_2|$.



Solution:

Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. Construct the parallelogram $OACB$. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent $z_1 + z_2$, $z_1 - z_2$ respectively. Since $\arg(z_1 - z_2) = \arg(z_1 + z_2) + \frac{\pi}{2}$, $\overrightarrow{BA}$ is obtained from $\overrightarrow{OC}$ by a rotation anticlockwise about O through $\frac{\pi}{2}$, followed by an enlargement in O. Therefore diagonals OC and AB of the parallelogram $OACB$ meet at right angle and $OACB$ is a rhombus. Hence $OA = OB$ and $|z_1| = |z_2|$.

Problem COM3_24.

If $z_1 = 24 + 7i$ and $|z_2| = 6$, find the greatest and least values of $|z_1 + z_2|$.

Answer: 19

Solution: $|z_1 + z_2| \le |z_1| + |z_2| = 25 + 6 = 31$ and this greatest value of 31 is attained when $z_2 = kz_1$ for some positive real k. But $|z_2| = 6$ and $z_2 = kz_1 \Rightarrow 6 = 25k$.

$\therefore |z_1 + z_2|$ attained the greatest value of 31 when $z_2 = \frac{6}{25}(24 + 7i) = \frac{144}{25} + \frac{42}{25}i$.

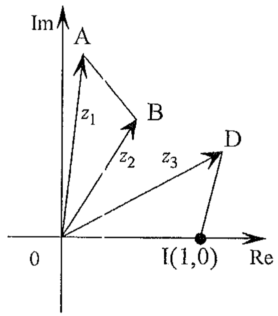
$|z_1 + z_2| \ge \big||z_1| + |z_2|\big| = 25 - 6 = 19$ and this least value of 19 is attained when $z_2 = -kz_1$ for some positive real k. But $|z_2| = 6$ and $z_2 = -kz_1 \Rightarrow 6 = 25k$.

$\therefore |z_1 + z_2|$ attained the least value of 19 when $z_2 = -\frac{6}{25}(24 + 7i) = -\frac{144}{25} + \frac{42}{25}i$.

Problem COM3_25.

On an Argand diagram the points A and B represent the numbers z_1 and z_2 respectively. I is the point (1; 0). D is the point such that triangle OID is similar to triangle OBA. Show that D represents $\frac{z_1}{z_2}$.

Solution:



$\Delta OID \equiv \Delta OBA$,

$\therefore \frac{OD}{OA} = \frac{OI}{OB} \Rightarrow \frac{|z_3|}{|z_1|} = \frac{1}{|z_2|}$ $\angle DOI = \angle AOB \Rightarrow \arg z_3 = \arg z_1 - \arg z_2$. Hence $|z_3| = \frac{|z_1|}{|z_2|}$ and $\arg z_3 = \arg z_1 - \arg z_2$.

$\therefore z_3 = \frac{z_1}{z_2}$ and $\overrightarrow{OD}$ represents the quotient of z_1 and z_2.

Problem COM3_26.

Express $-1 + \sqrt{3}i$ in modulus/argument form. Use *de Moivre's* theorem to show that $\left(-1 + \sqrt{3}i\right)^n + \left(-1 - \sqrt{3}i\right)^n = 2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$, n is a positive integer. Evaluate this expression in each of the cases $n = 3m$, $n = 3m \pm 1$, where m is a positive integer.

Answer: $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$; 2^{n+1}, -2^n.

Solution: Let $z=-1+\sqrt{3}i$. Then $z=2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ and

$-1-\sqrt{3}i=\bar{z}=2\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$. Using *de Moivre's* theorem $z^n=2^n\left(\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}\right)$. Now

$z^n+(\bar{z})^n=z^n+\left(\overline{z^n}\right)=2\operatorname{Re}\left(z^n\right)=2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$. Thus $\left(-1+\sqrt{3}i\right)^n+\left(-1-\sqrt{3}i\right)^n=2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$.

If $n=3m$, $\left(-1+\sqrt{3}i\right)^n+\left(-1-\sqrt{3}i\right)^n=2^{n+1}\cos\left(\frac{6m\pi}{3}\right)=2^{n+1}$.

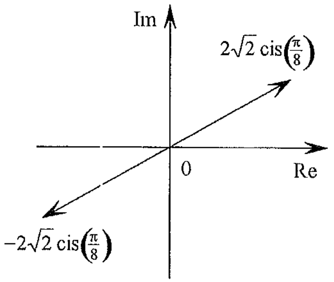
If $n=3m\pm1$, $\left(-1+\sqrt{3}i\right)^n+\left(-1-\sqrt{3}i\right)^n=2^{n+1}\cos\left(2m\pi\pm\frac{2\pi}{3}\right)=-2^n$.

Problem COM3_27.

Express $z=4\sqrt{2}(1+i)$ in modulus/argument form. Hence find the two square roots of z and mark their representations on an Argand diagram.

Answer: $z=8\operatorname{cis}\frac{\pi}{4}$, $\pm2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{8}\right)$.

Solution:



$z=8\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=8\operatorname{cis}\frac{\pi}{4}$,

$\therefore$ $|z|=8$ and $\arg z=\frac{\pi}{4}$. By *de Moivre's* theorem, one square root of z has modulus $2\sqrt{2}$ and argument $\frac{\pi}{8}$. Hence the two square roots of z are $\pm2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{8}\right)$.

Problem COM3_28.

Use *de Moivre's* theorem to find in modulus/argument form the cube roots of $-2-2i$.

Answer: $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{12}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$.

Solution: $z=-2-2i=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=8\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$.

$\therefore|z|=\sqrt{8}$ and $\arg z=-\frac{3\pi}{4}$. By *de Moivre's* theorem cube roots of z have modulus $\sqrt{2}$ and arguments $-\frac{\pi}{4}+\frac{2\pi k}{3}$, $k=-1,0,1$. Hence the three roots of z are

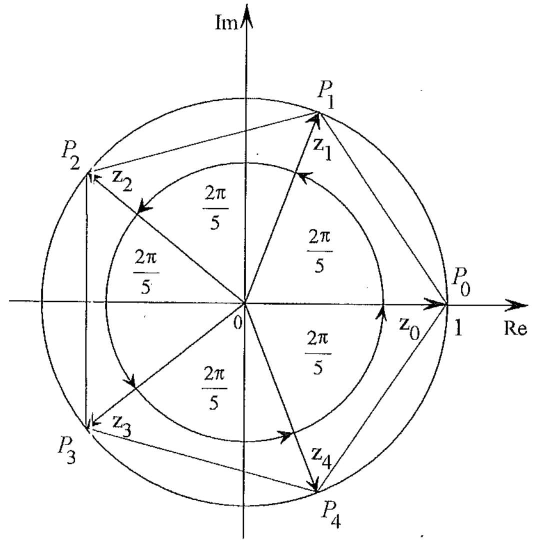
$\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{12}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$.

Problem COM3_29.

Use *de Moivre's* theorem to solve the equation $z^5=1$. Show that the points representing the five roots of this equation on an Argand diagram form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$ and perimeter $10\sin\frac{\pi}{5}$.

Answer: 1, $\operatorname{cis}\left(\pm\frac{2\pi}{5}\right)$, $\operatorname{cis}\left(\pm\frac{4\pi}{5}\right)$.

Solution:



$z^5=1\Rightarrow|z|=1$. Hence 5th roots of unity have modulus 1 and their representations $P_k(k=0,1,2,3,4)$ lie on the unit circle with the center in the origin. By *de Moivre's* theorem one root (z_0) has argument zero, the others being equally spaced around the unit circle in the

Argand diagram by an angle $\frac{2\pi}{5}$. Hence the complex 5th roots of unity are 1, $\text{cis}\left(\pm\frac{2\pi}{5}\right)$, $\text{cis}\left(\pm\frac{4\pi}{5}\right)$.

Since $\angle P_kOP_{k+1} = \frac{2\pi}{5}$ and $OP_k = OP_{k+1} = 1$, $P_kP_{k+1} = 2\sin\frac{\pi}{5}$ for any $k = 0,1,2,3,4 (P_5 := P_0)$. Therefore the points $P_k (k = 0,1,2,3,4)$ form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}(= 5\cdot(\text{area of } \Delta P_0OP_1))$ and perimeter $10\sin\frac{\pi}{5}(= 5\cdot P_0P_1)$.

Problem COM3_30.

Show that the roots of $z^6 + z^3 + 1 = 0$ are among the roots of $z^9 - 1 = 0$. Hence find the roots of $z^6 + z^3 + 1 = 0$ in modulus/argument form.

Answer: $\text{cis}\left(\pm\frac{2\pi}{9}\right)$, $\text{cis}\left(\pm\frac{4\pi}{9}\right)$, $\text{cis}\left(\pm\frac{8\pi}{9}\right)$.

Solution: $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$. Therefore, if $z^6 + z^3 + 1 = 0$, then $z^9 - 1 = 0$. Hence the roots of $z^6 + z^3 + 1 = 0$ are among the roots of $z^9 - 1 = 0$. Let $z = \cos\theta + i\sin\theta$ satisfy $z^9 = 1$.
Using *de Moivre's* theorem, $\cos(9\theta) + i\sin(9\theta) = 1 + 0i$

$\therefore \cos(9\theta) = 1$ and $\sin(9\theta) = 0$

$\therefore 9\theta = 2\pi k$, k integral.

$\therefore \theta = \frac{2\pi}{9}k$, k integral.

Taking $\theta = \frac{2\pi}{9}k$, $k = 0,1,\ldots,8$ gives 9 distinct numbers z with argument $\frac{2\pi}{9}k$.

If $z^6 + z^3 + 1 = 0$, then $z^9 = 1$ but $z^3 \neq 1$. Hence the roots of $z^6 + z^3 + 1 = 0$ are $\cos\left(\frac{2\pi}{9}k\right) + i\sin\left(\frac{2\pi}{9}k\right)$, $k = 1,2,4,5,7,8$.

$\therefore z^6 + z^3 + 1 = 0$ has the roots $\text{cis}\left(\pm\frac{2\pi}{9}\right)$, $\text{cis}\left(\pm\frac{4\pi}{9}\right)$, $\text{cis}\left(\pm\frac{8\pi}{9}\right)$.

Problem COM3_31.

If $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2i\sin n\theta$. Hence show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Solution: By *de Moivre's* theorem and $z^n = \cos n\theta + i\sin n\theta$ and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$. Then $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$.
$2i\sin\theta = z - z^{-1}$. Then $32i^5\sin^5\theta = (z - z^{-1})^5$. But $(z - z^{-1})^5 = z^5 - 5z^3 + 10z - 10z^{-1}$ $+ 5z^{-3} - z^{-5} = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z^1 - z^{-1}) = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$.
Hence $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Problem COM3_32.

1, ω and ω^2 are the cube roots of unity. State the values of ω^3 and $1 + \omega + \omega^2$. Hence show that $(1 + \omega^2)^{12} = 1$ and $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)(1 - \omega^7)(1 - \omega^8) = 27$.

Answer: $\omega^3 = 1$, $\omega^2 + \omega + 1 = 0$.

Solution: The cube roots of unity satisfy $x^3 - 1 = 0$. But $x^3 - 1 = (x - 1)(x^2 + x + 1)$. Hence $\omega \neq 1 \Rightarrow \omega^2 + \omega + 1 = 0$. Clearly, $\omega^3 = 1$. Therefore $(1 + \omega^2)^{12} = (-\omega)^{12} = (\omega^3)^4 = 1$.

Then $\omega^4 = \omega^3 \cdot \omega = \omega$,

$\omega^5 = \omega^3 \cdot \omega^2 = \omega^2$,

$\omega^7 = \omega^6 \cdot \omega = \omega$,

$\omega^8 = \omega^6 \cdot \omega^2 = \omega^2$.

Hence $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)(1 - \omega^7)(1 - \omega^8) =$

$((1 - \omega)(1 - \omega^2))^3 = (1 - \omega - \omega^2 + \omega^3)^3 = (2 - \omega - \omega^2)^3 =$

$(3 - (1 + \omega + \omega^2))^3 = 3^3 = 27$.

Problem COM3_33.

1, ω and ω^2 are the three cube roots of unity. Show that if the equations $z^3 - 1 = 0$ and $pz^5 + qz + r = 0$ have a common root,
then $(p + q + r)(p\omega^5 + q\omega + r)(p\omega^{10} + q\omega^2 + r) = 0$.

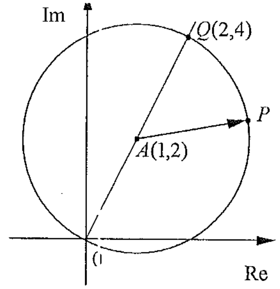
Solution: The cube roots of unity satisfy $z^3 - 1 = 0$. Therefore, if z is a common root of the equations $z^3 - 1 = 0$ and $pz^5 + qz + r = 0$, then z is one of the cube roots. Thus if $z = 1$, then $p + q + z = 0$; if $z = \omega$, then $p\omega^5 + q\omega + r = 0$; if $z = \omega^2$, then $p\omega^{10} + q\omega^2 + r = 0$.
Hence $(p + q + r)(p\omega^5 + q\omega + r)(p\omega^{10} + q\omega^2 + r) = 0$.

Problem COM3_34.

Express $z_1 = \frac{7 + 4i}{3 - 2i}$ in the form $a + ib$, where a and b are real. On an Argand diagram sketch the locus of the point representing the complex number z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition.

Answer: $z_1 = 1 + 2i$, $2\sqrt{5}$.

Solution: $z_1 = \frac{7 + 4i}{3 - 2i} = \frac{(7 + 4i)(3 + 2i)}{(3 - 2i)(3 + 2i)} = \frac{(21 - 8) + (14 + 12)i}{9 + 4} = 1 + 2i$.



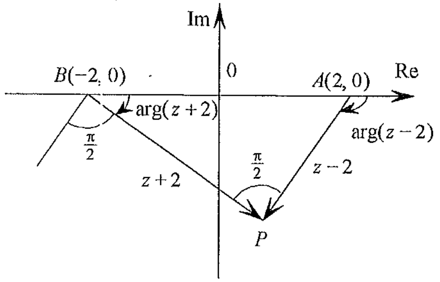
Let A represent $z_1 = 1+2i$. Then $\vec{AP}$ represents $z - z_1$ and $|z - z_1| = \sqrt{5} \Rightarrow AP = \sqrt{5}$. Hence P lies on the circle center $A(1,2)$ and radius $\sqrt{5}$. So the locus of P has equation $(x-1)^2 + (y-2)^2 = 5$. Let Q be the intersection of the line OA and the circle. Then the greatest value of $|z|$ is $OQ = 2\sqrt{5}$.

Problem COM3_35.

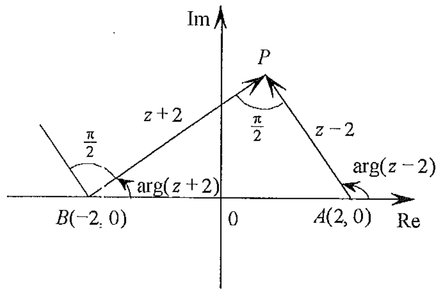
The point P represents the complex number z on an Argand diagram. Describe the locus of P when $\arg(z-2) = \arg(z+2) + \frac{\pi}{2}$.

Answer: $y = \sqrt{4 - x^2}$.

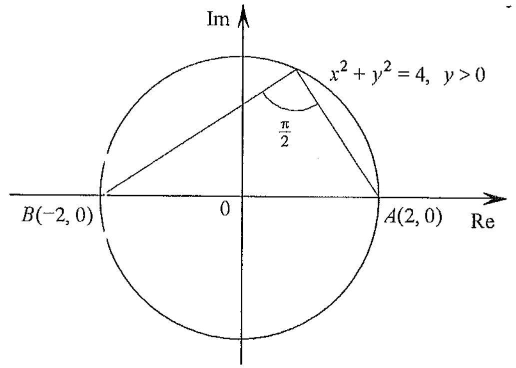
Solution: Let $A(2,0)$, $B(-2,0)$ represent 2, -2 respectively. Then $\vec{AP}$ and $\vec{BP}$ represent $z-2$ and $z+2$ respectively. $\arg(z-2) = \arg(z+2) + \frac{\pi}{2}$ requires $\vec{AP}$ to be parallel to the vector obtained by rotation of $\vec{BP}$ anticlockwise through an angle of $\frac{\pi}{2}$.



If P lies below the x-axis, AP must be parallel to a clockwise rotation of BP. This diagram shows $\arg(z-2) = \arg(z+2) - \frac{\pi}{2}$. Hence P must lie above the x-axis.



Since alternate angles between parallel lines are equal, $\angle BPA = \frac{\pi}{2}$ as P traces its locus. Hence P lies on the upper arc AB of a circle through A and B.



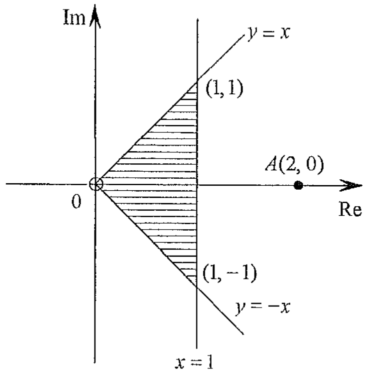
The center of this circle is the center of diameter AB. Hence the locus of P has equation $x^2 + y^2 = 4, y > 0$, or $y = \sqrt{4 - x^2}$.

Let $z = x + iy$ satisfies both $|z| = |z-2|$ and $\arg(z-2) = \arg(z+2) + \frac{\pi}{2}$. Then $x = 1$ and $y = \sqrt{4-1} = \sqrt{3}$. Hence $z = 1 + i\sqrt{3}$.

Problem COM3_36.

Indicate on an Argand diagram the region which contains the point P representing z when $|z| \le |z-2|$ and $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$.

Solution:



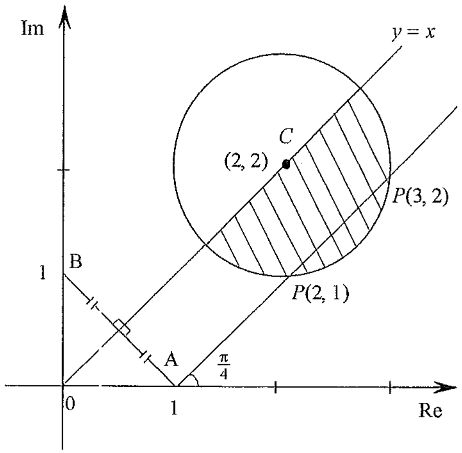
$|z| = |z-2|$ is the perpendicular bisector of OA. $\arg z = \frac{\pi}{4}$ is the ray $y = x$, $x > 0$. $\arg z = -\frac{\pi}{4}$ is the ray $y = -x$, $x > 0$.

Problem COM3_37.

$|z-1| \le |z-i|$ and $|z-2-2i| \le 1$. Sketch the region in the Argand diagram which contains the point P representing z. If P describes the boundary of this region, find the value of z when $\arg(z-1) = \frac{\pi}{4}$.

Answer: $2-i$, $3+2i$.

Solution:



Let A, B and Q represent 1, i, z respectively. If $|z-1| = |z-i|$, then $AQ = BQ$ and the locus of Q is the perpendicular bisector of AB. Since AB has midpoint $\left(\frac{1}{2}, \frac{1}{2}\right)$ and gradient -1, the locus of Q passes through $\left(\frac{1}{2}, \frac{1}{2}\right)$ with gradient 1 and has Cartesian equation $y = x$.

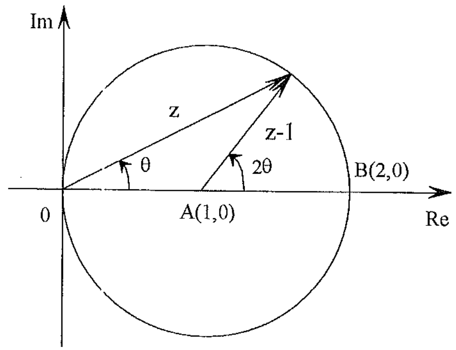
Let C represent $2+2i$. If $|z-2-2i| \le 1$, then $CQ = 1$ and Q lies on or inside the circle with center $(2,2)$ and radius 1.

Let now $|z-1| \le |z-i|$ and $|z-2-2i| \le 1$. Then $AQ \le BQ$ and $CQ \le 1$. Hence Q lies on the right-hand side of the perpendicular bisector of AB inside the circle centre C and radius 1, or Q lies on the boundary of this region. If P describes the boundary of this region and $\arg(z-1) = \frac{\pi}{4}$, then $CP = 1$ and $\vec{AP}$ makes the angle $\frac{\pi}{4}$ with the positive x-axis. Thus we must solve simultaneously two Cartesian equations $(x-2)^2 + (y-2)^2 = 1$ and $y = x-1$. Substituting the second equation into the first gives $(x-2)^2 + (x-3)^2 = 1 \Rightarrow 2x^2 - 10x + 12 = 0 \Rightarrow x = 2, 3 \Rightarrow y = 1$ (when $x = 2$), $y = 2$ (when $x = 3$). Therefore such P represents $z = 2+i$ and $z = 3+2i$.

Problem COM3_38.

$|z-1| = 1$. Sketch the locus of the point P representing z on an Argand diagram. Hence deduce that $\arg(z-1) = \arg(z^2)$.

Solution:



Let A represent 1. Then $\vec{AP}$ represents $z-1$ and $AP = 1$. Hence P lies on the circle center $A(1,0)$ and radius 1.

Let $\theta = \arg z$ and B represent 2. Then $\angle POB = \theta$ and $\angle PAB = \arg(z-1)$. But $\angle PAB = 2\angle POB$ and $\arg(z^2) = 2\arg z$. Therefore $\arg(z-1) = 2\theta = 2\arg z = \arg(z^2)$ as $\triangle OAP$ is isosceles with $AO = AP$.

Problem COM3_39.

The complex number z is given by $z = t + \frac{1}{t}$, where $t = r(\cos\theta + i\sin\theta)$. Find the equation of the locus of the point P which represents z on an Argand diagram when $\theta = \frac{\pi}{4}$ and r varies.

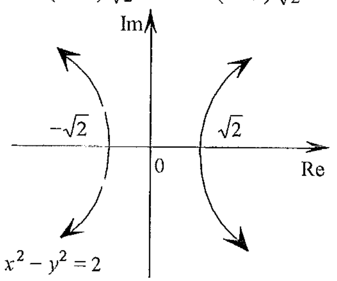
Answer: $x^2 - y^2 = 2$.

Solution: Let $P(x, y)$ represent $z = x + iy$. Then $x + iy = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) =$

$\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$,

$\therefore x = \left(r + \frac{1}{r}\right)\cos\theta$ and $y = \left(r - \frac{1}{r}\right)\sin\theta$.

$x = \left(r + \frac{1}{r}\right)\frac{1}{\sqrt{2}}$ and $y = \left(r - \frac{1}{r}\right)\frac{1}{\sqrt{2}}$. Hence $x^2 - y^2 = 2$.

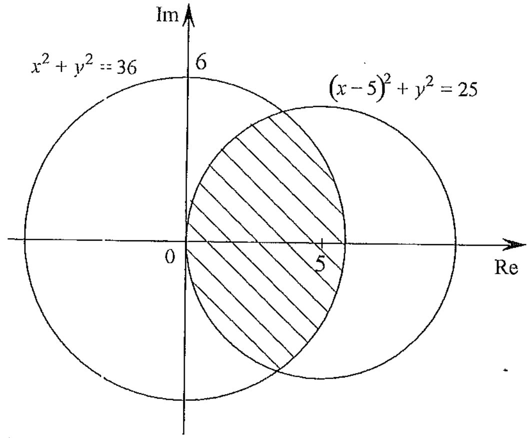


Problem COM3_40.

Indicate on an Argand diagram the region defined by the pair of inequalities $|z| \le 6$ and $|z - 5| \le 5$. Write down the range of values of $\arg z$ for such z. Find the values of z for which both $|z| = 6$ and $|z - 5| = 5$.

Answer: $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$, $z = \frac{18}{5} \pm i\frac{24}{5}$.

Solution:



$|z| = 6$ is the circle, center $(0,0)$ and radius 6. $|z - 5| = 5$ is the circle, center $(5,0)$ and the radius 5. Since y-axis is a tangent line to the circle $|z - 5| = 5$ at point $(0,0)$, if $|z| \le 6$ and $|z - 5| \le 5$, then $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$.

Let $z = x + iy$. Then $|z| = 6 \Rightarrow x^2 + y^2 = 36$, and $|z - 5| = 5 \Rightarrow (x-5)^2 + y^2 = 25$. Hence, if z such that both $|z| = 6$ and $|z - 5| = 5$, then both $x^2 + y^2 = 36$ and $x^2 + y^2 - 10x + 25 = 25$. Therefore $10x = 36$.

$\therefore x = \frac{18}{5}$.

$\therefore y = \pm\sqrt{36 - \left(\frac{18}{5}\right)^2} = \frac{24}{5}$.

Hence the values of z for which both $|z| = 6$ and $|z - 5| = 5$ are $\frac{18}{5} \pm i\frac{24}{5}$.