Topic 3 Complex Numbers
Level 2.

Problem CO2_1. Find (a) z_1+z_2; (b) z_1-z_2; (c) z_1z_2; (d) $\frac{z_1}{z_2}$, when $z_1=4+i$, $z_2=2+3i$.

Answer: (a) $6+4i$; (b) $2-2i$; (c) $5+14i$; (d) $\frac{11}{13}-\frac{10}{13}i$.

Explanation: (a) $z_1+z_2=(4+i)+(2+3i)=6+4i$

(b) $z_1-z_2=(4+i)-(2+3i)=2-2i$

(c) $z_1z_2=(4+i)\cdot(2+3i)=8+3i^2+12i+2i=5+14i$

(d) $\frac{z_1}{z_2}=\frac{4+i}{2+3i}=\frac{(4+i)(2-3i)}{(2+3i)(2-3i)}=\frac{(8+3)+(2-12)i}{4+9}=\frac{11}{13}-\frac{10}{13}i$.

Problem CO2_2. Prove the following results above complex conjugates (a) $\overline{z_1+z_2}=\bar z_1+\bar z_2$; (b) $\overline{z_1-z_2}=\bar z_1-\bar z_2$; (c) $\overline{z_1z_2}=\bar z_1\bar z_2$; (d) $\overline{\left(\frac{1}{z}\right)}=\frac{1}{(\bar z)}$; (e) $\overline{z_1\div z_2}=\bar z_1\div\bar z_2$; (f) $\overline{5z}=5\bar z$.

Explanation: Let $z_1=x_1+iy_1,\ z_2=x_2+iy_2,\ x_1,\ y_1,\ x_2,\ y_2\in\mathbf{R}$. Then

(a) $\overline{z_1+z_2}=\overline{(x_1+x_2)+i(y_1+y_2)}=(x_1+x_2)-i(y_1+y_2)=\bar z_1+\bar z_2$

(b) $\overline{z_1-z_2}=\overline{(x_1-x_2)+i(y_1-y_2)}=(x_1-x_2)-i(y_1-y_2)=\bar z_1-\bar z_2$

(c) $\overline{z_1z_2}=\overline{(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)}=$

$=(x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1)=$

$=\bar z_1\bar z_2$

(e) $\overline{z_1\div z_2}=\overline{\left\{\frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}\right\}}=\overline{\left(\frac{x_1}{x_2^2+y_2^2}+i\frac{y_1}{x_2^2+y_2^2}\right)\left(\frac{x_2}{x_2^2+y_2^2}-i\frac{y_2}{x_2^2+y_2^2}\right)}=$

$=\left(\frac{x_1}{x_2^2+y_2^2}-i\frac{y_1}{x_2^2+y_2^2}\right)\left(\frac{x_2}{x_2^2+y_2^2}+i\frac{y_2}{x_2^2+y_2^2}\right)=$

$=\frac{(x_1-iy_1)(x_2+iy_2)}{x_2^2+y_2^2}=\frac{x_1-iy_1}{x_2-iy_2}=\bar z_1\div\bar z_2$

(d) Identity $\overline{\left(\frac{1}{z}\right)}=\frac{1}{(\bar z)}$ follows from (e) with $z_1=1$ and $z_2=z$.

(f) Identity $\overline{5z}=5\bar z$ follows from (c) with $z_1=5$ and $z_2=z$.

Problem CO2_3. (a) $a(\alpha)^2+b\alpha+c=0$, where $a,\ b,\ c\in R$ and $\alpha\in C$. Show that $a(\bar\alpha)^2+b\bar\alpha+c=0$.

(b) Deduce that if α is a non-real root of $ax^2+bx^2+c=0$, where a,b,c are real, then $\bar\alpha$ is the other root of this quadratic equation.

Explanation: (a) Using the results in problem **CO2_2** gives

$a(\bar\alpha)^2-b\bar\alpha+c=\overline{(a\alpha^2)}+\overline{b\alpha}+\bar c=\overline{a\alpha^2+b\alpha+c}=\bar 0=0$.

(b) If α is a non-real number, then $\mathrm{Im}\alpha\neq 0$. Hence $\bar\alpha\neq\alpha$, since $\mathrm{Im}(\bar\alpha)=-\mathrm{Im}\alpha$. Thus if α is a non-real root of $ax^2+bx^2+c=0$, where $a,\ b,\ c$ are real, then $\bar\alpha$ is the other root of this quadratic equation (see (a)).

Problem CO2_4. $z\in\mathbf{C}$ such that $\mathrm{Re}\,z=2\mathrm{Im}\,z$, and z^2-4i is real. Find z.

Answer: $2-i;-2-i$.

Explanation: $\mathrm{Re}\,z=2\mathrm{Im}\,z\Rightarrow z=2y+iy$ and $z^2-4i=(4y^2-y^2)+i(4y^2-4),\ y\in\mathbf{R}$

z^2-4i real $\Rightarrow 4y^2-4=0\Rightarrow y=\pm1$,

$\therefore z=2+i$ or $z=-2-i$.

Problem CO2_5. Find real x and y such that $(x+iy)^2=3+4i$.

Answer: $\therefore\ x=2,\ y=1$ or $x=-2,\ y=-1$.

Explanation: $(x+iy)^2=3+4i\Rightarrow(x^2-y^2)+(2xy)i=3+4i$

Equating real and imaginary parts: $x^2-y^2=3$ and $2xy=4$

$\therefore\ x^4-x^2y^2=3x^2$ and $x^2y^2=4$

Then $x^4-3x^2-4=0\Rightarrow(x^2-4)(x^2+1)=0,\ x$ real,

$\therefore\ x=2,\ y=1$ or $x=-2,\ y=-1$.

Problem CO2_6. Find the square root of $z=-6i$.

Answer: $\sqrt{3}-i\sqrt{3},\ -\sqrt{3}+i\sqrt{3}$.

Explanation: Let $(a+ib)^2=-6i,\ a,\ b\in\mathbf{R}$. Then $(a^2-b^2)+i(2ab)=-6i$. Equating real and imaginary parts, $a^2-b^2=0$ and $2ab=-6$.

$a^2-\frac{9}{a^2}=0\Rightarrow a^4-9=0$

$(a^2-3)(a^2+3)=0,\ a$ real $\Rightarrow a=\sqrt{3},\ b=-\sqrt{3}$ or $a=-\sqrt{3},\ b=\sqrt{3}$. Hence $-6i$ has square roots $\sqrt{3}-i\sqrt{3}$; $-\sqrt{3}+i\sqrt{3}$.

Problem CO2_7. Find the square root of $z=-5-12i$.

Answer: $2-3i$; $-2+3i$.

Explanation: Let $(a+ib)^2=-5-12i,\ a,\ b\in\mathbf{R}$. Then $(a^2-b^2)+i(2ab)=-5-12i$. Equating real and imaginary parts, $a^2-b^2=-5$ and $2ab=-12$.

$a^2 - \frac{36}{a^2} = -5 \Rightarrow a^4 + 5a^2 - 36 = 0$

$(a^2 - 4)(a^2 + 9) = 0$, a real $\Rightarrow a = 2, b = -3$ or $a = -2$, $b = 3$. Hence $-5 - 12i$ has square roots $2 - 3i$; $-2 + 3i$.

Problem CO2_8. Solve the following quadratic equation $2x^2 - 4x + 3 = 0$.

Answer: $1 \pm i\frac{1}{\sqrt{2}}$.

Explanation: $\Delta = -8 = 8i^2$, $\therefore$ $x = \frac{4 \pm i\sqrt{8}}{4} = 1 \pm i\frac{1}{\sqrt{2}}$.

Problem CO2_9. Solve the following quadratic equation $ix^2 - 2(i+1)x + 10 = 0$.

Answer: $-1 - 3i$ or $3 + i$.

Explanation: Find Δ: $4(1+i)^2 - 40i = -32i$.

Find square roots of Δ: Let $(a+ib)^2 = -32i$, a, $b \in \mathbf{R}$. Then $(a^2 - b^2) + i(2ab) = -32i$.

Equating real and imaginary parts, $a^2 - b^2 = 0$ and $ab = -16$. $a^2 - \frac{16^2}{a^2} = 0 \Rightarrow a^4 - 16^2 = 0$

$(a^2 - 16)(a^2 + 16) = 0$, a real $\Rightarrow$ $a = 4$, $b = -4$ or $a = -4$, $b = 4$. Hence Δ has square roots $\pm(4 - 4i)$.

Use the quadratic formula: $ix^2 - 2(i+1)x + 10 = 0$ has solutions $x = \frac{2(1+i) \pm 4(1-i)}{2i}$,

$\therefore$ $x = -1 - 3i$ or $x = 3 + i$.

Problem CO2_10. $x^2 + 6x + k = 0$ has one root α where $\operatorname{Im} \alpha = 2$. If k is real, find both roots of the equation and the value of k.

Answer: $-3 \pm 2i$; $k = 13$.

Explanation: $\operatorname{Im} \alpha = 2 \Rightarrow \alpha = x + 2i, x \in \mathbf{R}$. k real $\Rightarrow \overline{\alpha} = x - 2i$ is the other root of $x^2 + 6x + k = 0$. Hence $k = (x + 2i)(x - 2i)$ and $-6 = (x + 2i) + (x - 2i)$.

$\therefore$ $k = x^2 + 4$ and $-6 = 2x$. Thus $x = -3$ and $k = 13$. Hence both roots of the equation are $-3 \pm 2i$.

Problem CO2_11. $1 - 2i$ is one root of $x^2 - (3+i)x + k = 0$. Find k and the other root of the equation.

Answer: $2 - 3i$; $k = 8 - i$.

Explanation: Let z be the other root of $x^2 - (3+i)x + k = 0$. Then $3 + i = (1 - 2i) + z$.

$\therefore$ $z = (3+i) - (1-2i) = 2 + 3i$. Hence $k = (1-2i)z = (1-2i)(2+3i) = (2+6) + i(-4+3) =$
$= 8 - i$.

Problem CO2_12. Express $(3+2i)(5+4i)$ and $(3-2i)(5-4i)$ in the form $a+ib$. Hence find the prime factors of $7^2 + 22^2$.

Answer: $7 - - 22i$; $7 - 22i$; $(3^2 + 2^2)(5^2 + 4^2)$.

Explanation: Let $z_1 = 3 + 2i$ and $z_2 = 5 + 4i$. Then

$z_1 z_2 = (3 + 2i)(5 + 4i) = (15 - 8) + i(12 + 10) = 7 + 22i$,

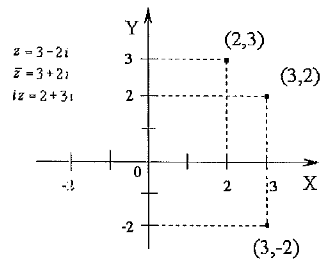
$\bar{z}_1 \bar{z}_2 = (3 - 2i)(5 - 4i) = (15 - 8) - i(12 + 10) = 7 - 22i$.

Hence $|z_1 z_2|^2 = 7^2 + 22^2$. But $|z_1 z_2|^2 = |z_1|^2 \cdot |z_2|^2 = (3^2 + 2^2)(5^2 + 4^2)$. Therefore

$7^2 + 22^2 = (3^2 + 2^2)(5^2 + 4^2)$.

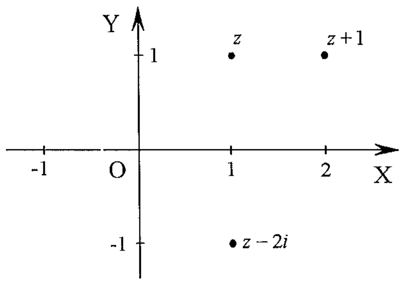
Problem CO2_13. If $z = 3 - 2i$, plot on an Argand diagram the points representing respectivly z, $\bar{z}$, iz.

Explanation:



Problem CO2_14. If $z = 1 + i$, mark on an Argand diagram the points representing $z + 1$, $z - 2i$.

Explanation:



(a) $z+1=2+i$
(b) $z-2i=1-i$

Problem CO2_15. Find $|z|$ and $\arg z$ when (a) $z=2i$; (b) $z=-\sqrt{3}-i$.

Answer: (a) $|z|=2, \arg z=\frac{\pi}{2}$; (b) $|z|=2, \arg z=-\frac{5\pi}{6}$.

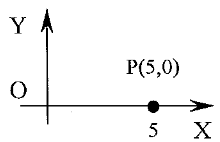
Explanation:
(a) $z=2i=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\Rightarrow |z|=2,\quad \arg z=\frac{\pi}{2}$
(b) $z=-\sqrt{3}-i=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)\Rightarrow |z|=2,\quad \arg z=-\frac{5\pi}{6}$.

Problem CO2_16. Find the modulus and principal argument of (a) 5; (b) -5; (c) $2-i$; (d) $-1-\sqrt{3}i$.

Answer: (a) $5, 0$; (b) $5, \pi$; (c) $\sqrt{5}, -\tan^{-1}(1/2)$; (d) $2, -\frac{2\pi}{3}$.

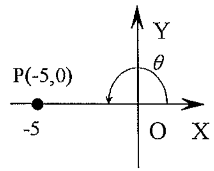
Explanation: In each case $P(a,b)$ represents the complex number $z=a+ib$ and θ is the principal argument of z

(a)



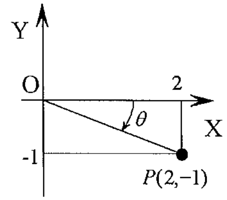
$z=5$
$|z|=5$
$\theta=0\Rightarrow \arg z=0$

(b)



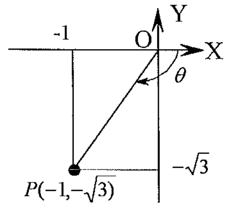
$z=-5$
$|z|=5$
$\arg z=\pi$

(c)



$z=2-i$
$|z|=\sqrt{4+1}=\sqrt{5}$
$\arg z=-\tan^{-1}(1/2)$

(d)



$z=-1-\sqrt{3}i$
$|z|=\sqrt{1+3}=2$
$\theta=-\pi+\frac{\pi}{3}\Rightarrow \arg z=-\frac{2\pi}{3}$

Problem CO2_17. Express $1-i$ in modulus/argument form.

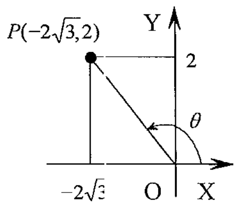
Answer: $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$.

Explanation: $z=1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$.

Problem CO2_18. Express $-2\sqrt{3}+2i$ in modulus/argument form.

Answer: $4\operatorname{cis}\frac{5\pi}{6}$.

Explanation:



Let $z=-2\sqrt{3}+2i$
$|z|=\sqrt{12+4}=4$
$\theta=\pi-\frac{\pi}{6}\Rightarrow \arg z=\frac{5\pi}{6}$
$z=4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=4\operatorname{cis}\frac{5\pi}{6}$

Problem CO2_19. Write z in the form $a+ib$ when $|z|=2$; $\arg z=-\frac{\pi}{6}$.

Answer: $\sqrt{3}-i$.

Explanation: $z=2\text{cis}\left(-\frac{\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\sqrt{3}-i$.

Problem CO2_20. Find $|z_1z_2|$ and $\arg(z_1z_2)$ when $z_1=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$, $z_2=\sqrt{2}[\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})]$.

Answer: $2\sqrt{2}$; $\frac{\pi}{12}$.

Explanation: $|z_1|=2$ and $\arg z_1=\frac{\pi}{3}$, $|z_2|=\sqrt{2}$ and $\arg z_2=-\frac{\pi}{4}$.

$|z_1z_2|=|z_1|\cdot|z_2|=2\sqrt{2}$ and $\arg(z_1z_2)=\arg z_1+\arg z_2=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$.

Problem CO2_21. Use the method of mathematical induction to prove that $|z^n|=|z|^n$ and $\arg(z^n)=n\arg z$ for all positive integers n.

Explanation: Define the statement $S(n)$: $|z^n|=|z|^n$ and $\arg(z^n)=n\arg z$, $n=1,2,\ldots$ Clearly $S(1)$ is true. If $S(k)$ is true, then $|z^k|=|z|^k$ and $\arg(z^k)=k\arg z$. Consider $S(k+1)$.

$|z^{k+1}|=|z^k\cdot z|=|z^k|\cdot|z|=|z|^k\cdot|z|$, if $S(k)$ is true.

$\therefore |z^{k+1}|=|z|^{k+1}$, if $S(k)$ is true.

$\arg(z^{k+1})=\arg(z^k\cdot z)=\arg(z^k)+\arg z=k\arg z+\arg z$, if $S(k)$ is true.

$\therefore \arg(z^{k+1})=(k+1)\arg z$, if $S(k)$ is true.

Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers n.

$\therefore |z^n|=|z|^n$ and $\arg(z^n)=n\arg z$ for all positive integers n.

Problem CO2_22. $z_1=4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$ and $z_2=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$. Write down the modulus and principal argument of (a) z_1^3; (b) $\frac{1}{z_2}$; (c) $\frac{z_1^3}{z_2}$.

Answer: (a) 64, π; (b) $\frac{1}{2}$, $-\frac{\pi}{6}$; 32, $\frac{5\pi}{6}$.

Explanation: (a) $|z_1|=4\Rightarrow|z_1^3|=4^3=64$, $\arg z_1=\frac{\pi}{3}\Rightarrow\arg(z_1^3)=3\cdot\frac{\pi}{3}=\pi$.

$\therefore z_1^3$ has modulus 64 and principal argument π.

(b) $|z_2|=2\Rightarrow\left|\frac{1}{z_2}\right|=\frac{1}{2}$, $\arg z_2=\frac{\pi}{6}\Rightarrow\arg\left(\frac{1}{z_2}\right)=-\frac{\pi}{6}$.

$\therefore \frac{1}{z_2}$ has modulus $\frac{1}{2}$ and principal argument $-\frac{\pi}{6}$.

(c) $\frac{z_1^3}{z_2}=z_1^3\cdot\left(\frac{1}{z_2}\right)\Rightarrow\begin{cases}\left|\frac{z_1^3}{z_2}\right|=|z_1^3|\cdot\left|\frac{1}{z_2}\right|=64\cdot\frac{1}{2}=32\\ \arg\left(\frac{z_1^3}{z_2}\right)=\arg(z_1^3)+\arg\left(\frac{1}{z_2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}\end{cases}$.

$\therefore \frac{z_1^3}{z_2}$ has modulus 32 and principal argument $\frac{5\pi}{6}$.

Problem CO2_23. Write down the moduli and arguments of $-\sqrt{3}+i$ and $4+4i$. Hence express in modulus/argument form $\frac{-\sqrt{3}+i}{4+4i}$.

Answer: 2, $\frac{5\pi}{6}$; $4\sqrt{2}$, $\frac{\pi}{4}$; $\frac{1}{2\sqrt{2}}\text{cis}\frac{7\pi}{12}$.

Explanation: Let $z_1=-\sqrt{3}+i$ and $z_2=4+4i$. Then

$z_1=2\left(\frac{-\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)\Rightarrow|z_1|=2, \arg z_1=\frac{5\pi}{6}$,

$z_2=4\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\Rightarrow|z_2|=4\sqrt{2}, \arg z_2=\frac{\pi}{4}$.

$\frac{-\sqrt{3}+i}{4+4i}=\frac{z_1}{z_2}$. But $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}=\frac{1}{2\sqrt{2}}$ and $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2=\frac{5\pi}{6}-\frac{\pi}{4}=\frac{7\pi}{12}$. Hence

$\frac{-\sqrt{3}+i}{4+4i}=\frac{1}{2\sqrt{2}}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=\frac{1}{2\sqrt{2}}\text{cis}\frac{7\pi}{12}$.

Problem CO2_24. z has modulus r and argument θ. Find in terms of r and θ the modulus and one argument of (a) $\frac{1}{z}$; (b) iz.

Answer: (a) $\left|\frac{1}{z}\right|=\frac{1}{r}$, $\arg\left(\frac{1}{z}\right)=-\theta$; (b) $|iz|=r$, $\arg(iz)=\frac{\pi}{2}+\theta$.

Explanation: (a) $\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{r}$ and $\arg\left(\frac{1}{z}\right) = -\arg z = -\theta$

(b) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Rightarrow |i| = 1$ and $\arg i = \frac{\pi}{2}$. Then $|iz| = |i|\cdot|z| = 1\cdot r = r$ and

$\arg(iz) = \arg(i) + \arg z = \frac{\pi}{2} + \theta$.

Problem CO2_25. Write $\sqrt{3}+i$ and $\sqrt{3}-i$ in modulus/argument form. Hence write $\left(\sqrt{3}+i\right)^{10} + \left(\sqrt{3}-i\right)^{10}$ in the form $a+ib$.

Answer: $2, \frac{\pi}{6}$; $2, -\frac{\pi}{6}$; 1024.

Explanation: Let $z = \sqrt{3}+i$. Then $z = 2\cdot\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \Rightarrow |z| = 2, \arg z = \frac{\pi}{6}$.

Let $z_1 = \sqrt{3} - i$. Then $z_1 = \bar{z}$ and $|z_1| = |z| = 2$, $\arg z_1 = -\arg z = -\frac{\pi}{6}$. Hence

$\left|z^{10}\right| = |z|^{10} = 2^{10} = 1024$, $\left|z_1^{10}\right| = |z_1|^{10} = |z|^{10} = 1024$ and $\arg\left(z^{10}\right) = 10\arg z = \frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$,

$\arg\left(z_1^{10}\right) = 10\arg z_1 = -\frac{5\pi}{3} = -2\pi + \frac{\pi}{3}$. Therefore

$$z^{10} + z_1^{10} = 1024\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) + 1024\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\cdot1024\cdot\cos\frac{\pi}{3} = 1024.$$

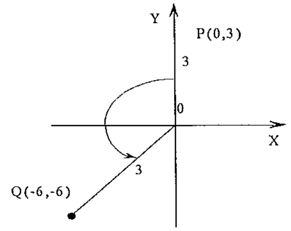
$\therefore \sqrt{3}+i = 2\operatorname{cis}\frac{\pi}{6}$, $\sqrt{3}-i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$, $\left(\sqrt{3}+i\right)^{10} + \left(\sqrt{3}-i\right)^{10} = 1024$.

Problem CO2_26. Describe geometrically the transformation $z \to \alpha z$, where $\alpha = -2+2i$. Illustrate on an Argand diagram for $z = 3i$.

Explanation: $\alpha = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$,

$\therefore \alpha = 2\sqrt{2}\beta$, where $\beta = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$. $z \to \alpha z$ can be expressed as $z \to \beta z \to 2\sqrt{2}\beta z$. Let P, Q_1, Q represent z, βz, $2\sqrt{2}\beta z$ respectively. Then $|\beta z| = |\beta|\cdot|z| = |z| \Rightarrow OQ_1 = OP$

$\arg(\beta z) = \frac{3\pi}{4} + \arg z \Rightarrow$ ray OQ_1 makes the angle $\frac{3\pi}{4}$ with ray OP. Hence $\beta \to \beta z$ is a rotation anticlockwise about P through $\frac{3\pi}{4}$ and $z \to \alpha z$ is the composition of this rotation followed by an enlargement about O by the factor $2\sqrt{2}$.



$z = 3i, |z| = 3$ and $\arg z = \frac{\pi}{2}$

$|\beta z| = 3$ and $\arg(\beta z) = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$

$|\alpha z| = 6\sqrt{2}$ and $\arg(\alpha z) = \frac{5\pi}{4}$

$\alpha z = -6 - 6i$

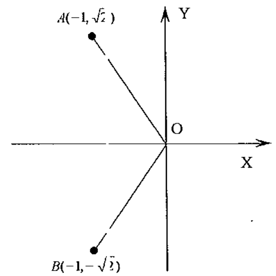
Problem CO2_27. Obtain in the form $a+ib$ the roots of the equation $x^2 + 2x + 3 = 0$. Find the modulus and argument of each root and represent the roots on an Argand diagram by the points A and B.

Answer: $x_1 = -1+\sqrt{2}i$, $x_2 = -1-\sqrt{2}i$; $\sqrt{3}$, $\pi - \tan^{-1}\sqrt{2}$; $\sqrt{3}$, $-\left(\pi - \tan^{-1}\sqrt{2}\right)$;

Explanation: Using the quadratic formula:

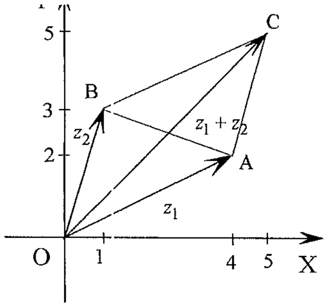
$x^2 + 2x + 3 = 0 \Rightarrow \Delta = -8 \Rightarrow x = \frac{-2 \pm i2\sqrt{2}}{2} = -1 \pm \sqrt{2}i$. Let $x_1 = -1+\sqrt{2}i$ and $x_2 = -1-\sqrt{2}i$.

Then $|x_1| = |x_2| = \sqrt{1+2} = \sqrt{3}$ and $\arg x_1 = \pi - \tan^{-1}\sqrt{2}$, $\arg x_2 = -\left(\pi - \tan^{-1}\sqrt{2}\right)$.



Problem CO2_28. Show geometrically how to construct the vectors representing (a) $z_1 + z_2$; (b) $z_1 - z_2$; (c) $z_2 - z_1$, when $z_1 = 4+2i$, $z_2 = 1+3i$.

Explanation:



Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2.

Then (a) $\overrightarrow{OC}$ represents z_1+z_2

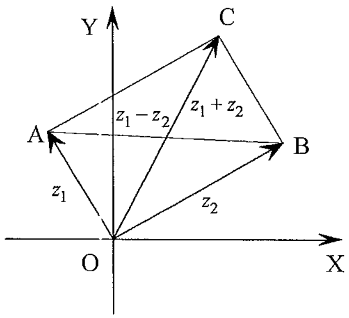
(b) $\overrightarrow{BA}$ represents z_1-z_2

(c) $\overrightarrow{AB}$ represents z_2-z_1.

Problem CO2_29. If $|z_1+z_2|=|z_1-z_2|$, find the possible values of $\arg\left(\frac{z_1}{z_2}\right)$.

Answer: $\pm\frac{\pi}{2}$.

Explanation:

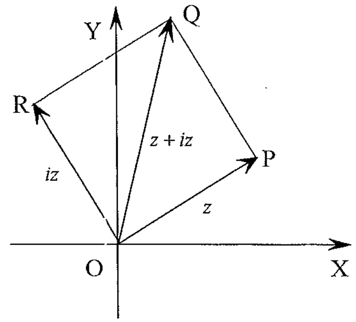


Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. Construct the parallelogram $OACB$. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent z_1+z_2, z_1-z_2 respectively. Since $|z_1+z_2|=|z_1-z_2|$, $OC=AB$. Hence $OACB$ is a rectangular. Therefore $\angle AOB=\frac{\pi}{2}$. But $\angle AOB=\arg z_1-\arg z_2$ (or $\angle AOB=\arg z_2-\arg z_1$). Thus

$$\arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}.$$

Problem CO2_30. On an Argand diagram the points P and Q represent z and $z+iz$ respectively. Show that OPQ is a right-angled triangle.

Explanation:



Let R represent iz. We know that the transformation $z\to iz$ corresponds to a rotation anticlockwise about O through the angle $\frac{\pi}{2}$ in the Argand diagram. Therefore $OPQP$ is a square. Hence OPQ is a right-angled triangle.

Problem CO2_31. Show that $||z_1|-|z_2||=|z_1+z_2|$. State the condition for equality.

Answer: $z_1=-kz_2, k>0$, or $z_1=0$, or $z_2=0$.

Explanation: If $z_1=0$ or $z_2=0$, $||z_1|-|z_2||=|z_1+z_2|$. Let now $z_1\neq 0$ and $z_2\neq 0$. Then $|z_1|-|z_2|=|z_1+z_2-z_2|-|z_2|\leq|z_1+z_2|+|-z_2|-|z_2|=|z_1+z_2|$ with equality if and only if

$z_1+z_2=k\cdot(-z_2)$, $k>0$.

$\therefore$ $|z_1|-|z_2|\leq|z_1+z_2|$ with equality if and only if $z_1=-(1+k)z_2, k>0$.

$|z_2|-|z_1|=|z_2+z_1-z_1|-|z_1|\leq|z_2+z_1|+|-z_1|-|z_1|=|z_2+z_1|$ with equality if and only if

$z_2-z_1=k\cdot(-z_1)$, $k>0$.

$\therefore$ $|z_2|-|z_1|\leq|z_1+z_2|$ with equality if and only if $z_1=-\frac{1}{1+k}z_2, k>0$.

Hence $||z_1|-|z_2||\leq|z_1+z_2|$ with equality if and only if $z_1=-kz_2, k>0$, or $z_1=0$, or $z_2=0$.

Problem CO2_32. Use *de Moivre's* theorem to solve $z^5=-1$. By grouping the roots in complex conjugate pairs, show that $z^5+1=(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$.

Answer: $\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$, -1.

Explanation: $|-1| = 1$ and $\arg(-1) = \pi$. Hence the complex 5th roots of -1 all have modulus 1 and by *de Moivre's* theorem one complex 5th root of -1 has argument $\frac{\pi}{5}$, the others being equally spaced around the unit circle in the Argand diagram by an angle $\frac{2\pi}{5}$. Therefore the complex 5th roots of -1 are $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$, and -1.

Then $z^5 + 1 = (z+1)\left(z - \operatorname{cis}\frac{\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{\pi}{5}\right)\right)\left(z - \operatorname{cis}\frac{3\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{3\pi}{5}\right)\right)$. But

$\left(z - \operatorname{cis}\frac{\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{\pi}{5}\right)\right) = \left(\left(z - \cos\frac{\pi}{5}\right) - i\sin\frac{\pi}{5}\right)\left(\left(z - \cos\frac{\pi}{5}\right) + i\sin\frac{\pi}{5}\right) = \left(z - \cos\frac{\pi}{5}\right)^2 + \left(\sin\frac{\pi}{5}\right)^2 =$

$= z^2 - 2z\cos\frac{\pi}{5} + 1$ and $\left(z - \operatorname{cis}\frac{3\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{3\pi}{5}\right)\right) = z^2 - 2z\cos\frac{3\pi}{5} + 1$.

$\therefore\ z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$.

Problem CO2_33. If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. Hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 2\cos 2\theta + 3)$.

Explanation: By *de Moivre's* theorem and $z^n = \cos n\theta + i\sin n\theta$ and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$. Then $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$.

$2\cos\theta = z + z^{-1}$. Then $16\cos^4\theta = \left(z + z^{-1}\right)^4$. But $\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$

$= \left(z^4 + z^{-4}\right) + 4\left(z^2 + z^{-2}\right) + 6$. Hence $16\cos^4\theta = 2\cos 4\theta + 4\cos 2\theta + 6$ and

$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 2\cos 2\theta + 3)$.

Problem CO2_34. 1, ω and ω^2 are the three cube roots of unity. Simplify each of the expressions $\left(1 + 3\omega + \omega^2\right)^2$ and $\left(1 + \omega + 3\omega^2\right)^2$ and show that their sum is -4 and their product is 16.

Explanation: $\omega^3 = 1$. Since ω is a non-real root of unity, $\omega^2 + \omega + 1 = 0$ (it follows from the factorization $\omega^3 - 1 = (\omega - 1)\left(\omega^2 + \omega + 1\right)$.

Let $z_1 = \left(1 + 3\omega + \omega^2\right)^2$ and $z_2 = \left(1 + \omega + 3\omega^2\right)^2$.

Then $z_1 = \left(1 + \omega + \omega^2 + 2\omega\right)^2 = (2\omega)^2$ $\left(\text{since } 1 + \omega + \omega^2 = 0\right)$

$= 4\omega^2$

and $z_2 = \left(1 + \omega + \omega^2 + 2\omega^2\right)^2 = \left(2\omega^2\right)^2$ $\left(\text{since } 1 + \omega + \omega^2 = 0\right)$

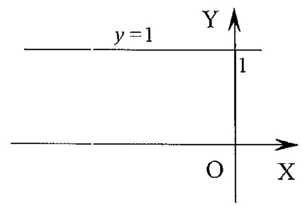
$= 4\omega^4 = 4\omega$ $\left(\text{since } \omega^3 = 1\right)$

Hence $z_1 + z_2 = 4\omega^2 + 4\omega = 4\left(\omega^2 + \omega + 1\right) - 4 = -4$ $\left(\text{since } \omega^2 + \omega + 1 = 0\right)$ and

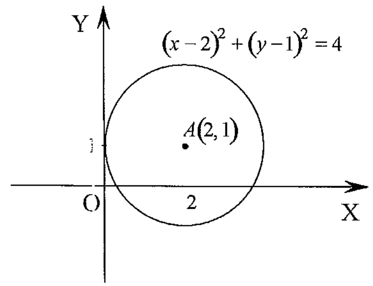
$z_1 \cdot z_2 = 4\omega^2 \cdot 4\omega = 16\omega^3 = 16$ $\left(\text{since } \omega^3 = 1\right)$.

Problem CO2_35. Indicate on an Argand diagram the locus of the point P representing z when (a) $\operatorname{Im} z = 1$; (b) $|z - 2 - i| = 2$; (c) $\arg(z + i) = \frac{3\pi}{4}$.

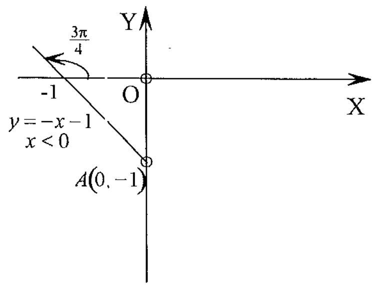
Explanation:



Let $z = x + iy$. Then
(a) $\operatorname{Im} z = 1 \Rightarrow y = 1$.



Let P represent z. Then
(b) Let A represent $2 + i$. Then $\vec{AP}$ represents $z - (2 + i)$ and $|z - 2 - i| = 2 \Rightarrow AP = 2$,
$\therefore$ P lies on the circle with the center $A(2,1)$ and radius 2.

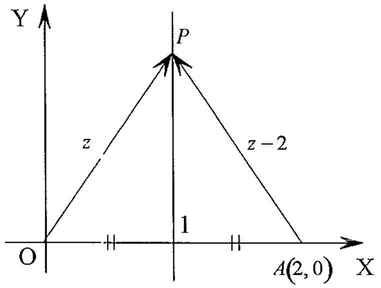


(c) Let A represent $-i$. Then $\overrightarrow{AP}$ represents $z+i$. AP has gradient $\tan\left(\frac{3\pi}{4}\right)=-1$. Hence the locus of P is the ray $y=-x-1,\ x<0$.

Problem CO2_36. The point P represent the complex number z on an Argand diagram. Describe the locus of P when $|z|=|z-2|$.

Answer: $x=1$.

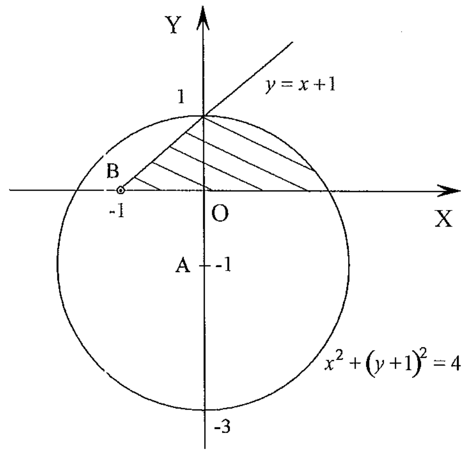
Explanation:



Let A represent 2. Then $\overrightarrow{AP}$ represents $z-2$, and $|z|=|z-2| \Rightarrow OP=AP$. The locus of P is the perpendicular bisector of OA. Therefore the locus of P has Cartesian equation $x=1$.

Problem CO2_37. $|z+i|\le 2$ and $0\le \arg(z+1)\le \frac{\pi}{4}$. Sketch the region in an Argand diagram which contains the point P representing z.

Explanation:

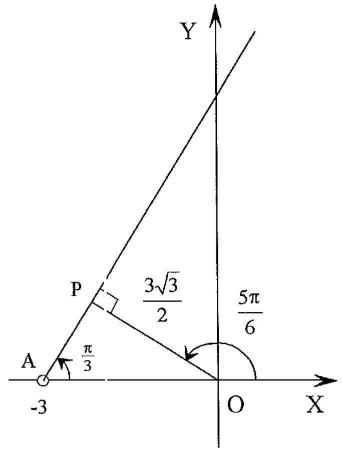


Let A represent $-i$ and B represent -1. Then, if P represents z, $\overrightarrow{AP}$ represents $z+i$ and $\overrightarrow{BP}$ represents $z+1$. Hence $AP\le 2$ and $\overrightarrow{BP}$ makes an angle between O and $\frac{\pi}{4}$ with the positive x-axis.

Problem CO2_38. $Arg(z+3)=\frac{\pi}{3}$. Sketch the locus of the point P representing z on an Argand diagram. Find the modulus and argument of z when $|z|$ takes its least value. Hence find in the form $a+ib$, the value of z for which $|z|$ is a minimum.

Answer: $\frac{3\sqrt{3}}{2},\ \frac{5\pi}{6};\ \frac{3}{4}\left(-3+i\sqrt{3}\right)$.

Explanation:

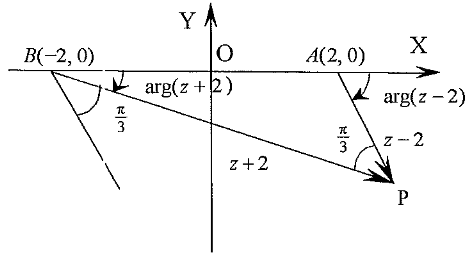


Let A represent -3. Then $\overrightarrow{AP}$ represents $z+3$. AP has gradient $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$. Hence the locus of P has Cartesian equation $y=\sqrt{3}x+3\sqrt{3}, x>-3$. Now $OP=|z|$. Hence the minimum value of $|z|$ is the perpendicular distance from $(0,0)$ to the locus of P. Therefore the minimum value of $|z|$ is $AO\cdot\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}$. Since AP has gradient $\tan\frac{\pi}{3}=\sqrt{3}$, OP has gradient $-\frac{1}{\sqrt{3}}=\tan\left(\frac{5\pi}{6}\right)$ when $|z|$ takes its least value. Hence modulus of z is $\frac{3\sqrt{3}}{2}$ and the argument of z is $\frac{5\pi}{6}$ when $|z|$ is a minimum. Therefore $z=\frac{3\sqrt{3}}{2}\operatorname{cis}\left(\frac{5\pi}{6}\right)=\frac{3\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\frac{3}{4}\left(-3+i\sqrt{3}\right)$.

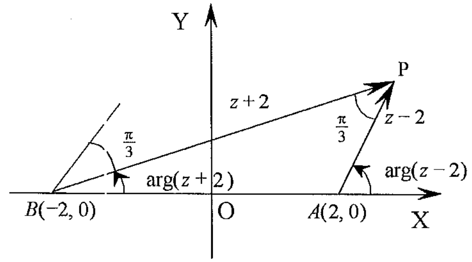
Problem CO2_39. If $\arg(z-2)=\arg(z+2)+\frac{\pi}{3}$, show that the locus of the point P representing z on an Argand diagram is an arc of a circle and find the center and radius of this circle.

Answer: $\left(0,\frac{2}{\sqrt{3}}\right)$, $\frac{4}{\sqrt{3}}$.

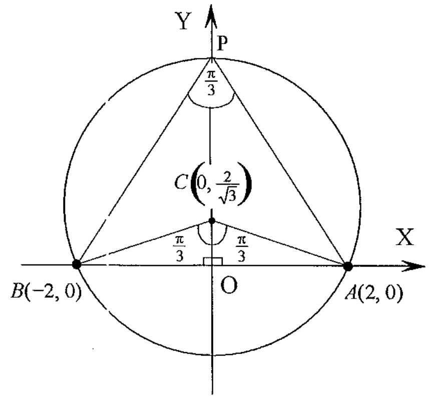
Explanation: Let $A(2,0)$, $B(-2,0)$ and P represent 2, -2, and z respectively. Then $\overrightarrow{AP}$ and $\overrightarrow{BP}$ represent $z-2$ and $z+2$ respectively, and $\arg(z-2)=\arg(z+2)+\frac{\pi}{3}$ requires $\overrightarrow{AP}$ to be parallel to the vector obtained by rotation of $\overrightarrow{BP}$ anticlockwise through the angle of $\frac{\pi}{3}$.



If P lies below the x-axis, AP must be parallel to a clockwise rotation of BP. This diagram shows $\arg(z-2)=\arg(z+2)-\frac{\pi}{3}$. Hence P must lie above the x-axis.



Since alternate angles between parallel lines are equal, $\angle BPA=\frac{\pi}{3}$ as P traces its locus. Hence P lies on the major arc AB of a circle through A and B.



The center C of this circle lies on the perpendicular bisector of AB, and the chord AB subtends an angle $2\cdot\frac{\pi}{3}=\frac{2\pi}{3}$ at C.

Therefore $OC = \frac{2}{\sqrt{3}}$ and $AC = \frac{4}{\sqrt{3}}$. Thus the center of this circle is $C\left(0, \frac{2}{\sqrt{3}}\right)$ and the radius is $\frac{4}{\sqrt{3}}$.

Problem CO2_40. The complex number is given by $z = t + \frac{1}{t}$, where $t = r(\cos\theta + i\sin\theta)$. Find the equation of the locus of the point P which represents z on an Argand diagram when $r = 2$ and θ varies.

Answer: $\frac{4x^2}{25} + \frac{4y^2}{9} = 1$.

Explanation: Let $P(x, y)$ represent $z = x + iy$. Then

$$x + iy = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta,$$

$\therefore x = \left(r + \frac{1}{r}\right)\cos\theta$ and $y = \left(r - \frac{1}{r}\right)\sin\theta$. $x = \frac{5}{2}\cos\theta$ and $y = \frac{3}{2}\sin\theta$. Hence $\frac{4x^2}{25} + \frac{4y^2}{9} = 1$.