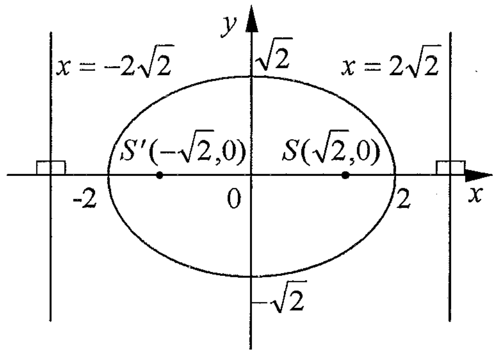
Topic 4. Conics.
Level 2.

Problem CON2_1. For the ellipse $x^2+2y^2=4$, find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices. Sketch the ellipse.

Answer: (a) $\frac{1}{\sqrt{2}}$; (b) $(\pm\sqrt{2},0)$; (c) $x=\pm 2\sqrt{2}$.

Explanation:



$x^2+2y^2=4$, $\frac{x^2}{4}+\frac{y^2}{2}=1$, $\begin{matrix} a=2, b=\sqrt{2}\Rightarrow b<a \\ b^2=a^2(1-e^2) \end{matrix}$,

eccentricity: $e=\sqrt{1-\frac{2}{4}}=\frac{1}{\sqrt{2}}$

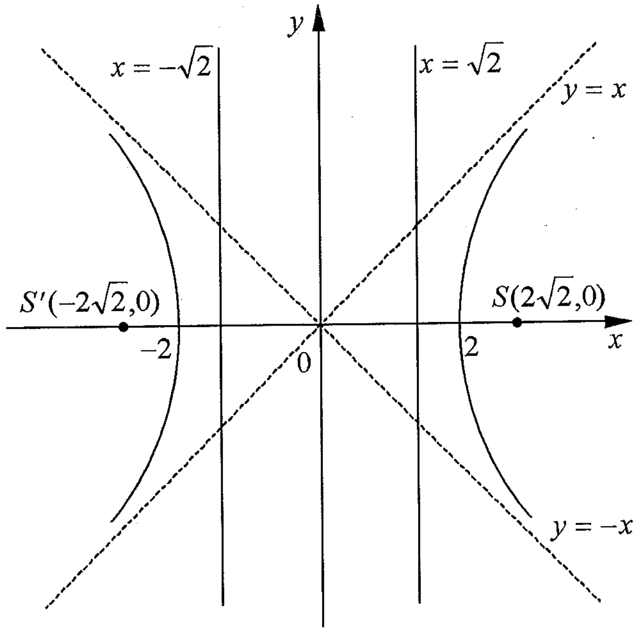
foci: $(\pm ae,0)\Rightarrow(\pm\sqrt{2},0)$

directrices: $x=\pm\frac{a}{e}\Rightarrow x=\pm 2\sqrt{2}$.

Problem CON2_2. For the hyperbola $x^2-y^2=4$, find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices, (d) the equations of the asymptotes. Sketch the hyperbola.

Answer: (a) $\sqrt{2}$; (b) $(\pm 2\sqrt{2},0)$; (c) $x=\pm\sqrt{2}$; (d) $y=\pm x$.

Explanation:



$x^2-y^2=4$, $\dfrac{x^2}{4}-\dfrac{y^2}{4}=1$, $\begin{matrix} a=2, b=2 \\ b^2=a^2(e^2-1)\end{matrix}$,

eccentricity: $e=\sqrt{1+\dfrac{4}{4}}=\sqrt{2}$

foci: $(\pm ae,0)\Rightarrow(\pm 2\sqrt{2},0)$

directrices: $x=\pm\dfrac{a}{e}\Rightarrow x=\pm\sqrt{2}$

asymptotes: $y=\pm\dfrac{b}{a}x\Rightarrow y=\pm x$

Problem CON2_3. The ellipse has eccentricity $\dfrac{2}{3}$ and directrices $x=-9$ and $x=9$. Find the equation of the ellipse.

Answer: $\dfrac{x^2}{36}+\dfrac{y^2}{20}=1$.

Explanation: We have the eccentricity $e=\dfrac{2}{3}$ and the directrices $x=\pm 9$ of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. But the directrices have equations $x=\pm\dfrac{a}{e}$. Therefore $a=9\cdot\dfrac{2}{3}=6$. Then $b^2=a^2(1-e^2)=36\cdot\left(1-\dfrac{4}{9}\right)=20$. Hence the Cartesian equation of the ellipse is $\dfrac{x^2}{36}+\dfrac{y^2}{20}=1$.

Problem CON2_4. The hyperbola has eccentricity $\dfrac{5}{4}$ and foci $(-5,0)$ and $(5,0)$. Find the equation of the hyperbola.

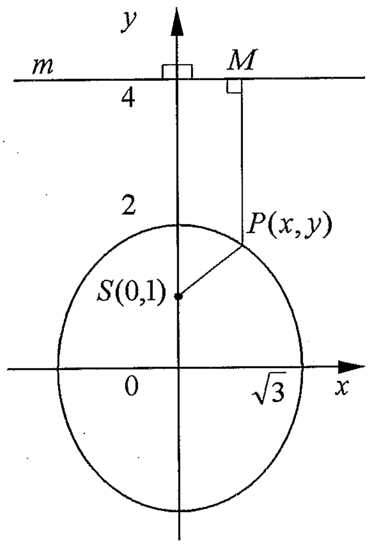
Answer: $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Explanation: We have the eccentricity $e=\frac{5}{4}$ and the foci $(\pm 5,0)$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. But the coordinates of the foci are $(\pm ae,0)$. Therefore $a=5\cdot\frac{4}{5}=4$. Then $b^2=a^2(e^2-1)=16\cdot\left(\frac{25}{16}-1\right)=9$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Problem CON2_5. A variable point $P(x,y)$ moves so that its distance from $(0,1)$ is one-half its distance from $y=4$. Find the locus of P.

Answer: $\frac{x^2}{3}+\frac{y^2}{4}=1$.

Explanation:



The locus of a variable point $P(x,y)$ is the ellipse with focus at $S(0,1)$, directrix $m: y=4$ and eccentricity $e=\frac{1}{2}$. Let M be the foot of the perpendicular from P to m. Then M has coordinates $(x,4)$.

$$PS=e\cdot PM \Rightarrow x^2+(y-1)^2=\left(\frac{1}{2}\right)^2(y-4)^2$$

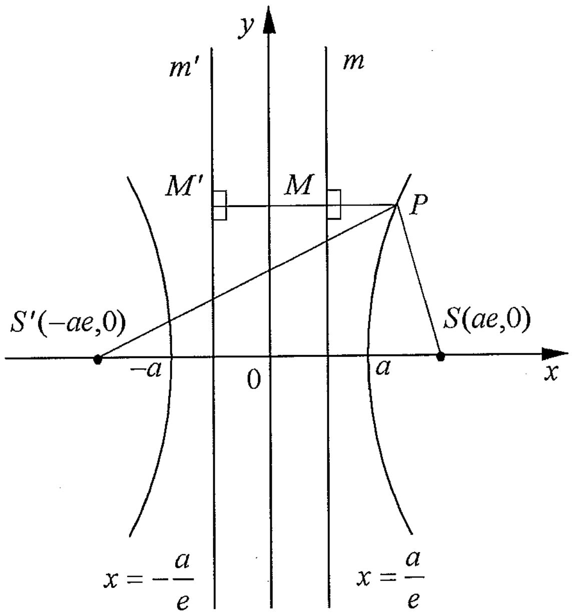
$$x^2+y^2\left(1-\frac{1}{4}\right)=4-1.$$

Therefore the Cartesian equation of the ellipse is $\frac{x^2}{3}+\frac{y^2}{4}=1$.

Problem CON2_6. A point P lies on the hyperbola $\frac{x^2}{9}-\frac{y^2}{72}=1$ with foci S and S'. Find PS' if $PS=8$.

Answer: 14 or 2.

Explanation:



Let m and m' be the directrices of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then for P on the curve, both $PS=e\cdot PM$ and $PS'=e\cdot PM'$, where M and M' are the feet of the perpendiculars from P to m and m' respectively. Therefore $|PS-PS'|=e|PM-PM'|=eMM'$. Thus $|PS-PS'|=2a$.

For the hyperbola $\frac{x^2}{9}-\frac{y^2}{72}=1$ $a=3$. Hence $|PS-PS'|=6$. Since $b^2=72$, $e=\sqrt{\frac{b^2}{a^2}+1}=\sqrt{\frac{72}{9}+1}=3$. Therefore the coordinates of the foci are $(\pm 9,0)$. If $PS=8$, then $|PS'-8|=6$. Thus $PS'=14$ or 2.

Problem CON2_7. A hyperbola has center at the origin and foci on the x-axis. The distance between the foci is 16 units and the distance between the directrices is 4 units. Find the equation of the hyperbola.

Answer: $\frac{x^2}{16}-\frac{y^2}{48}=1$.

Explanation: Since foci of a hyperbola are on the x-axes, then the equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Thus we need to find the parameters a and b. Coordinates of the foci are $(\pm ae,0)$. Therefore the distance between the foci is $2ae=16$. The equations of the directrices are $x=\pm\frac{a}{e}$.

Hence the distance between the directrices is $2\cdot\frac{a}{e}=4$. Thus we have two equations $ae=8$ and $\frac{a}{e}=2$. From the first equation we get $e=\frac{8}{a}$. Substituting the expression for the e to the second equation we obtain $a^2=16$. Therefore $a=4$ and $e=\frac{8}{4}=2$.

Then $b^2=a^2(e^2-1)=16\cdot(4-1)=48$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{48}=1$.

Problem CON2_8. Show that the equation $\frac{x^2}{29-k}+\frac{y^2}{4-k}=1$, where k is a real number, represents (i) an ellipse if $k<4$; (ii) a hyperbola if $4<k<29$. Show that the foci of each ellipse in (i) and each hyperbola in (ii) are independent of the value of k.

Explanation: (i) If $k<4$, then $29-k>0$ and $4-k>0$. Therefore $\frac{x^2}{29-k}+\frac{y^2}{4-k}=1$ is an ellipse with $a=\sqrt{29-k}$ and $b=\sqrt{4-k}$. Since $b<a$, then $b^2=a^2(1-e^2)$. Hence $e=\frac{\sqrt{a^2-b^2}}{a}$ and the foci have coordinates $(\pm ae,0)=\left(\pm\sqrt{a^2-b^2},0\right)=(\pm 5,0)$. Thus the foci of the ellipse are independent of the value of k.

(ii) If $4<k<29$, then $29-k>0$ and $4-k<0$. Therefore $\frac{x^2}{29-k}+\frac{y^2}{4-k}=1$ is a hyperbola with $a=\sqrt{29-k}$ and $b=\sqrt{k-4}$. For the hyperbola $b^2=a^2(e^2-1)$. Hence $e=\frac{\sqrt{a^2+b^2}}{a}$ and the foci have coordinates $(\pm ae,0)=\left(\pm\sqrt{a^2+b^2},0\right)=(\pm 5,0)$. Thus the foci of the hyperbola are independent of the value of k.

Problem CON2_9. Find the parametric equations of:

(a) The ellipse $x^2+4y^2=4$;

(b) The hyperbola $x^2-y^2=4$.

Answer: (a) $x=2\cos\theta$, $y=\sin\theta$; (b) $x=2\sec\theta$, $y=2\tan\theta$.

Explanation: (a) Cartesian equation of the ellipse is $x^2+4y^2=4$. Then $\frac{x^2}{4}+\frac{y^2}{1}=1$. Hence $a=2$ and $b=1$. Therefore the ellipse has parametric equations $x=2\cos\theta$ and $y=\sin\theta$, $-\pi<\theta\le\pi$.

(b) Cartesian equation of the hyperbola is $x^2-y^2=4$. Then $\frac{x^2}{4}-\frac{y^2}{4}=1$. Hence $a=2$ and $b=2$. Therefore the hyperbola has parametric equations $x=2\sec\theta$ and $y=2\tan\theta$, $-\pi<\theta\le\pi; \theta\ne\pm\frac{\pi}{2}$.

Problem CON2_10. Find the Cartesian equations of:
(a) The ellipse $x = 5\cos\theta$, $y = 4\sin\theta$;
(b) The hyperbola $x = 2\sec\theta$, $y = 5\tan\theta$.

Answer: (a) $\frac{x^2}{25}+\frac{y^2}{16}=1$; (b) $\frac{x^2}{4}-\frac{y^2}{25}=1$.

Explanation:

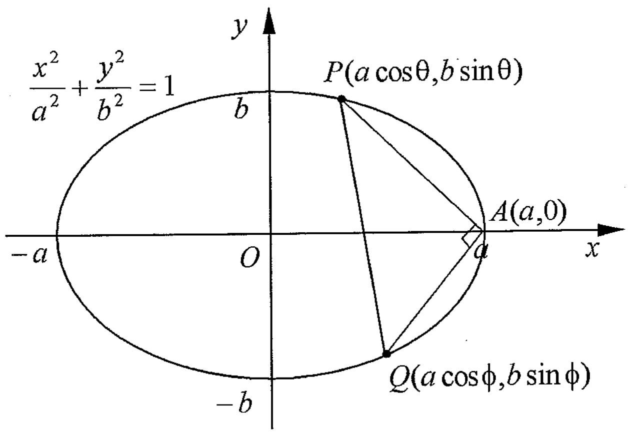
(a) The ellipse has parametric equations $x = 5\cos\theta$, $y = 4\sin\theta$. Therefore $\frac{x^2}{25}+\frac{y^2}{16}=\cos^2\theta+\sin^2\theta=1$. Hence the Cartesian equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$.
(b) The hyperbola has parametric equations $x = 2\sec\theta$, $y = 5\tan\theta$. Therefore $\frac{x^2}{4}-\frac{y^2}{25}=\sec^2\theta-\tan^2\theta=1$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{25}=1$.

Problem CON2_11. The points $P(a\sec\theta, b\tan\theta)$ and $Q[a\sec(\pi-\theta), b\tan(\pi-\theta)]$ lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Show that PQ passes through $(0,0)$.

Explanation: The equation of the chord PQ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$, where P, Q have parameters θ, ϕ. We have $\phi=\pi-\theta$. Hence the equation of the chord PQ transforms into $\frac{x}{a}\cos\left(\frac{2\theta-\pi}{2}\right)-\frac{y}{b}\sin\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)$. Thus $\frac{x}{a}\sin\theta-\frac{y}{b}=0$. Therefore $(0,0)$ lies on the chord PQ.

Problem CON2_12. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If PQ subtends a right angle at $(a,0)$. Show that $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

Explanation:



PAQ is a right-angled triangle. Therefore $AP^2 + AQ^2 = PQ^2$.

$a^2(\cos\theta - 1)^2 + b^2\sin^2\theta + a^2(\cos\phi - 1)^2 + b^2\sin^2\phi = a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2$.

Then $-2a^2\cos\theta + a^2 - 2a^2\cos\phi + a^2 = -2a^2\cos\theta\cos\phi - 2b^2\sin\theta\sin\phi$,

$\cos\theta + \cos\phi - 1 - \cos\theta\cos\phi = \frac{b^2}{a^2}\sin\theta\sin\phi$,

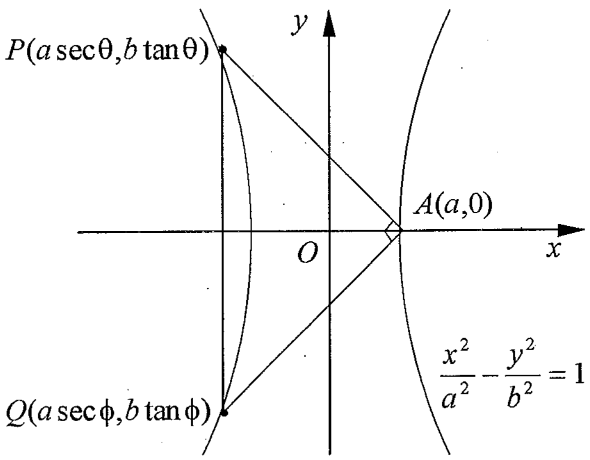
$\left(1-2\sin^2\frac{\theta}{2}\right)+\left(1-2\sin^2\frac{\phi}{2}\right)-1-\left(1-2\sin^2\frac{\theta}{2}\right)\left(1-2\sin^2\frac{\phi}{2}\right)=\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$,

$-4\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2}=\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$.

Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{b^2}{a^2}$.

Problem CON2_13. The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If PQ subtends a right angle at $(a, 0)$. Show that $\tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{b^2}{a^2}$.

Explanation:



PAQ is a right-angled triangle. Therefore $AP^2 + AQ^2 = PQ^2$.

$a^2(\sec\theta - 1)^2 + b^2\tan^2\theta + a^2(\sec\phi - 1)^2 + b^2\tan^2\phi = a^2(\sec\theta - \sec\phi)^2 + b^2(\tan\theta - \tan\phi)^2$

Then $-2a^2\sec\theta + a^2 - 2a^2\sec\phi + a^2 = -2a^2\sec\theta\sec\phi - 2b^2\tan\theta\tan\phi$,

$\cos\theta + \cos\phi - 1 - \cos\theta\cos\phi = \frac{b^2}{a^2}\sin\theta\sin\phi$,

$\left(1-2\sin^2\frac{\theta}{2}\right)+\left(1-2\sin^2\frac{\phi}{2}\right)-1-\left(1-2\sin^2\frac{\theta}{2}\right)\left(1-2\sin^2\frac{\phi}{2}\right)=\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$,

$-4\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2}=\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$.

Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

Problem CON2_14. The points $P(a\sec\theta, b\tan\theta)$ and $Q[a\sec(-\theta), b\tan(-\theta)]$ are the extremeties of the latus rectum $x = ae$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Show that (a) $\sec\theta = e$; (b) PQ has length $\frac{2b^2}{a}$.

Explanation: (a) Chord PQ has equation $x = ae$, P has coordinates $(a\sec\theta, b\tan\theta)$. Hence $a\sec\theta = ae$. Thus $\sec\theta = e$.
(b) Length of the chord PQ is $|b\tan\theta - b\tan(-\theta)| = 2b|\tan\theta| = 2b\sqrt{\sec^2\theta - 1} = 2b\sqrt{e^2-1}$. But for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ we have $b^2 = a^2(e^2-1)$. Therefore the length of the chord PQ is $2b\cdot\frac{b}{a}=\frac{2b^2}{a}$.

Problem CON2_15. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci $S(al, 0)$ and $S'(-al, 0)$. Show that (a) $PS = a(1-e\cos\theta)$ and $PS' = a(1+e\cos\theta)$;
(b) $PS + PS' = 2a$.

Explanation: (a) Length of PS is $\sqrt{(a\cos\theta - ae)^2 + (b\sin\theta)^2} = \sqrt{a^2(\cos\theta - e)^2 + b^2\sin^2\theta}$. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have $b^2 = a^2(1-e^2)$. Therefore the length of PS is
$\sqrt{a^2(\cos\theta - e)^2 + a^2(1-e^2)\sin^2\theta} = a\sqrt{\cos^2\theta - 2e\cos\theta + e^2 + \sin^2\theta - e^2\sin^2\theta} =$
$a\sqrt{(\cos^2\theta + \sin^2\theta) - 2e\cos\theta + e^2(1-\sin^2\theta)} = a\sqrt{1 - 2e\cos\theta + e^2\cos^2\theta} = a\sqrt{(1-e\cos\theta)^2}$
Hence the length of PS is $a(1-e\cos\theta)$.
Length of PS' is $\sqrt{(a\cos\theta + ae)^2 + (b\sin\theta)^2} = \sqrt{a^2(\cos\theta + e)^2 + b^2\sin^2\theta}$. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have $b^2 = a^2(1-e^2)$. Therefore the length of PS' is
$\sqrt{a^2(\cos\theta + e)^2 + a^2(1-e^2)\sin^2\theta} = a\sqrt{\cos^2\theta + 2e\cos\theta + e^2 + \sin^2\theta - e^2\sin^2\theta} =$
$a\sqrt{(\cos^2\theta + \sin^2\theta) + 2e\cos\theta + e^2(1-\sin^2\theta)} = a\sqrt{1 + 2e\cos\theta + e^2\cos^2\theta} = a\sqrt{(1+e\cos\theta)^2}$
Hence the length of PS' is $a(1+e\cos\theta)$.
(b) $PS + PS' = a(1-e\cos\theta) + a(1+e\cos\theta) = 2a$.

Problem CON2_16. Find the equations of the tangent and the normal to the ellipse $3x^2 + 4y^2 = 48$ at the point $(2,-3)$.

Answer: $x - 2y = 8$, $2x + y = 1$.

Explanation: $3x^2+4y^2=48 \Rightarrow \frac{x^2}{16}+\frac{y^2}{12}=1$. The tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ at the point (2,–3) has equation $\frac{2x}{16}+\frac{-3y}{12}=1 \Rightarrow x-2y=8$. The normal to the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ at the point (2,–3) has equation $\frac{16x}{2}-\frac{12y}{-3}=16-12 \Rightarrow 2x+y=1$.

Problem CON2_17. Find the equation of the tangent and the normal to the hyperbola $9x^2-2y^2=18$ at the point $(2,-3)$.

Answer: $3x+y=3$, $x-3y=11$.

Explanation: $9x^2-2y^2=18 \Rightarrow \frac{x^2}{2}-\frac{y^2}{9}=1$. The tangent to the hyperbola $\frac{x^2}{2}-\frac{y^2}{9}=1$ at the point (2,–3) has equation $\frac{2x}{2}-\frac{-3y}{9}=1 \Rightarrow 3x+y=3$. The normal to the hyperbola $\frac{x^2}{2}-\frac{y^2}{9}=1$ at the point (2,–3) has equation $\frac{2x}{2}+\frac{9y}{-3}=2+9 \Rightarrow x-3y=11$.

Problem CON2_18. Find the equations of the tangent and the normal to the ellipse $x=4\cos\theta, y=2\sin\theta$ at the point where $\theta=-\frac{\pi}{4}$.

Answer: $x-2y=4\sqrt{2}$, $2x+y=3\sqrt{2}$.

Explanation: The tangent to the ellipse $x=4\cos\theta, y=2\sin\theta$ at the point where $\theta=-\frac{\pi}{4}$ has equation $\frac{x\cos\left(-\frac{\pi}{4}\right)}{4}+\frac{y\sin\left(-\frac{\pi}{4}\right)}{2}=1 \Rightarrow x-2y=4\sqrt{2}$. The normal to the ellipse $x=4\cos\theta, y=2\sin\theta$ at the point where $\theta=-\frac{\pi}{4}$ has equation

$$\frac{4x}{\cos\left(-\frac{\pi}{4}\right)}-\frac{2y}{\sin\left(-\frac{\pi}{4}\right)}=16-4 \Rightarrow 2x+y=3\sqrt{2}.$$

Problem CON2_19. Find the equation of the tangent and the normal to the hyperbola $x=2\sec\theta, y=4\tan\theta$ at the point where $\theta=-\frac{\pi}{4}$.

Answer: $2\sqrt{2}x+y=4$, $x-2\sqrt{2}y=10\sqrt{2}$.

Explanation: The tangent to the hyperbola $x=2\sec\theta, y=4\tan\theta$ at the point where $\theta=-\frac{\pi}{4}$ has equation $\frac{x\sec\left(-\frac{\pi}{4}\right)}{2}-\frac{y\tan\left(-\frac{\pi}{4}\right)}{4}=1 \Rightarrow 2\sqrt{2}x+y=4$. The normal to the hyperbola

$x = 2\sec\theta, y = 4\tan\theta$ at the point where $\theta = -\frac{\pi}{4}$ has equation

$$\frac{2x}{\sec\left(-\frac{\pi}{4}\right)} + \frac{4y}{\tan\left(-\frac{\pi}{4}\right)} = 4 + 16 \Rightarrow x - 2\sqrt{2}y = 10\sqrt{2}.$$

Problem CON2_20. Find the equation of the chord of contact of tangents to the ellipse $3x^2 + 4y^2 = 48$ from the point $(6,4)$.

Answer: $9x + 8y = 24$.

Explanation: $3x^2 + 4y^2 = 48 \Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1$. The chord of contact of tangents from the point $(6,4)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$ has equation $\frac{6x}{16} + \frac{4y}{12} = 1 \Rightarrow 9x + 8y = 24$.

Problem CON2_21. Find the equation of the chord of contact of tangents to the hyperbola $9x^2 - 2y^2 = 18$ from the point $(1,2)$.

Answer: $9x - 4y = 18$.

Explanation: $9x^2 - 2y^2 = 18 \Rightarrow \frac{x^2}{2} - \frac{y^2}{9} = 1$. The chord of contact of tangents from the point $(1,2)$ to the hyperbola $\frac{x^2}{2} - \frac{y^2}{9} = 1$ has equation $\frac{x}{2} - \frac{2y}{9} = 1 \Rightarrow 9x - 4y = 18$.

Problem CON2_22. The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that $\frac{PX}{PY} = \sin^2\theta$ and deduce that if P is an extremity of a latus rectum, then $\frac{PX}{PY} = \frac{e^2 - 1}{e^2}$.

Explanation: The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. Point X has coordinates $(a\cos\theta, 0)$ and point Y has coordinates $(0, -b\cot\theta)$. Hence

$PX^2 = (a\sec\theta - a\cos\theta)^2 + b^2\tan^2\theta = a^2\cos^2\theta\tan^4\theta + b^2\tan^2\theta$,

$PY^2 = a^2\sec^2\theta + (b\tan\theta + b\cot\theta)^2 = a^2\sec^2\theta + b^2\sec^2\theta\,\mathrm{cosec}^2\theta$.

Therefore $\frac{PX}{PY} = \frac{\sqrt{\sin^2\theta(a^2\tan^2\theta + b^2\sec^2\theta)}}{\sqrt{\mathrm{cosec}^2\theta(a^2\tan^2\theta + b^2\sec^2\theta)}} = \sin^2\theta$. If P is an extremity of a latus rectum, then $a\sec\theta = \pm ae$. Thus $\cos\theta = \pm\frac{1}{e}$. But $\frac{PX}{PY} = 1 - \cos^2\theta$. Hence $\frac{PX}{PY} = 1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2}$.

Problem CON2_23. The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The tangent at P meets the asymptotes at the points M and N. Show that $PM = PN$.

Explanation: The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. This tangent meets the asymptote $y=\frac{b}{a}x$ at the point $M\left(a\frac{\cos\theta}{1-\sin\theta}, b\frac{\cos\theta}{1-\sin\theta}\right)$ and meets the asymptote $y=-\frac{b}{a}x$ at the point $N\left(a\frac{\cos\theta}{1+\sin\theta}, -b\frac{\cos\theta}{1+\sin\theta}\right)$. Hence

$$PM^2=\left(a\sec\theta-a\frac{\cos\theta}{1-\sin\theta}\right)^2+\left(b\tan\theta-b\frac{\cos\theta}{1-\sin\theta}\right)^2=a^2\tan^2\theta+b^2\sec^2\theta,$$

$$PN^2=\left(a\sec\theta-a\frac{\cos\theta}{1+\sin\theta}\right)^2+\left(b\tan\theta+b\frac{\cos\theta}{1+\sin\theta}\right)^2=a^2\tan^2\theta+b^2\sec^2\theta.$$

Therefore $PM = PN$.

Problem CON2_24. Show that the tangents at the endpoints of a focal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meet on the corresponding directrix.

Explanation: Let PQ be a focal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If tangents at P and Q meet in $T(x_0, y_0)$, then PQ has equation $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$. Hence if $S(ae,0)$ lies on PQ, then $x_0=\frac{a}{e}$ and T lies on the directrix $x=\frac{a}{e}$; if $S'(-ae,0)$ lies on PQ, then $x_0=-\frac{a}{e}$ and T lies on the directrix $x=-\frac{a}{e}$.

Problem CON2_25. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus. Deduce that if P is the point $\left(1, \frac{2\sqrt{2}}{3}\right)$ lies on the ellipse $\frac{x^2}{9}+y^2=1$ with foci S and S', then Q, S, R, S' are concyclic, and find the equation of the circle through these points.

Answer: $x^2+\left(y-\frac{3}{2\sqrt{2}}\right)^2=\frac{73}{8}$.

Explanation:

Explanation: The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. Point X has coordinates $(a\sec\theta, 0)$ and point Y has coordinates $(0, b\,\text{cosec}\,\theta)$. Hence

$PX^2=(a\cos\theta-a\sec\theta)^2+b^2\sin^2\theta=a^2\sin^2\theta\tan^2\theta+b^2\sin^2\theta$,

$PY^2=a^2\cos^2\theta+(b\sin\theta-b\,\text{cosec}\,\theta)^2=a^2\cos^2\theta+b^2\cos^2\theta\cot^2\theta$.

Therefore $\frac{PX}{PY}=\frac{\sqrt{\tan^2\theta(a^2\sin^2\theta+b^2\cos^2\theta)}}{\sqrt{\cot^2\theta(a^2\sin^2\theta+b^2\cos^2\theta)}}=\tan^2\theta$. If P is an extremity of a latus rectum, then $a\cos\theta=\pm ae$. Thus $\cos\theta=\pm e$. Hence $\frac{PX}{PY}=\frac{1-e^2}{e^2}$.

Problem CON2_27. The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The normal at P cuts the x-axis at X and the y-axis at Y. Show that $\frac{PX}{PY}=\frac{b^2}{a^2}$.

Explanation: The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$. Point X has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$ and point Y has coordinates $\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$. Hence

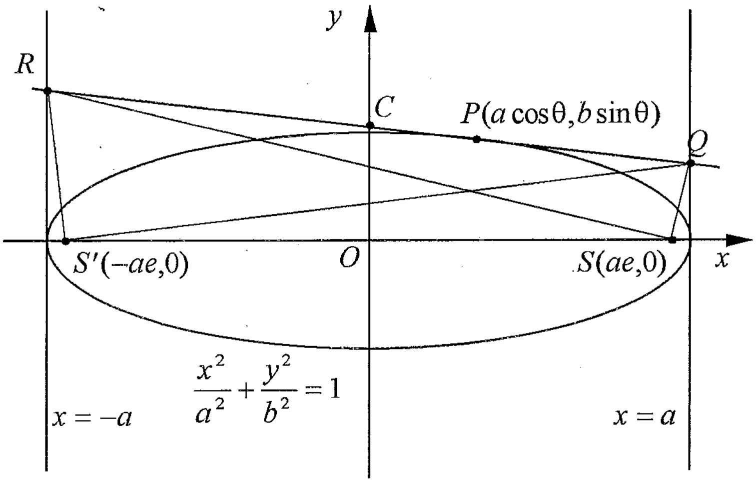
$$PX^2=\left(a-\frac{a^2+b^2}{a}\right)^2\sec^2\theta+b^2\tan^2\theta=\frac{b^4}{a^2}\sec^2\theta+b^2\tan^2\theta=\frac{b^2}{a^2}\left(b^2\sec^2\theta+a^2\tan^2\theta\right),$$

$$PY^2=a^2\sec^2\theta+\left(b-\frac{a^2+b^2}{b}\right)^2\tan^2\theta=a^2\sec^2\theta+\frac{a^4}{b^2}\tan^2\theta=\frac{a^2}{b^2}\left(b^2\sec^2\theta+a^2\tan^2\theta\right).$$

Therefore $\frac{PX}{PY}=\frac{\frac{b}{a}\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}{\frac{a}{b}\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}=\frac{b^2}{a^2}$.

Problem CON2_28. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the y-axis at B and Y is the foot of the perpendicular from P to the y-axis. Show that $OY\cdot OB=b^2$.

Explanation: The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.



Let the tangent at P meet $x=a$, $x=-a$ in Q, R respectively. Let QR meet the y-axis in C.

Tangent PR has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

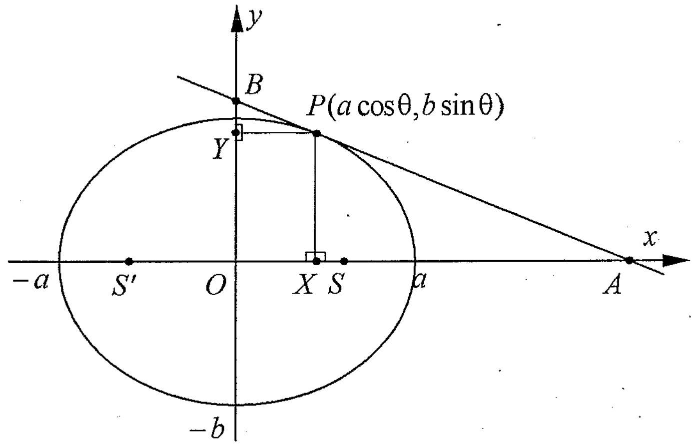
Hence Q has coordinates $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$ and R has coordinates $\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$.

Gradient $QS \times$ gradient $RS = \frac{b(1-\cos\theta)}{a(1-e)\sin\theta}\cdot\frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} = -\frac{b^2}{a^2(1-e^2)}\cdot\frac{1-\cos^2\theta}{\sin^2\theta}$.

Then $b^2 = a^2(1-e^2) \Rightarrow$ gradient $QS \times$ gradient $RS = -1 \therefore QS \perp RS$. Similarly, replacing e by $-e$, $QS' \perp RS'$. Hence QR subtends angles of $90°$ at each of S and S', and Q, S, R, S' are concyclic, with QR the diameter of the circle through the points. The y-axis is the perpendicular bisector of the chord SS', hence the center of this circle is the point C where the diameter QR meets the y-axis.

If $P\left(1, \frac{2\sqrt{2}}{3}\right)$ lies on the ellipse $\frac{x^2}{9}+y^2=1$, then QR has equation $\frac{x}{9}+\frac{2\sqrt{2}y}{3}=1$ and meets the y-axis in $C\left(0, \frac{3}{2\sqrt{2}}\right)$. Also $b^2=a^2(1-e^2)$ gives $e^2=\frac{8}{9}$, and S has coordinates $(2\sqrt{2},0)$. Hence $CS^2=\frac{73}{8}$ and the circle through Q, S, R, S' has equation $x^2+\left(y-\frac{3}{2\sqrt{2}}\right)^2=\frac{73}{8}$.

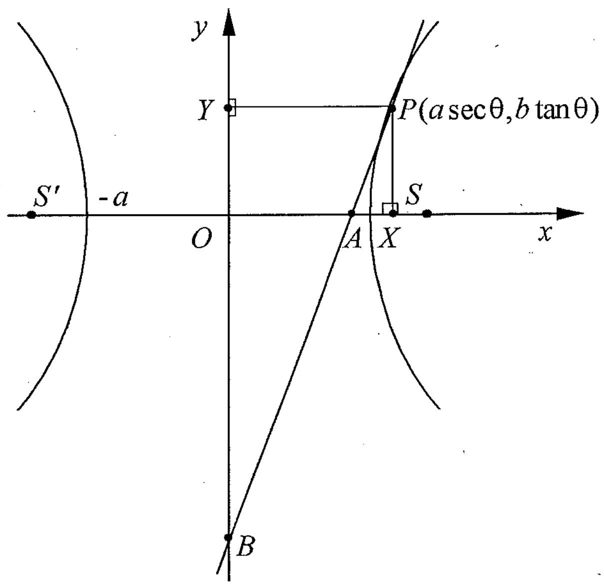
Problem CON2_26. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that $\frac{PX}{PY}=\tan^2\theta$ and deduce that if P is an extremity of a latus rectum, then $\frac{PX}{PY}=\frac{1-e^2}{e^2}$.



The point B has coordinates $(0, b\,\mathrm{cosec}\,\theta)$ and the point Y has coordinates $(0, b\sin\theta)$. Hence $OY \cdot OB = b\sin\theta \cdot b\,\mathrm{cosec}\,\theta = b^2$.

Problem CON2_29. The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts the y-axis at B and Y is the foot of the perpendicular from P to the y-axis. Show that $OY \cdot OB = b^2$.

Explanation: The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

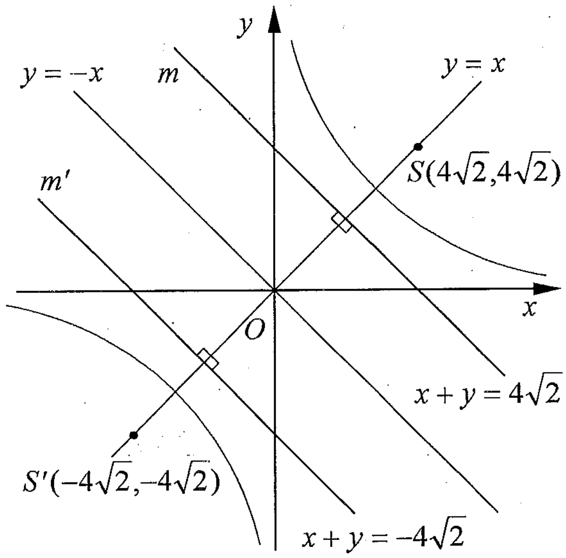


The point B has coordinates $(0, -b\cot\theta)$ and the point Y has coordinates $(0, b\tan\theta)$. Hence $OY \cdot OB = b\tan\theta \cdot b\cot\theta = b^2$.

Problem 2_30. For the rectangular hyperbola $xy = 16$, find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices; (d) the equations of the asymptotes. Sketch the hyperbola.

Answer: (a) $\sqrt{2}$; (b) $(4\sqrt{2}, 4\sqrt{2})$, $(-4\sqrt{2}, -4\sqrt{2})$; (c) $x + y = \pm 4\sqrt{2}$; (d) $x = 0$, $y = 0$.

Explanation:



For the hyperbola $xy = 16$ we have $c^2 = 16 \Rightarrow c = 4$. Hence the hyperbola $xy = 16$ has
eccentricity $e = \sqrt{2}$,
foci $S(c\sqrt{2}, c\sqrt{2}) = S(4\sqrt{2}, 4\sqrt{2})$
and $S'(-c\sqrt{2}, -c\sqrt{2}) = S(-4\sqrt{2}, -4\sqrt{2})$,
directrices $x + y = \pm c\sqrt{2} \Rightarrow x + y = \pm 4\sqrt{2}$,
asymptotes $x = 0$ and $y = 0$.

Problem 2_31. Find the parametric equation of the rectangular hyperbola $xy = 25$.

Answer: $x = 5t,\ y = \dfrac{5}{t}$.

Explanation: For the hyperbola $xy = 25$ we have $c^2 = 25 \Rightarrow c = 5$. Hence the hyperbola $xy = 25$ has parametric equations $x = ct, y = \dfrac{c}{t} \Rightarrow x = 5t, y = \dfrac{5}{t}$.

Problem CON2_32. Find the Cartesian equation of the rectangular hyperbola $x = 3t, y = \dfrac{3}{t}$.

Answer: $xy = 9$.

Explanation: The hyperbola $x = 3t, y = \dfrac{3}{t}$ has Cartesian equation $xy = 3t \cdot \dfrac{3}{t} \Rightarrow xy = 9$.

Problem CON2_33. Find the equations of the tangent and the normal to the rectangular hyperbola $xy = 12$ at the point $(-3,-4)$.

Answer: $4x + 3y = -24$, $3x - 4y = 7$.

Explanation: For the hyperbola $xy = 12$ we have $c^2 = 12$. Hence the tangent to the hyperbola $xy = 12$ at the point $P(x_1, y_1) = P(-3,-4)$ has equation $xy_1 + yx_1 = 2c^2 \Rightarrow 4x + 3y = -24$ and the normal has equation $xx_1 - yy_1 = x_1^2 - y_1^2 \Rightarrow 3x - 4y = 7$.

Problem CON2_34. Find the equations of the tangent and the normal to the rectangular hyperbola $x = 3t, y = \frac{3}{t}$ at the point $t = -1$.

Answer: $x + y = -6$, $x - y = 0$.

Explanation: For the hyperbola $x = 3t, y = \frac{3}{t}$ we have $c = 3$. Hence the tangent to the hyperbola $x = 3t, y = \frac{3}{t}$ at the point where $t = -1$ has equation $x + t^2 y = 2ct \Rightarrow x + y = -6$ and the normal has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow x - y = 0$.

Problem CON2_35. Find the equation of the chord of contact of tangents from the point $(1,-2)$ to $xy = 6$.

Answer: $2x - y = -12$.

Explanation: For the hyperbola $xy = 6$ we have $c^2 = 6$. Hence the chord of contact of tangents from the point $T(x_0, y_0) = T(1,-2)$ to the hyperbola $xy = 6$ has equation $xy_0 + yx_0 = 2c^2 \Rightarrow 2x - y = -12$.

Problem CON2_36. Find the equation of the chord of contact of tangents from the point $(-1,-3)$ to the rectangular hyperbola $xy = 4$. Hence find the coordinates of their points of contact and the equations of these tangents.

Answer: $3x + y = -8$, $(-2,-2)$, $\left(-\frac{2}{3},-6\right)$.

Explanation: The chord of contact of tangents from the point $(-1,-3)$ to the hyperbola $xy = 4$ has equation $3x + y = -8$. Let $T(x_0, y_0)$ be a point of contact. Then

T lies on the chord $\Rightarrow 3x_0 + y_0 = -8$,

T lies on the hyperbola $\Rightarrow x_0 y_0 = 4$.

Hence $x_0(-8 - 3x_0) = 4 \Rightarrow 3x_0^2 + 8x_0 + 4 = 0 \Rightarrow (3x + 2)(x + 2) = 0$

$\therefore x_0 = -\frac{2}{3}, y_0 = -8-3x_0 = -6$ or $x_0 = -2, y_0 = -8-3x_0 = -2$.

Equation of tangent at the point $T(x_0, y_0)$ is $xy_0 + yx_0 = 2c^2$. Therefore the tangents from the point $(-1,-3)$ to the hyperbola $xy = 4$ are $y = -x-4$, with point of contact $P(-2,-2)$ and $y = -9x-12$, with point of contact $P\left(-\frac{2}{3},-6\right)$.

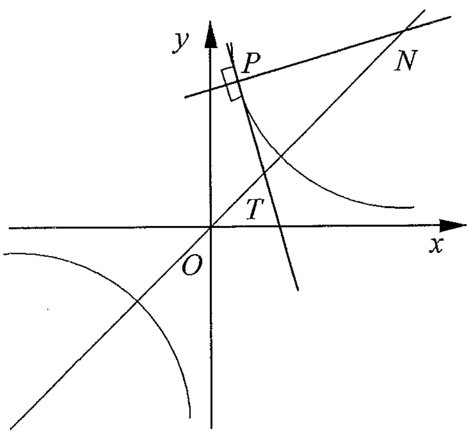
Problem CON2_37. The points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. The chord PQ subtends a right angle at the another point $R\left(cr,\frac{c}{r}\right)$ on the hyperbola. Show that the normal at R is parallel to PQ.

Explanation: The gradient of PR is $\frac{c\left(\frac{1}{p}-\frac{1}{r}\right)}{c(p-r)} = -\frac{1}{pr}$, the gradient of QR is $\frac{c\left(\frac{1}{q}-\frac{1}{r}\right)}{c(q-r)} = -\frac{1}{qr}$.

Therefore $PR \perp QR \Rightarrow$ gradient $PR \times$ gradient $QR = -1 \Rightarrow \frac{1}{pqr^2} = -1 \Rightarrow r^2 = -\frac{1}{pq}$. The normal at the point $R\left(cr,\frac{c}{r}\right)$ has gradient r^2, the gradient of PQ is $\frac{c\left(\frac{1}{p}-\frac{1}{q}\right)}{c(p-q)} = -\frac{1}{pq}$. Since $r^2 = -\frac{1}{pq}$, then gradient of the normal at R equals to gradient of PQ. Thus the normal at the point R is parallel to the chord PQ.

Problem CON2_38. The point $P\left(ct,\frac{c}{t}\right)$, where $t \neq 1$ lies on the rectangular hyperbola $xy = c^2$. The tangent and the normal at P meet the line $y = x$ at T and N respectively. Show that $OT \cdot ON = 4c^2$.

Explanation:



The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. As $y = x$ the point T has coordinates $\left(\frac{2ct}{1+t^2}, \frac{2ct}{1+t^2}\right)$. The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. Therefore the point N has coordinates $\left(c\frac{t^2+1}{t}, c\cdot\frac{t^2+1}{t}\right)$.

$OT = \frac{2ct}{1+t^2}\sqrt{2}, ON = c\frac{t^2+1}{t}\sqrt{2} \Rightarrow OT \times ON = 4c^2$.

Problem CON2_39. On the rectangular hyperbola $xy = c^2$ there are variable points P and Q. The tangents at P and Q meet at R. Find the equation of the locus of R if PQ passes through the point $(a,0)$

Answer: $y = \frac{2c^2}{a}$.

Explanation: Let R has coordinates (x_0, y_0). PQ is the chord of contact of tangents from R to the hyperbola $xy = c^2$. Hence PQ has equation $xy_0 + yx_0 = 2c^2$. Then $(a,0)$ lies on PQ. Therefore $ay_0 = 2c^2$. Thus the locus of R has equation $y = \frac{2c^2}{a}$.

Problem CON2_40. The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that $PX = PY$.

Explanation: The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. Hence the point X has coordinates $(2ct, 0)$ and the point Y has coordinates $\left(0, \frac{2c}{t}\right)$. $PY^2 = (ct - 2ct)^2 + \left(\frac{c}{t}\right)^2 = c^2\left(t^2 + \frac{1}{t^2}\right)$ and $PX^2 = (ct)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2 = c^2\left(t^2 + \frac{1}{t^2}\right)$. Therefore $PX = PY$.