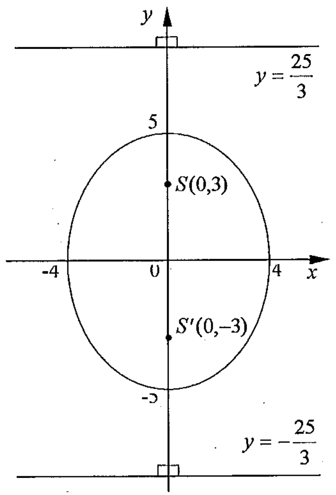
Topic 4. Conics.
Level 3.
Problem CON3_01.

For the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices. Sketch the ellipse.

Answer: (a) $\frac{3}{5}$; (b) $(0,\pm 3)$; (c) $y=\pm\frac{25}{3}$.

Solution:



$\frac{x^2}{16}+\frac{y^2}{25}=1$; $\begin{array}{l} a=4, b=5 \Rightarrow b>a \\ a^2=b^2(1-e^2) \end{array}$,

eccentricity: $e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$,

foci: $(0,\pm be)\Rightarrow(0,\pm 3)$,

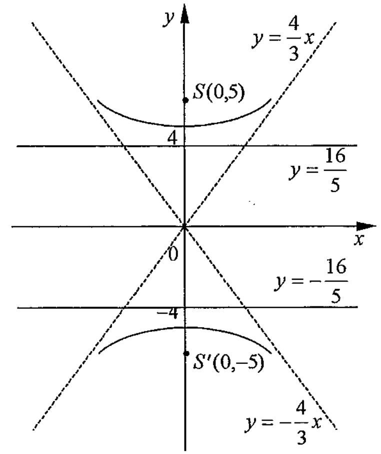
directrices: $y=\pm\frac{b}{e}\Rightarrow y=\pm\frac{25}{3}$.

Problem CON3_02.

For the hyperbola $\frac{y^2}{16}-\frac{x^2}{9}=1$ find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices. Sketch the hyperbola.

Answer: (a) $\frac{5}{4}$; (b) $(0,\pm 5)$; (c) $y=\pm\frac{16}{5}$.

Solution:



$\frac{y^2}{16}-\frac{x^2}{9}=1$; $\begin{array}{l} a=3, b=4 \\ a^2=b^2(e^2-1) \end{array}$,

eccentricity: $e=\sqrt{1+\frac{9}{16}}=\frac{5}{4}$,

foci: $(0,\pm be)\Rightarrow(0,\pm 5)$,

directrices: $y=\pm\frac{b}{e}\Rightarrow y=\pm\frac{16}{5}$,

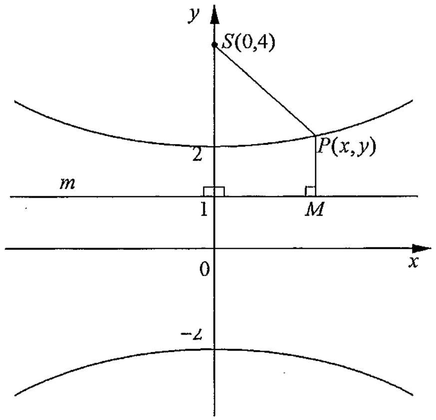
asymptotes: $x=\pm\frac{a}{b}y\Rightarrow x=\pm\frac{3}{4}y\Rightarrow y=\pm\frac{4}{3}x$.

Problem CON3_03.

A variable point $P(x,y)$ moves so that its distance from $(0,4)$ is two times its distance from $y=1$. Find the locus of *P*.

Answer: $\frac{y^2}{4}-\frac{x^2}{12}=1$.

Solution:



The locus of a variable point $P(x,y)$ is the hyperbola with focus at $S(0,4)$, directrix $m: y=1$ and eccentricity $e=2$. Let M be the foot of the perpendicular from P to m. Then M has coordinates $(x,1)$.

$PS = e\cdot PM \Rightarrow x^2+(y-4)^2 = 2^2(y-1)^2$. Therefore the Cartesian equation of the hyperbola is

$x^2+y^2(1-4)=4-16$.

$\frac{y^2}{4}-\frac{x^2}{12}=1$.

Problem CON3_04.

The asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are inclined to each other at an angle α. Show that $\tan\alpha = \frac{2ab}{\left|a^2-b^2\right|}$.

Solution: Let φ denote the smallest angle from positive x-axis to the asymptote $y=\frac{b}{a}x$. Then $\alpha = 2\varphi$ when $\varphi \le \frac{\pi}{4}$, or $\alpha = \pi - 2\varphi$ when $\varphi > \frac{\pi}{4}$. Therefore $\tan\alpha = \left|\tan 2\varphi\right|$. Since $\tan\varphi = \frac{b}{a}$, then

$\tan\alpha = \left|\frac{2\tan\varphi}{1-\tan^2\varphi}\right| = \left|\frac{2b}{a}\cdot\left(1-\frac{b^2}{a^2}\right)^{-1}\right| = \frac{2ab}{\left|a^2-b^2\right|}$.

Problem CON3_05.

A point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with foci $S(al,0)$ and $S'(-al,0)$.

(a) Show that $PS = a\left|e\sec\theta-1\right|$ and $PS' = a\left|e\sec\theta+1\right|$.

(b) Deduce that $\left|PS-PS'\right| = 2a$.

Solution: (a) Length of PS is $\sqrt{(a\sec\theta-ae)^2+(b\tan\theta)^2} = \sqrt{a^2(\sec\theta-e)^2+b^2\tan^2\theta}$. For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ we have $b^2=a^2(e^2-1)$. Therefore the length of PS is

$\sqrt{a^2(\sec\theta-e)^2+a^2(e^2-1)\tan^2\theta} = a\sqrt{\sec^2\theta-2e\sec\theta+e^2+e^2\tan^2\theta-\tan^2\theta} =$

$a\sqrt{e^2(1+\tan^2\theta)-2e\sec\theta+(\sec^2\theta-\tan^2\theta)} = a\sqrt{e^2\sec^2\theta-2e\sec\theta+1} = a\sqrt{(e\sec\theta-1)^2}$

Hence the length of PS is $a\left|e\sec\theta-1\right|$.

Length of PS' is $\sqrt{(a\sec\theta+ae)^2+(b\tan\theta)^2} = \sqrt{a^2(\sec\theta+e)^2+b^2\tan^2\theta}$. For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ we have $b^2=a^2(e^2-1)$. Therefore the length of PS' is

$\sqrt{a^2(\sec\theta+e)^2+a^2(e^2-1)\tan^2\theta} = a\sqrt{\sec^2\theta+2e\sec\theta+e^2+e^2\tan^2\theta-\tan^2\theta} =$

$a\sqrt{e^2(1+\tan^2\theta)+2e\sec\theta+(\sec^2\theta-\tan^2\theta)} = a\sqrt{e^2\sec^2\theta+2e\sec\theta+1} = a\sqrt{(e\sec\theta+1)^2}$

Hence the length of PS' is $a\left|e\sec\theta+1\right|$.

(b) If P lies on the right-hand branch of the hyperbola, then $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Since for hyperbola $e>1$, $PS = a(e\sec\theta-1)$ and $PS' = a(e\sec\theta+1)$. Therefore $PS-PS' = -2a$. If P lies on the left-hand branch of the hyperbola, then $-\pi<\theta<-\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$. Since for hyperbola $e>1$, $PS = -a(e\sec\theta-1)$ and $PS' = -a(e\sec\theta+1)$. Therefore $PS-PS' = +2a$. Hence $\left|PS-PS'\right| = 2a$.

Problem CON3_06.

Points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. (a) Use the result that the chord PQ has equation $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$ to show that if PQ is a local chord, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ takes one of the values $\frac{1-e}{1+e}$ or $\frac{1+e}{1-e}$. (b) The point $P\left(2\sqrt{3},3\sqrt{3}\right)$ is one extremity of a focus chord on the hyperbola $\frac{x^2}{3}-\frac{y^2}{9}=1$. Find the coordinates of the other extremity Q.

Answer: $\left(2\sqrt{3},-3\sqrt{3}\right)$ or $\left(-\frac{14\sqrt{3}}{13},\frac{9\sqrt{3}}{13}\right)$.

Solution: (a) If PQ is a focal chord through $S(ae,0)$, then $e\cos\left(\frac{\theta-\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$. Expanding both cosines gives $(e-1)\cos\frac{\theta}{2}\cos\frac{\phi}{2} = -(e+1)\sin\frac{\theta}{2}\sin\frac{\phi}{2}$. Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{1-e}{1+e}$. Similarly, if PQ is a focal chord through $S'(-ae,0)$. Then replacing e by $-e$, $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{1+e}{1-e}$.

(b) $\frac{x^2}{3}-\frac{y^2}{9}=1 \Rightarrow a=\sqrt{3}$ and $b=3$, $\therefore P\left(2\sqrt{3},3\sqrt{3}\right)\equiv P\left(\sqrt{3}\sec\frac{\pi}{3},3\tan\frac{\pi}{3}\right)$.

Also $b^2 = a^2(e^2 - 1)$ $\therefore e = \sqrt{\left(1 + \frac{9}{3}\right)} = 2$. P has parameter $\frac{\pi}{3}$. Let Q have parameter ϕ. Hence

$\tan\frac{\pi}{6}\tan\frac{\phi}{2} = \frac{1-2}{1+2}$, or $\tan\frac{\pi}{6}\tan\frac{\phi}{2} = \frac{1+2}{1-2}$, $\therefore \tan\frac{\phi}{2} = -\frac{1}{\sqrt{3}}$, $\tan\frac{\phi}{2} = -3\sqrt{3}$,

$\sec\phi = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$ or $\sec\phi = \frac{1+27}{1-27} = -\frac{14}{13}$, and $\tan\phi = \frac{2\left(-\frac{1}{\sqrt{3}}\right)}{1-\frac{1}{3}} = -\sqrt{3}$ or $\tan\phi = \frac{2\left(-3\sqrt{3}\right)}{1-27} = \frac{3\sqrt{3}}{13}$.

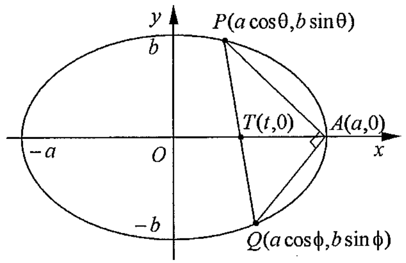
Q has coordinates $(\sqrt{3}\sec\phi, 3\tan\phi) \Rightarrow Q\left(2\sqrt{3}, -3\sqrt{3}\right)$ or $Q\left(-\frac{14\sqrt{3}}{13}, \frac{9\sqrt{3}}{13}\right)$.

Problem CON3_07.

Points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equation of the chord PQ. Hence show that if PQ subtends a right angle at the point $A(a,0)$ then PQ passes through a fixed point on the x-axis.

Answer: $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$.

Solution:



The chord PQ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has equation $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$, where P, Q have parameters θ, ϕ. The chord PQ cuts the x-axis at point $T(t,0)$. So $t = a\cos\left(\frac{\theta-\phi}{2}\right)\sec\left(\frac{\theta+\phi}{2}\right) = a\left(1+\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)\left(1-\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)^{-1}$. The gradient of AP is $\frac{b\sin\theta}{a(\cos\theta - 1)} = -\frac{b}{a}\cot\frac{\theta}{2}$ and the gradient of AQ is $\frac{b\sin\theta}{a(\cos\phi - 1)} = -\frac{b}{a}\cot\frac{\phi}{2}$. If the chord PQ subtends a right angle at the point A, then gradient $AP \times$ gradient $AQ = -1$. Therefore $\frac{b^2}{a^2}\cot\frac{\theta}{2}\cot\frac{\phi}{2} = -1 \Rightarrow \tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{b^2}{a^2}$. Hence $t = a\left(1-\frac{b^2}{a^2}\right)\left(1+\frac{b^2}{a^2}\right)^{-1} = a\frac{a^2-b^2}{a^2+b^2}$. But for the

ellipse $b^2 = a^2(1-e^2)$. Thus $t = \frac{ae^2}{2+e^2}$. So PQ passes through a fixed point $T\left(\frac{ae^2}{2+e^2}, 0\right)$ on the x-axis.

Problem CON3_08.

A point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N. Show that ON is perpendicular to the asymptote.

Solution: The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$. So the point N has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$. Since the asymptotes have equations $y = \pm\frac{b}{a}x$, then the point Q has coordinates $(a\sec\theta, \pm b\sec\theta)$. Thus the gradient of QN is $\mp b\sec\theta \cdot \left[\left(\frac{a^2+b^2}{a} - a\right)\sec\theta\right]^{-1} = \mp\frac{a}{b}$. Therefore QN is perpendicular to the asymptote.

Problem CON3_09.

A point $P(a\sec\theta, a\tan\theta)$ lies on the rectangular hyperbola $x^2 - y^2 = a^2$. A is the point $(a,0)$. M is the midpoint of AP. Find the equation of the locus of M.

Answer: $(2x-a)^2 - (2y)^2 = a^2$.

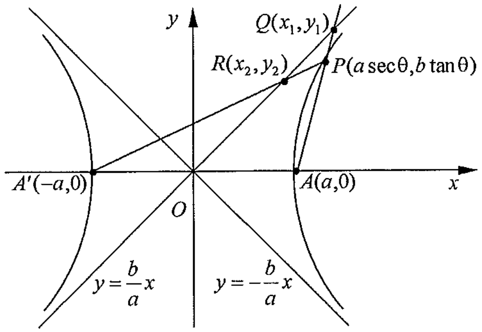
Solution: If $M(x,y)$ is the midpoint of AP, then $x = \frac{a}{2}(\sec\theta + 1)$ and $y = \frac{a}{2}\tan\theta$. Therefore $(2x-a)^2 - (2y)^2 = a^2(\sec^2\theta - \tan^2\theta) = a^2$. Hence the locus of M is hyperbola $(2x-a)^2 - (2y)^2 = a^2$.

Problem CON3_10.

The point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is joined to the vertices $A(a,0)$ and $A'(-a,0)$. The lines AP and $A'P$ meet he asymptote $y = \frac{b}{a}x$ at Q and R respectively. (i) Find the coordinetes of Q and R. (ii) Hence find the length QR, showing that it is independent of θ, and show that the area of triangle PQR is $\frac{1}{2}\left|ab(\sec\theta - \tan\theta)\right|$ square units.

Answer: (i) $\left(\frac{a\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}, \frac{b\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right)$, $\left(\frac{a\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}, \frac{b\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right)$; (ii) $\sqrt{a^2+b^2}$.

Solution:



(i) The line AP has equation $y=\frac{b\tan\theta}{a(\sec\theta-1)}(x-a)$. Since the point Q lies on the line AP, then $y_1=\frac{b\tan\theta}{a(\sec\theta-1)}(x_1-a)$. Since the point Q lies on the asymptote $y=\frac{b}{a}x$, then $y_1=\frac{b}{a}x_1$.

Therefore $x_1=\frac{\tan\theta}{(\sec\theta-1)}(x_1-a)\Rightarrow x_1=\frac{a\tan\theta}{\tan\theta-\sec\theta+1}=\frac{a\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$ and $y_1=\frac{b\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$.

Thus the point Q has coordinates $\left(\frac{a\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}},\frac{b\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)$. Similarly the line $A'P$ has equation $y=\frac{b\tan\theta}{a(\sec\theta+1)}(x+a)$. Since the point R lies on the line $A'P$, then $y_2=\frac{b\tan\theta}{a(\sec\theta+1)}(x_2+a)$. Since the point R lies on the asymptote $y=\frac{b}{a}x$, then $y_2=\frac{b}{a}x_2$. So

$x_2=\frac{\tan\theta}{(\sec\theta+1)}(x_2+a)\Rightarrow x_2=\frac{-a\tan\theta}{\tan\theta-\sec\theta-1}=\frac{a\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$ and $y_2=\frac{b\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$. Thus the point R has coordinates $\left(\frac{a\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}},\frac{b\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)$.

(ii) $QR^2=(x_1-x_2)^2+(y_1-y_2)^2=\frac{a^2\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}+\frac{b^2\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}=a^2+b^2$. Thus the length of QR is $\sqrt{a^2+b^2}$ and hence is independent of θ. The area of the triangle PQR is

$\frac{1}{2}\cdot QR\cdot h$ where h is the height of the triangle. Since h is the distance from $P(a\sec\theta,b\tan\theta)$ to the line $y=\frac{b}{a}x$, then $h=\frac{\left|\frac{b}{a}\cdot a\sec\theta-b\tan\theta\right|}{\sqrt{\left(\frac{b}{a}\right)^2+1}}=\frac{ba|\sec\theta-\tan\theta|}{\sqrt{a^2+b^2}}$. Therefore the area of the triangle PQR is $\frac{1}{2}\cdot\sqrt{a^2+b^2}\cdot\frac{ba|\sec\theta-\tan\theta|}{\sqrt{a^2+b^2}}=\frac{1}{2}ab|\sec\theta-\tan\theta|$.

Problem CON3_11.

Find the equation of the tangent and normal to (a) the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$ at the point $(2,1)$; (b) the ellipse $x=4\cos\theta, y=2\sin\theta$ at the point where $\theta=\frac{\pi}{3}$; (c) the hyperbola $\frac{x^2}{12}-\frac{y^2}{27}=1$ at the point $(4,3)$; (d) the hyperbola $x=3\sec\theta, y=6\tan\theta$ at the point where $\theta=\frac{\pi}{6}$.

Answer: (a) $x+2y=4$, $2x-y=3$; (b) $x+2\sqrt{3}y=8$, $6x-\sqrt{3}y=9$; (c) $3x-y=9$, $x+3y=13$; (d) $4x-y=6\sqrt{3}$, $x+4y=10\sqrt{3}$.

Solution: (a) The tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$ at the point $(2,1)$ has equation $\frac{2x}{8}+\frac{y}{2}=1\Rightarrow x+2y=4$. The normal to the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$ at the point $(2,1)$ has equation $\frac{8x}{2}-\frac{2y}{1}=8-2\Rightarrow 2x-y=3$.

(b) The tangent to the ellipse $x=4\cos\theta, y=2\sin\theta$ at the point where $\theta=\frac{\pi}{3}$ has equation $\frac{x\cos\frac{\pi}{3}}{4}+\frac{y\sin\frac{\pi}{3}}{2}=1\Rightarrow x+2\sqrt{3}y=8$. The normal to the ellipse $x=4\cos\theta, y=2\sin\theta$ at the point where $\theta=\frac{\pi}{3}$ has equation $\frac{4x}{\cos\frac{\pi}{3}}-\frac{2y}{\sin\frac{\pi}{3}}=16-4\Rightarrow 6x-\sqrt{3}y=9$.

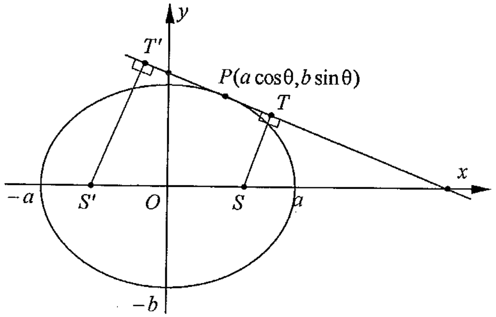
(c) The tangent to the hyperbola $\frac{x^2}{12}-\frac{y^2}{27}=1$ at the point $(4,3)$ has equation $\frac{4x}{12}-\frac{3y}{27}=1\Rightarrow 3x-y=9$. The normal to the hyperbola $\frac{x^2}{12}-\frac{y^2}{27}=1$ at the point $(4,3)$ has equation $\frac{12x}{4}+\frac{27y}{3}=12+27\Rightarrow x+3y=13$.

(d) The tangent to the hyperbola $x=3\sec\theta, y=6\tan\theta$ at the point where $\theta=\frac{\pi}{6}$ has equation $\frac{x\sec\frac{\pi}{6}}{3}-\frac{y\tan\frac{\pi}{6}}{6}=1\Rightarrow 4x-y=6\sqrt{3}$. The normal to the hyperbola $x=3\sec\theta, y=6\tan\theta$ at the point where $\theta=\frac{\pi}{6}$ has equation $\frac{3x}{\sec\frac{\pi}{6}}+\frac{6y}{\tan\frac{\pi}{6}}=9+36\Rightarrow x+4y=10\sqrt{3}$.

Problem CON3_12.

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent. Show that $ST\cdot S'T'=b^2$.

Solution: The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.



Since S has coordinates $(ae,0)$, then $ST=\frac{|e\cos\theta-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$ (ST is the distance from S to the line $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$). Since S' has coordinates $(-ae,0)$, then $S'T'=\frac{|-e\cos\theta-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$. Therefore

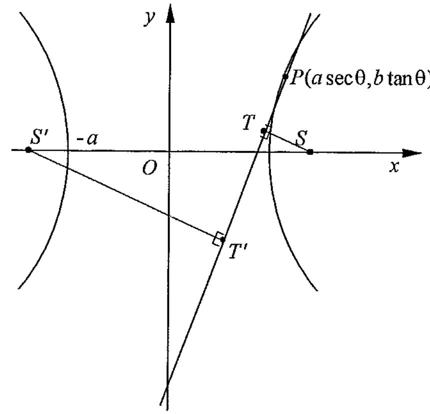
$ST\cdot S'T'=\frac{1-e^2\cos^2\theta}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}$. But for the ellipse $b^2=a^2(1-e^2)\Rightarrow e^2=1-\frac{b^2}{a^2}$. Hence

$$ST\cdot S'T'=\frac{1-\cos^2\theta+\frac{b^2}{a^2}\cos^2\theta}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}=b^2.$$

Problem CON3_13.

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent. Show that $ST\cdot S'T'=b^2$.

Solution: The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.



Since S has coordinates $(ae,0)$, then $ST=\frac{|e\sec\theta-1|}{\sqrt{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}}$ (ST is the distance from S to the line $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$). Since S' has coordinates $(-ae,0)$, then $S'T'=\frac{|-e\sec\theta-1|}{\sqrt{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}}$. Hence

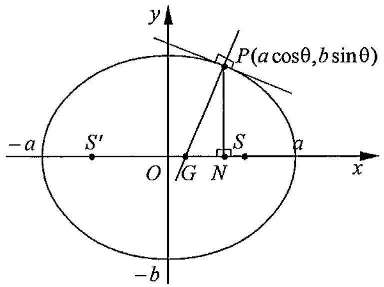
$ST\cdot S'T'=\frac{e^2\sec^2\theta-1}{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}$. But for the hyperbola $b^2=a^2(e^2-1)\Rightarrow e^2=\frac{b^2}{a^2}+1$. Thus

$$ST\cdot S'T'=\frac{\frac{b^2}{a^2}\sec^2\theta+\sec^2\theta-1}{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}=b^2.$$

Problem CON3_14.

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at the P cuts the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. Show that $SG = eSP$, and $S'G = eS'P$.

Solution:



The normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation

$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. The point G has coordinates $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$. Since the focus S has coordinates $(ae,0)$, then $SG=\left|ae-\frac{a^2-b^2}{a}\cos\theta\right|=ae(1-e\cos\theta)$ and

$$SP=\sqrt{(ae-a\cos\theta)^2+b^2\sin^2\theta}=a\sqrt{(e-\cos\theta)^2+(1-e^2)\sin^2\theta}$$
$$=a\sqrt{1-2e\cos\theta+e^2\cos^2\theta}=a(1-e\cos\theta).$$

Hence $SG=eSP$. Since the focus S' has coordinates $(-ae,0)$,

then $S'G=\left|-ae-\frac{a^2-b^2}{a}\cos\theta\right|=ae(1+e\cos\theta)$

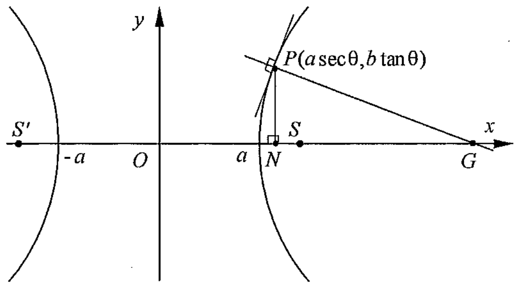
and $S'P=\sqrt{(-ae-a\cos\theta)^2+b^2\sin^2\theta}=a\sqrt{(e+\cos\theta)^2+(1-e^2)\sin^2\theta}$
$$=a\sqrt{1+2e\cos\theta+e^2\cos^2\theta}=a(1+e\cos\theta).$$

Hence $S'G=eS'P$.

Problem CON3_15.

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The normal at P cuts the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. Show that $SG = eSP$, and $S'G = eS'P$.

Solution:



The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation

$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$. The point G has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$.

Since the focus S has coordinates $(ae,0)$,

then $SG=\left|ae-\frac{a^2+b^2}{a}\sec\theta\right|=ae|1-e\sec\theta|$

and $SP=\sqrt{(ae-a\sec\theta)^2+b^2\tan^2\theta}=a\sqrt{(e-\sec\theta)^2+(e^2-1)\tan^2\theta}$
$$=a\sqrt{1-2e\sec\theta+e^2\sec^2\theta}=a|1-e\sec\theta|.$$

Hence $SG=eSP$. Since the focus S' has coordinates $(-ae,0)$,

then $S'G=\left|-ae-\frac{a^2+b^2}{a}\sec\theta\right|=ae|1+e\sec\theta|$

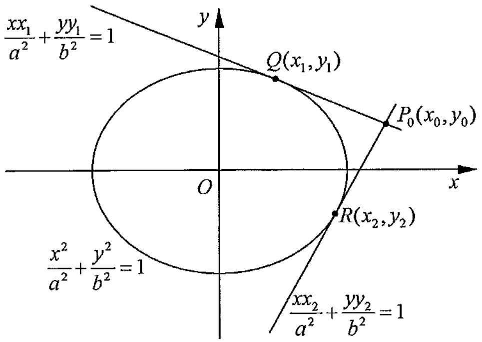
and $S'P=\sqrt{(-ae-a\sec\theta)^2+b^2\tan^2\theta}=a\sqrt{(e+\sec\theta)^2+(e^2-1)\tan^2\theta}$
$$=a\sqrt{1+2e\sec\theta+e^2\sec^2\theta}=a|1+e\sec\theta|.$$

Hence $S'G=eS'P$.

Problem CON3_16.

Show that the chord of contact of the tangents from the point $P_0(x_0, y_0)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has equation $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$.

Solution:



Since $P_0(x_0,y_0)$ lies on the tangent P_0Q, then $\frac{x_0x_1}{a^2}+\frac{y_0y_1}{b^2}=1$. Since $P_0(x_0,y_0)$ lies on the tangent P_0R, then $\frac{x_0x_2}{a^2}+\frac{y_0y_2}{b^2}=1$. Hence both $Q(x_1,y_1)$ and $R(x_2,y_2)$ satisfy $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$. But this is the equation of a straight line and is thus the equation of the chord of contact of tangents from $P_0(x_0,y_0)$.

Problem CON3_17.

Write down the equation of the chord of contact of the tangents from the point $(4,-1)$ to the ellipse $x^2+2y^2=6$. Hence find the coordinates of the points of contact and the equations of these tangents.

Answer: $2x-y=3$; $\left(\frac{2}{3},-\frac{5}{3}\right)$, $x-5y=9$; $(2,1)$, $x+y=3$.

Solution: $x^2+2y^2=6\Rightarrow\frac{x^2}{6}+\frac{y^2}{3}=1$. The chord of contact of tangents from the point $(4,-1)$ to the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ has equation $\frac{4x}{6}-\frac{y}{3}=1\Rightarrow 2x-y=3$. Let $T(x',y')$ be the extremity of the chord, then $2x'-y'=3\Rightarrow y'=2x'-3$. Since the point $T(x',y')$ lies on the ellipse, then $x'^2+2y'^2=6$. Hence $x'^2+2(2x'-3)^2=6\Rightarrow 9x'^2-24x'+12=0\Rightarrow(3x'-2)(x'-2)=0$.
Therefore the tangents to the ellipse $x^2+2y^2=6$ from the point $(4,-1)$ are
$\frac{2}{3}x-\frac{10}{3}y=6\Rightarrow x-5y=9$, with point of contact $T\left(\frac{2}{3},-\frac{5}{3}\right)$ and $2x+2y=6\Rightarrow x+y=3$, with point of contact $T(2,1)$.

Problem CON3_18.

Show that if $y=mx+k$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $m^2a^2-b^2=k^2$. Hence find the equation of the tangents from the point $(1,3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ and the coordinates of their points of contact.

Answer: $y=2x+1$, $(-8,-15)$; $y=-4x+7$, $\left(\frac{16}{7},-\frac{15}{7}\right)$.

Solution: The hyperbola has parametric equations $x=a\sec\theta$ and $y=b\tan\theta$. Hence $\frac{dy}{dx}=\frac{b\sec\theta}{a\tan\theta}$. If $y=mx+k$ is a tangent to the hyperbola at $P(a\sec\phi,b\tan\phi)$, then

$m=\frac{dy}{dx}$ at $P\Rightarrow ma\tan\phi-b\sec\phi=0$ (1)

P lies on $y=mx+k\Rightarrow ma\sec\phi-b\tan\phi=-k$ (2)

$(2)^2-(1)^2\Rightarrow m^2a^2(\sec^2\phi-\tan^2\phi)+b^2(\tan^2\phi-\sec^2\phi)=k^2\Rightarrow m^2a^2-b^2=k^2$.

$(2)\times\sec\phi-(1)\times\tan\phi\Rightarrow ma(\sec^2\phi-\tan^2\phi)=-k\sec\phi\Rightarrow a\sec\phi=-\frac{ma^2}{k}$,

$(2)\times\tan\phi-(1)\times\sec\phi\Rightarrow b(\sec^2\phi-\tan^2\phi)=-k\tan\phi\Rightarrow b\tan\phi=-\frac{b^2}{k}$.

Therefore the point of contact of the tangent $y=mx+k$ is $P\left(-\frac{ma^2}{k},-\frac{b^2}{k}\right)$. Now tangents from the point $(1,3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ have equations of the form $y-3=m(x-1)$, that is, $y=mx+(3-m)$.

Hence $m^2a^2-b^2=k^2\Rightarrow 4m^2-15=(3-m)^2\Rightarrow 3m^2+6m-24=0\Rightarrow(m-2)(m+4)=0$.

$\therefore m=2, k=3-m=1$ and $P\left(-\frac{ma^2}{k},-\frac{b^2}{k}\right)\equiv P(-8,-15)$,

or $m=-4, k=3-m=7$ and $P\left(-\frac{ma^2}{k},-\frac{b^2}{k}\right)\equiv P\left(\frac{16}{7},-\frac{15}{7}\right)$.

Hence the tangents from the point $(1,3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ are $y=2x+1$, with point of contact $P(-8,-15)$ and $y=-4x+7$, with point of contact $P\left(\frac{16}{7},-\frac{15}{7}\right)$.

Problem CON3_19.

Find the equations and the coordinates of the points of contact of the tangents to $x^2+2y^2=19$ which are parallel to $x+6y=5$.

Answer: $x+6y=19$, $(1,3)$; $x+6y=-19$, $(-1,-3)$.

Solution: The tangent to the ellipse $x^2+2y^2=19$ at the point $P(x_0,y_0)$ has equation $xx_0+2yy_0=19$. If this tangent is parallel to $x+6y=5$, then $\frac{2y_0}{x_0}=6\Rightarrow y_0=3x_0$. Since the point $P(x_0,y_0)$ lies on the ellipse, then $x_0^2+2y_0^2=19$. Therefore $x_0^2+2\cdot 9x_0^2=19\Rightarrow x_0^2=1$. Hence the tangents to the ellipse $x^2+2y^2=19$ are $x+6y=19$, with point of contact $P(1,3)$ and $x+6y=-19$, with point of contact $P(-1,-3)$.

Problem CON3_20.

Find the equations and the coordinates of the points of contact of the tangents to $2x^2-3y^2=5$ which are parallel to $8x=9y$.

Answer: $8x-9y=5$, $(4,3)$; $8x-9y=-5$, $(-4,-3)$.

Solution: The tangent to the hyperbola $2x^2-3y^2=5$ at the point $P(x_0,y_0)$ has equation $2xx_0-3yy_0=5$. If this tangent is parallel to $8x=9y$, then $\frac{2x_0}{3y_0}=\frac{8}{9}\Rightarrow y_0=\frac{3}{4}x_0$. Since the point $P(x_0,y_0)$ lies on the hyperbola, then $2x_0^2-3y_0^2=5$. Therefore $2x_0^2-3\cdot\frac{9}{16}x_0^2=5\Rightarrow x_0^2=16$.
Hence the tangents to the hyperbola $2x^2-3y^2=5$ are $8x-9y=5$, with point of contact $P(4,3)$ and $8x-9y=-5$, with point of contact $P(-4,-3)$.

Problem CON3_21.
Find the equations and the coordinates of the points of contact of the tangents to $x^2-y^2=7$ which are parallel to $3y=4x$.

Answer: $4x-3y=7$, $(4,3)$; $4x-3y=-7$, $(-4,-3)$.

Solution: The tangent to the hyperbola $x^2-y^2=7$ at the point $P(x_0,y_0)$ has equation $xx_0-yy_0=7$. If this tangent is parallel to $3y=4x$, then $\frac{x_0}{y_0}=\frac{4}{3}\Rightarrow y_0=\frac{3}{4}x_0$. Since the point $P(x_0,y_0)$ lies on the hyperbola, then $x_0^2-y_0^2=7$. Therefore $x_0^2-\frac{9}{16}x_0^2=7\Rightarrow x_0^2=16$. Hence the tangents to the hyperbola $x^2-y^2=7$ are $4x-3y=7$, with point of contact $P(4,3)$ and $4x-3y=-7$, with point of contact $P(-4,-3)$.

Problem CON3_22.
Find the equations and the coordinates of the points of contact of the tangents to $8x^2+3y^2=35$ from the point $\left(\frac{5}{4},5\right)$.

Answer: $16x+3y=35$, $(2,1)$; $-8x+9y=35$, $(-1,3)$.

Solution: The tangent to the ellipse $8x^2+3y^2=35$ at the point $P(x_0,y_0)$ has equation $8xx_0+3yy_0=35$. The point $\left(\frac{5}{4},5\right)$ lies on this tangent. So $10x_0+15y_0=35\Rightarrow y_0=\frac{7}{3}-\frac{2}{3}x_0$.
Since the point $P(x_0,y_0)$ lies on the ellipse, then $8x_0^2+3y_0^2=35$.
Therefore $8x_0^2+3\cdot\left(\frac{7}{3}-\frac{2}{3}x_0\right)^2=35\Rightarrow 28x_0^2-28x_0-56=0\Rightarrow (x_0-2)(x_0+1)=0$. Hence the tangents to the ellipse $8x^2+3y^2=35$ from the point $\left(\frac{5}{4},5\right)$ are $16x+3y=35$, with point of contact $P(2,1)$ and $-8x+9y=35$, with point of contact $P(-1,3)$.

Problem CON3_23.

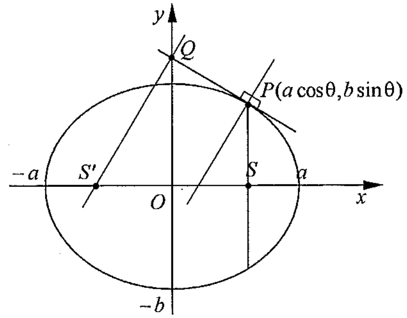
Find the equations and the coordinates of the points of contact of the tangents to $x^2-9y^2=9$ from the point $(3,2)$.

Answer: $x=3$, $(3,0)$; $-5x+12y=9$, $\left(-5,-\frac{4}{3}\right)$.

Solution: The tangent to the hyperbola $x^2-9y^2=9$ at the point $P(x_0,y_0)$ has equation $xx_0-9yy_0=9$. The point $(3,2)$ lies on this tangent. So $3x_0-18y_0=9\Rightarrow x_0=3+6y_0$. Since the point $P(x_0,y_0)$ lies on the hyperbola, then $x_0^2-9y_0^2=9$.
Therefore $(3+6y_0)^2-9y_0^2=9\Rightarrow 3y_0^2+4y_0=0\Rightarrow y_0(3y_0+4)=0$. Hence the tangents to the hyperbola $x^2-9y^2=9$ from the point $(3,2)$ are $x=3$, with point of contact $P(3,0)$ and $-5x+12y=9$, with point of contact $P\left(-5,-\frac{4}{3}\right)$.

Problem CON3_24.
The point $P(a\cos\theta, b\sin\theta)$ lies on an extremity of a latus rectum through one focus S of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the y-axis at Q. Show that the normal at P is parallel to QS', where S' is the other focus.

Solution:

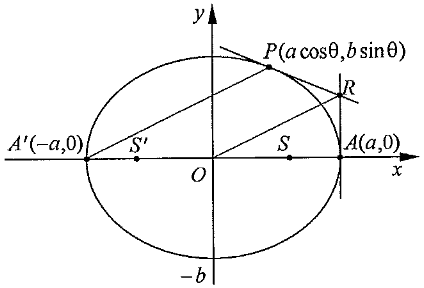


The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. Hence the point Q has coordinates $(0, b\operatorname{cosec}\theta)$. Thus the gradient of QS' is $\frac{b\operatorname{cosec}\theta}{ae}$. The gradient of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ is $\frac{a\sin\theta}{b\cos\theta}$. Since P lies at an extremity of a latus rectum through the focus $S(ae,0)$, then $\cos\theta=e$ and $\sin\theta=\sqrt{1-e^2}=\frac{b}{a}$. Therefore the gradient of QS' is $\frac{b}{ae}\cdot\frac{a}{b}=\frac{1}{e}$ and the gradient of the normal at P is $\frac{a}{be}\cdot\frac{b}{a}=\frac{1}{e}$. Hence the normal at P is parallel to QS'.

Problem CON3_25.

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the tangent at $A(a,0)$ at R. Show that OR is parallel to $A'P$, where A' is the point $(-a,0)$.

Solution:



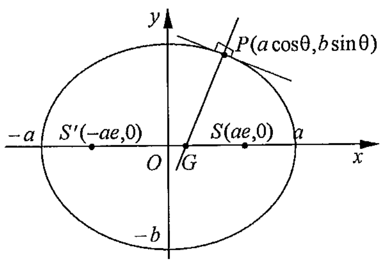
The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. Hence the point R has coordinates $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$. Thus the gradient of OR is $\frac{b(1-\cos\theta)}{a\sin\theta}$. The gradient of $A'P$ is

$\frac{b\sin\theta}{a(\cos\theta+1)}=\frac{b\sin\theta(1-\cos\theta)}{a(\cos\theta+1)(1-\cos\theta)}=\frac{b\sin\theta(1-\cos\theta)}{a(1-\cos^2\theta)}=\frac{b(1-\cos\theta)}{a\sin\theta}$. Therefore OR is parallel to $A'P$.

Problem CON3_26.

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S and S'. The normal at P meets SS' at G. Show that $PG^2=(1-e^2)PS\cdot PS'$.

Solution:



The normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. The point G has coordinates $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$. Therefore

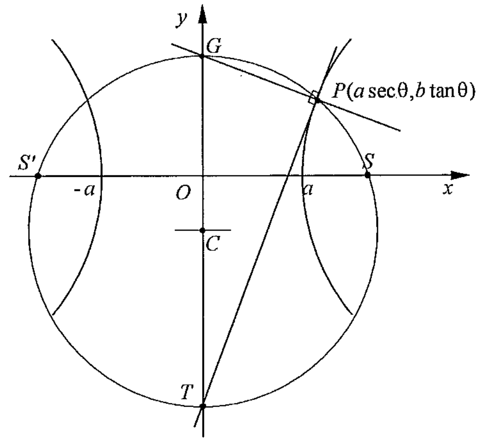
$$PG^2=\left(a-\frac{a^2-b^2}{a}\right)^2\cos^2\theta+b^2\sin^2\theta=\frac{b^2}{a^2}\left(b^2\cos^2\theta+a^2\sin^2\theta\right).$$

But for the ellipse $b^2=a^2(1-e^2)$. Hence $PG^2=a^2(1-e^2)(1-e^2\cos^2\theta)$.
From the other side
$PS^2=a^2(e-\cos\theta)^2+b^2\sin^2\theta=a^2(1-2e\cos\theta+e^2\cos^2\theta)=a^2(1-e\cos\theta)^2$,
$PS'^2=a^2(e+\cos\theta)^2+b^2\sin^2\theta=a^2(1+2e\cos\theta+e^2\cos^2\theta)=a^2(1+e\cos\theta)^2$.
Thus $PG^2=(1-e^2)\cdot a(1-e\cos\theta)\cdot a(1+e\cos\theta)=(1-e^2)PS\cdot PS'$.

Problem CON3_27.

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The tangent and the normal at P cut the y-axis at T and G respectively. Show that the circle on GT as diameter passes through the foci S and S'.

Solution:



The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. The point T has coordinates $(0,-b\cot\theta)$. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$. The point G has coordinates $\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$. So gradient $SG \times$ gradient $ST = \frac{a^2+b^2}{-bae}\tan\theta \cdot \frac{-b\cot\theta}{-ae}$. Since for the hyperbola $b^2 = a^2(e^2-1)$, then gradient $SG \times$ gradient $ST = -\frac{a^2+b^2}{a^2e^2} = -1$. Thus $SG \perp ST$ and consequently GT subtends a right angle at focus S.

Similarly gradient $S'G \times$ gradient $S'T = \frac{a^2+b^2}{bae}\tan\theta \cdot \frac{-b\cot\theta}{ae} = -\frac{a^2+b^2}{a^2e^2} = -1$. Thus $S'G \perp S'T$ and consequently GT subtends a right angle at focus S'. Therefore S, G, S', T are concyclic with GT the diameter of the circle through the points.

Problem CON3_28.

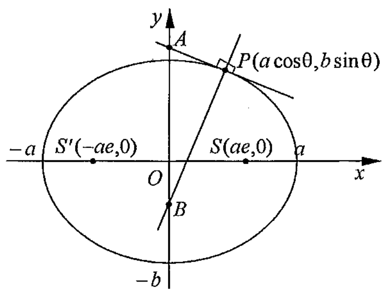
Show that the gradient of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the extremity in the first quadrant of its latus rectum is equal to the eccentricity of the hyperbola.

Solution: The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has gradient $\frac{b\sec\theta}{a\tan\theta}$. If P is an extremity in the first quadrant of a latus rectum, then $a\sec\theta = ae$. Thus $\sec\theta = e \Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{e^2-1}$. Since for the hyperbola $b^2 = a^2(e^2-1)$, then $\sqrt{e^2-1} = \frac{b}{a}$. Hence the gradient of the tangent is $\frac{be}{a\left(\frac{b}{a}\right)} = e$.

Problem CON3_29.

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$. The tangent and the normal at P cut the y-axis at A and B respectively, and S is a focus of the ellipse. (i) Show that $ASB = 90^0$. (ii) Hence show that A, P, S and B are concyclic and state the location of the center of the circle through A, P, S and B.

Solution:



The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. Therefore the point A has coordinates $(0, b\,\mathrm{cosec}\,\theta)$. The normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$. Hence the point B has coordinates $\left(0, \frac{b^2-a^2}{b}\sin\theta\right)$.

(i) Gradient $AS \times$ gradient $BS = \frac{b\,\mathrm{cosec}\,\theta}{-ae} \cdot \frac{(b^2-a^2)\sin\theta}{b(-ae)} = \frac{(b^2-a^2)}{a^2e^2}$. Since for the ellipse $b^2 = a^2(1-e^2)$, then gradient $AS \times$ gradient $BS = -1$. Hence AB subtends a right angle at S.
(ii) Since AB subtends a right angle at P, then A, P, S, B are concyclic with AB the diameter of the circle through the points. The center of the circle is the midpoint of AB.

Problem CON3_30.

Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ meet at the right angles. Find the equation of the circle through the points of intersection of these two curves.

Answer: $x^2 + y^2 = 5$.

Solution: Let $P(x_0, y_0)$ be the point of intersection. Then

P lies on the ellipse: $4x_0^2 + 9y_0^2 = 36$, (1)

P lies on the hyperbola: $4x_0^2 - y_0^2 = 4$. (2)

$(1)-(2) \Rightarrow 10y_0^2 = 32 \Rightarrow y_0^2 = 3.2$, (3)

$(1) + 9\times(2) \Rightarrow 40x_0^2 = 72 \Rightarrow x_0^2 = 1.8$. (4)

Since $x_0^2 + y_0^2 = 5$, then the points of intersection of the ellipse and the hyperbola lie on the circle $x^2 + y^2 = 5$. The tangent to the ellipse at P has gradient $g_e = -\frac{4x_0}{9y_0}$ and the tangent to the hyperbola at P has gradient $g_h = \frac{4x_0}{y_0}$. Therefore, using (3), (4) we obtain $g_e \cdot g_h = -\frac{16x_0^2}{9y_0^2} = -\frac{16\cdot 1.8}{9\cdot 3.2} = -1$. Hence the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ meet at right angles.

Problem CON3_31.

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$. The tangent at P passes through a focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that it is parallel to one of the lines $y = x$ and $y = -x$ and that its point of contact with the hyperbola lies on a directrix of the ellipse.

Solution: The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. Let e be the eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the tangent to the hyperbola passes through the focus $S(\pm ae,0)$ of the ellipse. Then $\pm e\sec\theta=1$ and consequently $|\tan\theta|=\sqrt{\sec^2\theta-1}=\sqrt{\frac{1}{e^2}-1}=\frac{\sqrt{1-e^2}}{e}=\frac{b}{ae}$. Hence the tangent to the hyperbola has equation $\pm\frac{x}{ae}-\frac{y}{ae}=1$ or $\pm\frac{x}{ae}+\frac{y}{ae}=1$. So the tangent is parallel to the line $y=x$ or to the line $y=-x$.

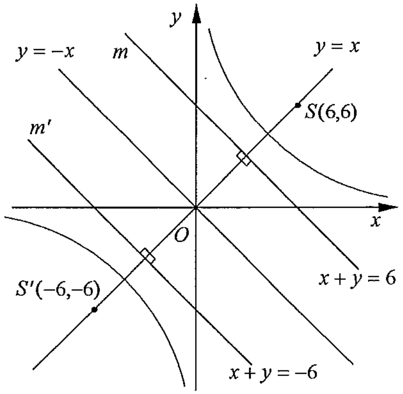
Then the point $P(a\sec\theta, b\tan\theta)$ has coordinates $\left(\pm\frac{a}{e},\frac{b^2}{ae}\right)$ or $\left(\pm\frac{a}{e},-\frac{b^2}{ae}\right)$. Therefore the point P lies on the directrix $x=\frac{a}{e}$ or $x=-\frac{a}{e}$ of the ellipse.

Problem CON3_32.

For the rectangular hyperbola $xy=18$, find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices, (d) the equations of the asymptotes. Sketch the rectangular hyperbola.

Answer: (a) $\sqrt{2}$; (b) $(6,6), (-6,-6)$; (c) $x+y=\pm 6$; (d) $x=0, y=0$.

Solution:



For the hyperbola $xy=18$ we have $c^2=18 \Rightarrow c=3\sqrt{2}$. Hence the hyperbola $xy=18$ has eccentricity $e=\sqrt{2}$,
foci $S(c\sqrt{2}, c\sqrt{2})=S(6,6)$ and $S'(-c\sqrt{2},-c\sqrt{2})=S(-6,-6)$,
directrices $x+y=\pm c\sqrt{2} \Rightarrow x+y=\pm 6$,
asymptotes $x=0$ and $y=0$.

Problem CON3_33.

Show that if $y=mx+k$ is a tangent to the rectungular hyperbola $xy=c^2$, then $k^2+4mc^2=0$. Hence find the equation of the tangents from the point $(-1,-3)$ to the rectangular hyperbola $xy=4$ and find the coordinates of their points of contact.

Answer: $y=-x-4, (-2,-2)$; $y=-9x-12, \left(-\frac{2}{3},-6\right)$.

Solution: The hyperbola $xy=c^2$ has parametric equations $x=ct$ and $y=\frac{c}{t}$. Hence $\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=-\frac{1}{t^2}$. If $y=mx+k$ is a tangent to the hyperbola at $P\left(cp,\frac{c}{p}\right)$, then

$m=\frac{dy}{dx}$ at P $\Rightarrow mp^2+1=0$ (1)

P lies on $y=mx+k \Rightarrow mcp-\frac{c}{p}=-k$ (2)

$\therefore$ (1) $\Rightarrow p^2=-\frac{1}{m}$. Thus squaring (2) we get $m^2c^2p^2-2mc^2+\frac{c^2}{p^2}=k^2 \Rightarrow 4mc^2+k^2=0$.

$(1)\times\frac{c}{p}+(2) \Rightarrow 2mcp=-k \Rightarrow cp=-\frac{k}{2m}$,

$(1)\times\frac{c}{p}-(2) \Rightarrow \frac{2c}{p}=k \Rightarrow \frac{c}{p}=\frac{k}{2}$.

Therefore the point of contact of the tangent $y=mx+k$ is $P\left(-\frac{k}{2m},\frac{k}{2}\right)$. Now tangents from the point $(-1,-3)$ to the hyperbola $xy=4$ have equations of the form $y+3=m(x+1)$, i.e. $y=mx+(m-3)$.

Hence $4mc^2+k^2=0 \Rightarrow 16m+(m-3)^2=0 \Rightarrow m^2+10m+9=0 \Rightarrow (m+1)(m+9)=0$.

$\therefore m=-1, k=m-3=-4$ and $P\left(-\frac{k}{2m},\frac{k}{2}\right)\equiv P(-2,-2)$,

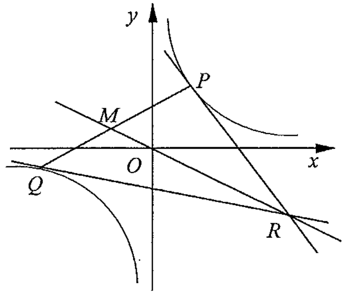
or $m=-9, k=m-3=-12$ and $P\left(-\frac{k}{2m},\frac{k}{2}\right)\equiv P\left(-\frac{2}{3},-6\right)$.

Hence the tangents from the point $(-1,-3)$ to the hyperbola $xy=4$ are $y=-x-4$, with point of contact $P(-2,-2)$ and $y=-9x-12$, with point of contact $P\left(-\frac{2}{3},-6\right)$.

Problem CON3_34.

The points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ lie on the rectangular hyperbola $xy=c^2$. The tangents at *P* and *Q* meet at *R*, and *OR* cuts *PQ* at *M*. Show that *M* is the midpoint of *PQ*.

Solution:



Since $R(x_0, y_0)$ lies on the tangent at the point $P\left(cp, \frac{c}{p}\right)$, then $x_0 + p^2 y_0 = 2cp$. Since $R(x_0, y_0)$ lies on the tangent at the point $Q\left(cq, \frac{c}{q}\right)$, then $x_0 + q^2 y_0 = 2cq$.

$\begin{matrix} x_0 + p^2 y_0 = 2cp \\ x_0 + q^2 y_0 = 2cq \end{matrix} \Rightarrow x_0 = \frac{2cpq}{p+q}$ and $y_0 = \frac{2c}{p+q}$. Then OR has equation $y = \frac{y_0}{x_0}x = \frac{x}{pq}$. The point $M(x_1, y_1)$ lies on OR. Therefore $y_1 = \frac{x_1}{pq}$. Since PQ is the chord of contact of tangents from the point $R(x_0, y_0)$, then PQ has equation $xy_0 + yx_0 = 2c^2$ or substituting the values of $x_0 = \frac{2cpq}{p+q}$ and $y_0 = \frac{2c}{p+q}$ have $\frac{x}{pq} + y = c\frac{p+q}{pq}$. $M(x_1, y_1)$ lies on PQ. Hence $\frac{x_1}{pq} + y_1 = c\frac{p+q}{pq}$. Thus $y_1 = \frac{1}{2}\left(\frac{c}{p} + \frac{c}{q}\right)$ and $x_1 = \frac{1}{2}(cp + cq)$. Therefore M is the midpoint of PQ.

Problem CON3_35.

The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the rectangular hyperbola $x^2 - y^2 = a^2$ at Q and R. Show that P is the midpoint of QR.

Solution: The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. Let the point Q, R have coordinates (x_1, y_1) and (x_2, y_2) respectively. Since Q, R lie on the hyperbola $x^2 - y^2 = a^2$, then subtracting $x_2^2 - y_2^2 = a^2$ from $x_1^2 - y_1^2 = a^2$ we get:

$$(x_1^2 - x_2^2) - (y_1^2 - y_2^2) = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2) = (y_1 - y_2)(y_1 + y_2). \qquad (1)$$

The points Q, R lie on the normal to the hyperbola. Therefore subtracting $tx_2 - \frac{y_2}{t} = c\left(t^2 - \frac{1}{t^2}\right)$ from $tx_1 - \frac{y_1}{t} = c\left(t^2 - \frac{1}{t^2}\right)$ we have:

$$t(x_1 - x_2) - \frac{y_1 - y_2}{t} = 0, \qquad (2)$$

$$t(x_1 + x_2) - \frac{y_1 + y_2}{t} = 2c\left(t^2 - \frac{1}{t^2}\right). \qquad (3)$$

Substituting (2) into (1), we obtain

$$x_1 + x_2 = t^2(y_1 + y_2). \qquad (4)$$

Then (3), (4) $\Rightarrow t^2(y_1 + y_2) - \frac{1}{t^2}(y_1 + y_2) = \frac{2c}{t}\left(t^2 - \frac{1}{t^2}\right)$.

Hence $y_1 + y_2 = \frac{2c}{t}$. (5)

Using (5) we get from (4)

$$x_1 + x_2 = 2ct. \qquad (6)$$

Thus, according to (5) and (6), the midpoint of QR the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ has coordinates $\left(ct, \frac{c}{t}\right)$. Hence the point $P\left(ct, \frac{c}{t}\right)$ is the midpoint of QR.

Problem CON3_36.

The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the hyperbola again at Q. The circle on PQ as diameter meets the hyperbola again at R. Find the coordinates of Q and R.

Answer: $\left(-\frac{c}{t^3}, -ct^3\right)$, $\left(-ct, -\frac{c}{t}\right)$.

Solution: The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. The point $Q\left(cq, \frac{c}{q}\right)$ lies on the normal. Hence $tcq - \frac{c}{tq} = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow (tq - t^2)\left(1 + \frac{1}{t^3 q}\right) = 0$. Since $Q \neq P$, then $q \neq t$. Therefore $q = -\frac{1}{t^3}$ and Q has coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. The point $R\left(cr, \frac{c}{r}\right)$ lies on the circle on PQ as diameter. Hence gradient $RP \times$ gradient $RQ = -1$. But gradient of RP is $c\left(\frac{1}{r} - \frac{1}{t}\right)\cdot\frac{1}{c(r-t)} = -\frac{1}{rt}$ and gradient of RQ is $c\left(\frac{1}{r} - \frac{1}{q}\right)\cdot\frac{1}{c(r-q)} = -\frac{1}{rq}$. Thus $\frac{1}{r^2 tq} = -1 \Rightarrow r^2 = -\frac{1}{tq}$. Since $q = -\frac{1}{t^3}$, then $r^2 = t^2$. Therefore $r = -t$, because $R \neq P$. So the point R has coordinates $\left(-ct, -\frac{c}{t}\right)$.

Problem CON3_37.

The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the hyperbola again at Q. M is the midpoint of PQ. Find the equation of the locus of M.

Answer: $4x^3y^3 + c^2(x^2 - y^2)^2 = 0$.

Solution: The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. The point $Q\left(cq, \frac{c}{q}\right)$ lies on the normal.

Hence $tcq - \frac{c}{tq} = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow (tq - t^2)\left(1 + \frac{1}{t^3q}\right) = 0$. Since $Q \neq P$, then $q \neq t$. Therefore $q = -\frac{1}{t^3}$ and Q has coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. If $M(x,y)$ is the midpoint of PQ, then

$$x = \frac{c}{2}(t+q) = \frac{c}{2t}\left(t^2 - \frac{1}{t^2}\right) \qquad (1)$$

and

$$y = \frac{c}{2}\left(\frac{1}{t} + \frac{1}{q}\right) = \frac{ct}{2}\left(\frac{1}{t^2} - t^2\right). \qquad (2)$$

We obtain from (1), (2) that $\frac{2tx}{c} = -\frac{2y}{ct} \Rightarrow t^2 = -\frac{y}{x}$. Substituting this formula for t^2 into (1), we get $x = \frac{c}{2\sqrt{-y/x}}\left(-\frac{y}{x} + \frac{x}{y}\right) \Rightarrow x^2 = \frac{-c^2x}{4y} \cdot \frac{(x^2 - y^2)}{x^2y^2} \Rightarrow 4x^3y^3 + c^2(x^2 - y^2)^2 = 0$. Therefore the locus of M has equation $4x^3y^3 + c^2(x^2 - y^2)^2 = 0$.

Problem CON3_38.

The point $P\left(ct, \frac{c}{t}\right)$, where $t \neq 1, t \neq -1$, lies on the rectangular hyperbola $xy = c^2$. The tangent at P meets the x-axis and the y-axis at Q and R respectively. The normal at P meets the lines $y = x$ and $y = -x$ at S and T respectively. Show that $QSRT$ is a rhombus.

Solution: The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$. Hence the tangent meets the x-axis at $Q(2ct, 0)$ and the y-axis at $R\left(0, \frac{2c}{t}\right)$. The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. Thus the normal meets the line $y = x$ at $S\left(c\left(t + \frac{1}{t}\right), c\left(t + \frac{1}{t}\right)\right)$ and the line $y = -x$ at $T\left(c\left(t - \frac{1}{t}\right), -c\left(t - \frac{1}{t}\right)\right)$. Therefore

$$QS^2 = c^2\left(t + \frac{1}{t} - 2t\right)^2 + c^2\left(t + \frac{1}{t}\right)^2 = 2c^2\left(t^2 + \frac{1}{t^2}\right),$$

$$SR^2 = c^2\left(t + \frac{1}{t}\right)^2 + c^2\left(\frac{2}{t} - t - \frac{1}{t}\right)^2 = 2c^2\left(t^2 + \frac{1}{t^2}\right),$$

$$RT^2 = c^2\left(t - \frac{1}{t}\right)^2 + c^2\left(-t + \frac{1}{t} - \frac{2}{t}\right)^2 = 2c^2\left(t^2 + \frac{1}{t^2}\right),$$

$$TQ^2 = c^2\left(2t - t + \frac{1}{t}\right)^2 + c^2\left(t - \frac{1}{t}\right)^2 = 2c^2\left(t^2 + \frac{1}{t^2}\right).$$

So $QS = SR = RT = TQ$ and, consequently, $QSRT$ is a rhombus.

Problem CON3_39.

The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. Show that the normal at P cuts the hyperbola again at the point Q with coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again.

Answer: $\left(ct^9, \frac{c}{t^9}\right)$.

Solution: The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. The point $Q\left(cq, \frac{c}{q}\right)$ lies on the normal. Hence $tcq - \frac{c}{tq} = c\left(t^2 - \frac{1}{t^2}\right)$.

Therefore $(tq - t^2)\left(1 + \frac{1}{t^3q}\right) = 0$. Since $Q \neq P$, then $q \neq t$. Thus $q = -\frac{1}{t^3}$ and Q has coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. Similarly the normal at Q cuts the hyperbola again at $R\left(cr, \frac{c}{r}\right)$ with $r = -\frac{1}{q^3} = t^9$.

So R has coordinates $\left(ct^9, \frac{c}{t^9}\right)$.

Problem CON3_40.

The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the x-axis at A and the tangent at P meets the y-axis at B. M is the midpoint of AB. Find the equation of the locus of M as P moves on the hyperbola.

Answer: $2c^2xy = c^4 - y^4$.

Solution: The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. The normal at P meets the x-axis at $A\left(\frac{c}{t}\left(t^2 - \frac{1}{t^2}\right), 0\right)$. The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$. Hence the tangent meets the y-axis at $B\left(0, \frac{2c}{t}\right)$. If $M(x,y)$ is the midpoint of AB, then $x = \frac{c}{2t}\left(t^2 - \frac{1}{t^2}\right)$ and $y = \frac{c}{t}$. Thus

$t=\frac{c}{y}$ and, consequently, $x=\frac{y}{2}\left(\frac{c^2}{y^2}-\frac{y^2}{c^2}\right)$. Therefore the locus of M has equation $2c^2xy=c^4-y^4$.