Topic 4. Exercises on Complex Numbers IV
Level 1

1. Use *de Moivre's* theorem to solve $z^5 = -1$. By grouping the roots in complex conjugate pairs, show that $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$.

   Indicate on an Argand diagram the locus of the point P representing z when

2. If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$.

Hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

3. 1, ω and ω^2 are the three cube roots of unity. Simplify each of the expressions $(1+3\omega+\omega^2)^2$ and $(1+\omega+3\omega^2)^2$ and show that their sum is -4 and their product is 16.

4. Use *de Moivre's* theorem to solve the equation $z^5 = 1$. Show that the points representing the five roots of this equation on an Argand diagram form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$ and perimeter $10\sin\frac{\pi}{5}$.

$$1, \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$$

5. Show that the roots of $z^6 + z^3 + 1 = 0$ are among the roots of $z^9 - 1 = 0$. Hence find the roots of $z^6 + z^3 + 1 = 0$ in modulus/argument form.

$$\boxed{\text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right)}$$

6. If $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2i\sin n\theta$. Hence show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

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1. Use *de Moivre's* theorem to solve $z^5 = -1$. By grouping the roots in complex conjugate pairs, show that $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$.
Indicate on an Argand diagram the locus of the point P representing z when

$z^5 = -1$

let $z = r\,\text{cis}\,\theta$

$r^5 \text{cis}\, 5\theta = \text{cis}\,\pi$

$r^5 = 1 \quad r = 1$ ①

$\text{cis}\, 5\theta = \text{cis}\,\pi$

$5\theta = \pi + 2n\pi$

$\theta = \frac{\pi(1+2n)}{5}$ ②

$\therefore z = \text{cis}\,\frac{\pi(1+2n)}{5}$

$z_0 = \text{cis}\,\frac{-3\pi}{5} \qquad z_3 = \text{cis}\,3\pi/5$

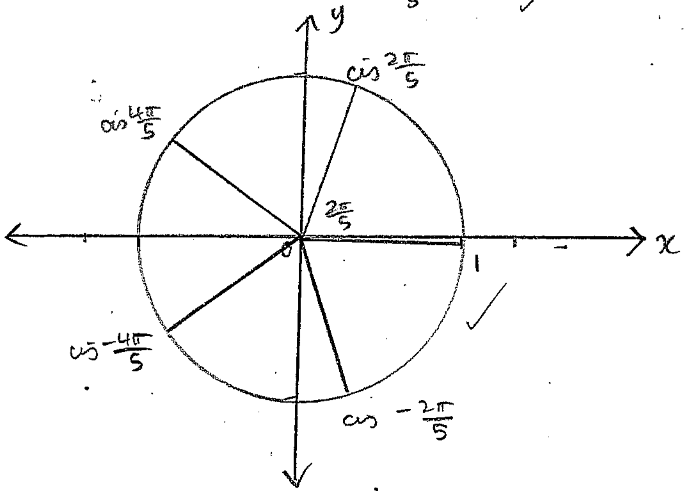
$z_1 = \text{cis}\,\frac{-\pi}{5} \qquad z_4 = \text{cis}\,\pi = 1$

$z_2 = \text{cis}\,\pi/5$

$z^5+1 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$

$z^5+1 = (z - \text{cis}\,\frac{-3\pi}{5})(z - \text{cis}\,\frac{-\pi}{5})(z - \text{cis}\,\frac{\pi}{5})(z - \text{cis}\,\frac{3\pi}{5})(z-1)$

$\therefore z^5+1 = (z-1)(z^2 - 2\cos\frac{3\pi}{5} + 1)(z^2 - 2\cos\frac{\pi}{5} + 1)$



2. If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$.
Hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

$z^n = \cos n\theta + i\sin n\theta \qquad z^{-n} = \cos -n\theta + i\sin -n\theta$
$= \cos n\theta - i\sin n\theta$

$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= 2\cos n\theta$

$(z^1 + z^{-1})^4$

$(2\cos\theta)^4 = z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4}$
$= (z^4 + z^{-4}) + 4z^2 + 6 + 4z^{-2}$
$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{2\cos 4\theta + 8\cos 2\theta + 6}{16}$

$\cos^4\theta = \frac{\cos 4\theta + 4\cos 2\theta + 3}{8}$

3. 1, ω and ω^2 are the three cube roots of unity. Simplify each of the expressions $(1+3\omega+\omega^2)^2$ and $(1+\omega+3\omega^2)^2$ and show that their sum is -4 and their product is 16.

~~$(1+3w+w^2)^2$~~

$1+w+w^2=0$ $\quad$ $1+w=-w^2$

$w^3=1$ $\quad$ $w^2+w=-1$

$(1+3w+w^2)^2 = (3(1+w+w^2)-2-2w^2)^2$

$= (2+2w^2)^2$

$= 4(1+w^2)^2$

$= 4w^2$

$(1+w+3w^2)^2 = (2w^2)^2$

$= 4w^4$

$= 4w$.

sum $= 4w^2+4w$

$= 4(-1) = -4$

product $= 16w^3$

$= 16$.

4. Use *de Moivre's* theorem to solve the equation $z^5=1$. Show that the points representing the five roots of this equation on an Argand diagram form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$ and perimeter $10\sin\frac{\pi}{5}$.

let $z = r\,\text{cis}\,\theta$

$r^5\,\text{cis}\,5\theta = \text{cis}\,0$

$r^5=1$

$r=1$

$5\theta = 2n\pi$

$\theta = \frac{2n\pi}{5}$ $\quad$ $z = \text{cis}\,\frac{2n\pi}{5}$.

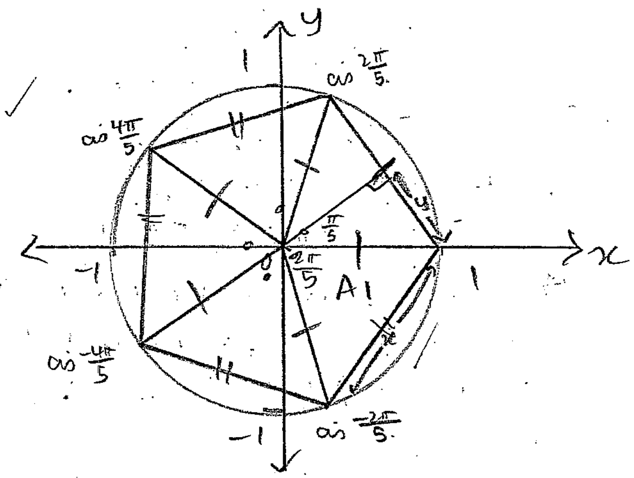
$z_1 = \text{cis}\,\frac{-4\pi}{5}$

$z_2 = \text{cis}\,\frac{-2\pi}{5}$

$z_3 = 1$

$z_4 = \text{cis}\,\frac{2\pi}{5}$

$z_5 = \text{cis}\,\frac{4\pi}{5}$



$A = 5\times A_1$

$= 5\times\frac{1}{2}ab\sin C$

$= 5\times\frac{1}{2}\times 1\times 1\times\sin\frac{2\pi}{5}$

$\therefore$ Area $= \frac{5}{2}\sin\frac{2\pi}{5}$ units^2.

$\sin\frac{\pi}{5} = \frac{y}{1}$

$y = \sin\frac{\pi}{5}$.

$\therefore P = 10\sin\frac{\pi}{5}$ units.

$P = 5x = 10y$.

~~$a^2 = b^2+c^2-2bc\cos A$~~

~~$x^2 = 1+1-2\cos\frac{2\pi}{5}$~~

~~$= 2-2\cos\frac{2\pi}{5}$~~

$1, \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$

5. Show that the roots of $z^6 + z^3 + 1 = 0$ are among the roots of $z^9 - 1 = 0$. Hence find the roots of $z^6 + z^3 + 1 = 0$ in modulus/argument form.

$(z^3)^3 - 1^3 = 0$

$(z^3 - 1)(z^6 + z^3 + 1) = 0$ (2)

$(z-1)(z^2+z+1)$ $z^3 = 1$ or $z^6 + z^3 + 1 = 0$ (1)

$z = 1$

$z^9 = 1$ let $z = r\,\text{cis}\,\theta$.

$r^9 \text{cis}\, 9\theta = \text{cis}\, 0$.

$r^9 = 1$, $r = 1$ (1) $\quad 9\theta = 2n\pi$, $\theta = \frac{2n\pi}{9}$ $\quad z = \text{cis}\,\frac{2n\pi}{9}$

$z_1 = \text{cis}\, -\frac{8\pi}{9}$

$z_2 = \text{cis}\, -\frac{6\pi}{9} = \text{cis}\, -\frac{2\pi}{3}$ $\quad \text{cis}\, -\frac{2\pi}{3}$

[redacted]

$z_5 = 1$

[redacted]

$z_8 = \text{cis}\,\frac{2\pi}{3}$

$z_9 = \text{cis}\,\frac{8\pi}{9}$

$\therefore z^9 - 1 = (z-1)(z - \text{cis}\,-\frac{8\pi}{9})(z - \text{cis}\,\frac{8\pi}{9})(z - \text{cis}\,-\frac{2\pi}{3})(z - \text{cis}\,\frac{2\pi}{3})(z - \text{cis}\,\frac{4\pi}{9})$
$(z - \text{cis}\,\frac{4\pi}{9})(z - \text{cis}\,\frac{2\pi}{9})(z - \text{cis}\,-\frac{4\pi}{9})$

$= (z-1)(z^2 - 2\cos\frac{8\pi}{9} + 1)(z^2 - 2\cos\frac{2\pi}{3} + 1)(z^2 - 2\cos\frac{4\pi}{9} + 1)$
$(z^2 - 2\cos\frac{2\pi}{9} + 1)$ (1)

fq (1) & (2):

$z^6 + z^3 + 1 = (z^2 - 2\cos\frac{8\pi}{9} + 1)(z^2 - 2\cos\frac{2\pi}{3} + 1)(z^2 - 2\cos\frac{4\pi}{9} + 1)(z^2 - 2\cos\frac{2\pi}{9} + 1)$

$\therefore$ roots of $z^6 + z^3 + 1$,

$z = \text{cis}\,\pm\frac{2\pi}{3}$, $\text{cis}\,\pm\frac{8\pi}{9}$, $\text{cis}\,\pm\frac{4\pi}{9}$, $\text{cis}\,\pm\frac{2\pi}{9}$.

but $\text{cis}\,\pm\frac{2\pi}{3}$ are roots of $z^3 = 1$.

$\therefore z = \text{cis}\,\pm\frac{8\pi}{9}$, $\text{cis}\,\pm\frac{4\pi}{9}$, $\text{cis}\,\pm\frac{2\pi}{9}$ only.

$\text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right)$

6. If $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2i\sin n\theta$. Hence show that $\sin^5\theta = \frac{i}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

$z^n = \cos n\theta + i\sin n\theta$ $\quad z^{-n} = \cos n\theta - i\sin n\theta$.

$z^n - z^{-n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$

$(z^1 - z^{-1})^5$

$(2i\sin\theta)^5 = z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5}$

$32i^5\sin^5\theta = (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$

$32i\sin^5\theta := 2i\sin 5\theta + 10i\sin 3\theta + 20i\sin\theta$

$32\sin^5\theta = 2\sin 5\theta + 10\sin 3\theta + 20\sin\theta$.

$\sin^5\theta = \dfrac{\sin 5\theta + 5\sin 3\theta + 10\sin\theta}{16}$