Topic 5. Harder Unit Topics.

Level 3.

Problem HAR3_01.

If $a>0$, show that $a^2+\frac{1}{a^2}\ge a+\frac{1}{a}\ge 2$.

Solution: $\left(a+\frac{1}{a}\right)^2=\left(a-\frac{1}{a}\right)^2+4\Rightarrow\left(a+\frac{1}{a}\right)^2\ge 4\Rightarrow a+\frac{1}{a}\ge 2$ (equality iff $a=\frac{1}{a}$, i.e., $a=1$);

$a^2+\frac{1}{a^2}-\left(a+\frac{1}{a}\right)=\left(a^2-a\right)+\left(\frac{1}{a^2}-\frac{1}{a}\right)=a(a-1)-\frac{1}{a^2}(a-1)=(a-1)\left(a-\frac{1}{a^2}\right)=$

$(a-1)\frac{\left(a^3-1\right)}{a^2}=(a-1)(a-1)\frac{\left(a^2+a+1\right)}{a^2}=(a-1)^2\frac{\left(a^2+a+1\right)}{a^2}\ge 0\Rightarrow a^2+\frac{1}{a^2}\ge a+\frac{1}{a}$ with equality iff $a=1$.

Problem HAR3_02.

If $a\ge 0$, $b\ge 0$ and $c\ge 0$, show that $ab+bc+ca\ge a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}$.

Solution: $(x+y)^2=(x-y)^2+4xy\ge 4xy\Rightarrow$ if $x\ge 0$ and $y\ge 0$ we get $\sqrt{xy}\le\frac{x+y}{2}$. Hence

$a\sqrt{bc}=\sqrt{ab}\sqrt{ca}\le\frac{ab+ca}{2}$,

Similarly $b\sqrt{ac}=\sqrt{ab}\sqrt{cb}\le\frac{ab+cb}{2}$,

and $c\sqrt{ab}=\sqrt{ac}\sqrt{cb}\le\frac{ac+cb}{2}$.

By addition $a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\le ab+bc+ca$ (equality iff $a=b=c$).

Problem HAR3_03.

If $a_1\ge 0$, $a_2\ge 0$, $b_1\ge 0$ and $b_2\ge 0$, show that $\sqrt{(a_1+b_1)(a_2+b_2)}\ge\sqrt{a_1a_2}+\sqrt{b_1b_2}$.

Solution: By squaring the sides of the inequality to be proved we get:

$(a_1+b_1)(a_2+b_2)\ge a_1a_2+b_1b_2+2\sqrt{a_1a_2b_1b_2}$.

Use the inequality $\sqrt{xy}\le\frac{x+y}{2}$ with $x=a_1b_2$ and $y=a_2b_1$:

$\sqrt{a_1a_2b_1b_2}=\sqrt{a_1b_2}\sqrt{a_2b_1}\le\frac{a_1b_2+a_2b_1}{2}$.

Problem HAR3_04.

Show that if a, b and c are the sides of a triangle, then $(a+b-c)(b+c-a)(c+a-b)\le abc$.
Hint: Show that if $x\ge 0$, $y\ge 0$ and $z\ge 0$, then $(x+y)(y+z)(z+x)\ge 8xyz$.

Solution: $(x+y)^2=(x-y)^2+4xy\ge 4xy$.
Hence $(x+y)\ge 2\sqrt{xy}$,
similarly $(z+y)\ge 2\sqrt{zy}$,
and $(x+z)\ge 2\sqrt{xz}$.
By multiplication $(x+z)(y+z)(z+x)\ge 8xyz$.
The substitution $x=a+b-c$, $y=b+c-a$, $z=c+a-b$ gives
$2b2c2a\ge 8(a+b-c)(b+c-a)(c+a-b)$.
Hence $abc\ge(a+b-c)(b+c-a)(c+a-b)$.
Note that since a, b and c are sides of a triangle, x, y and z are non-negative.

Problem HAR3_05.

If $a_1, a_2, \cdots, a_n$ are positive numbers such that $a_1a_2\cdot\ldots\cdot a_n=1$, show that $(1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n)\ge 2^n$.

Solution: $(1+x)^2=(1-x)^2+4x\ge 4x\Rightarrow 1+x\ge 2\sqrt{x}$ for $x\ge 0$.
Hence $(1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n)\ge 2\sqrt{a_1}\,2\sqrt{a_2}\cdot\ldots\cdot 2\sqrt{a_n}=2^n\sqrt{a_1a_2\cdot\ldots\cdot a_n}=2^n$.

Problem HAR3_06.

If $a>0$, $b>0$ and $c>0$, show that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9$.

Solution: Deduce that $(a+b+c)(ab+bc+ca)\ge 9abc$ and hence deduce that $a^2b+b^2c+c^2a+ab^2+bc^2+ca^2\ge 6abc$.

Problem HAR3_07.

If $a>0$, $b>0$, $c>0$ and $d>0$, show that

$$\frac{16}{a+b+c+d}\le\frac{3}{b+c+d}+\frac{3}{a+b+c}+\frac{3}{a+b+d}+\frac{3}{a+c+d}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

Hint: Use the inequality $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge 9$ with positive x, y and z.

Solution: It is clear that $\frac{9}{x+y+z}\le\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. Hence

$\frac{9}{a+b+c}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, $\quad\frac{9}{a+c+d}\le\frac{1}{a}+\frac{1}{c}+\frac{1}{d}$,

$\frac{9}{a+b+d}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{d}$, $\quad\frac{9}{b+c+d}\le\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$.

By addition

$$\frac{3}{b+c+d}+\frac{3}{a+b+c}+\frac{3}{a+b+d}+\frac{3}{a+c+d}\le\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(equality iff $a=b=c=d$).

Furthermore, if we employ the inequality $t+\frac{1}{t}\ge 2$, $t>0$, We have

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=1+1+1+1+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{d}+\frac{d}{b}\right)+$$

$$\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{d}+\frac{d}{c}\right)+\left(\frac{d}{a}+\frac{a}{d}\right)\geq 4+2+2+2+2+2+2=16.$$

Hence $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16$ (equality iff $a=b=c=d$).

The substitution $a\Rightarrow a+b+c,\quad b\Rightarrow b+c+d,\quad c\Rightarrow c+d+a,\quad d\Rightarrow d+a+b$ in this inequality permits to obtain the desired factor $\frac{1}{b+c+d}+\frac{1}{a+b+c}+\frac{1}{a+b+d}+\frac{1}{a+c+d}$, and we get

$$3(a+b+c+d)\left(\frac{1}{a+b+c}+\frac{1}{a+b+d}+\frac{1}{b+c+d}+\frac{1}{a+c+d}\right)\geq 16,$$

(equality iff $a=b=c=d$).

Then

$$\frac{16}{a+b+c+d}\leq\frac{3}{b+c+d}+\frac{3}{a+b+c}+\frac{3}{a+b+d}+\frac{3}{a+c+d}.$$

Finally, using the inequalities given above

$$\frac{3}{b+c+d}\leq\frac{1}{3}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right),\qquad \frac{3}{a+b+c}\leq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

$$\frac{3}{a+b+d}\leq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right),\qquad \frac{3}{a+c+d}\leq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right),$$

we come to the desired result

$$\frac{16}{a+b+c+d}\leq\frac{3}{b+c+d}+\frac{3}{a+b+c}+\frac{3}{a+b+d}+\frac{3}{a+c+d}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

(equality iff $a=b=c=d$).

Problem HAR3_08.

Show that $\frac{a^2+b^2+c^2}{3}\geq\left(\frac{a+b+c}{3}\right)^2$.

Solution: $a^2+b^2=(a-b)^2+2ab\geq 2ab$

Hence $2ab\leq a^2+b^2$ (equality iff $a=b$),

Similarly $2bc\leq c^2+b^2$ and $2ac\leq a^2+c^2$.

By addition $2ab+2bc+2ac\leq 2\left(a^2+b^2+c^2\right)$

But $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$

So $(a+b+c)^2\leq 3\left(a^2+b^2+c^2\right)$.

Hence $\left(\frac{a+b+c}{3}\right)^2\leq\frac{\left(a^2+b^2+c^2\right)}{3}$ (equality iff $a=b=c$).

Problem HAR3_09.

If $x\neq 0$ and $y\neq 0$, show that $x^4+y^4\leq\frac{x^6}{y^2}+\frac{y^6}{x^2}$.

Solution: Let $a=x^2$ and $b=y^2$. Then we can rewrite the inequality to be proved in the equivalent form $a^2+b^2\leq\frac{a^3}{b}+\frac{b^3}{a}$, $a>0$ and $b>0$.

Multiplication by $ab>0$ gives $ab\left(a^2+b^2\right)\leq a^4+b^4$.

Clearly $a^2+b^2=(a-b)^2+2ab\Rightarrow ab\leq\frac{a^2+b^2}{2}$ (1)

Hence $ab\left(a^2+b^2\right)\leq\frac{\left(a^2+b^2\right)^2}{2}=\frac{a^4+b^4+2a^2b^2}{2}$.

The substitution $a\to a^2, b\to b^2$ in (1) gives $a^2b^2\leq\frac{a^4+b^4}{2}\Rightarrow 2a^2b^2\leq a^4+b^4$.

Hence $ab\left(a^2+b^2\right)\leq\frac{a^4+b^4+2a^2b^2}{2}\leq\frac{a^4+b^4+a^4+b^4}{2}$.

So $ab\left(a^2+b^2\right)\leq a^4+b^4$ and $a^2+b^2\leq\frac{a^3}{b}+\frac{b^3}{a}\Rightarrow x^4+y^4\leq\frac{x^6}{y^2}+\frac{y^6}{x^2}$.

Problem HAR3_10.

If $a>1, b>1$ and $c>1$ such that $\frac{a}{b}\geq\frac{c}{a}$, show that $\frac{\lg a}{\lg b}\geq\frac{\lg c}{\lg a}$.

Solution: $\frac{a}{b}\geq\frac{c}{a}\Rightarrow a^2\geq bc\Rightarrow\lg a^2\geq\lg(bc)\Rightarrow 2\lg a\geq\lg b+\lg c\Rightarrow\lg a\geq\frac{\lg b+\lg c}{2}$.

Clearly $(x+y)^2=(x-y)^2+4xy\geq 4xy\Rightarrow(x+y)^2\geq 4xy$. Hence for $x\geq 0$ and $y\geq 0$ $\frac{x+y}{2}\geq\sqrt{xy}$. The substitution $x=\lg b$ and $y=\lg c$ gives $\frac{\lg b+\lg c}{2}\geq\sqrt{\lg b\lg c}$.

Hence $\lg a\geq\sqrt{\lg b\lg c}$. Squaring, $(\lg a)^2\geq\lg b\lg c\Rightarrow\frac{\lg a}{\lg b}\geq\frac{\lg c}{\lg a}$.

Problem HAR3_11.

If $x>y$ and $xy=1$, show that $\frac{x^2+y^2}{x-y}\geq 2\sqrt{2}$.

Solution: The multiplication of the inequality to be proved by $x-y>0$ gives $x^2+y^2\geq 2\sqrt{2}(x-y)$.

Squaring, $\left(x^2+y^2\right)^2\geq 8\left(x^2+y^2-2xy\right)=8\left(x^2+y^2-2\right)$.

Hence $\left(x^2+y^2\right)^2-8\left(x^2+y^2\right)+16\geq 0$. Let $t=x^2+y^2$ then it suffices to show that $t^2-8t+16\geq 0$. This is the case as the discriminant of this quadratic $\Delta=8^2-4\cdot 16=0$.

Problem HAR3_12.

Show that the geometric mean of n positive numbers can not exceed their arithmetic mean, i.e., if $x_1, x_2, \ldots, x_n$ are positive numbers, show that $(x_1x_2\ldots x_n)^{1/n}\leq\frac{x_1+x_2+\ldots+x_n}{n}$.

Hint: Show that $x\leq e^{x-1}$ for all x.

Solution: Let $f(x)=e^{x-1}-x$. Then $f'(x)=e^{x-1}-1$. Clearly $f'(x)=0$ when $x=1$, $f'(x)>0$ when $x>1$ and $f'(x)<0$ when $x<1$. Hence $f(x)$ has an absolute minimum of 0 when $x\le e^{x-1}$. Let $A=\frac{x_1+x_2+\dots+x_n}{n}$. The substitution $x\Rightarrow\frac{x_k}{A}$ gives $\frac{x_k}{A}\le e^{\frac{x_k}{A}-1}$, $k=1,\dots,n$.

By multiplication $\frac{x_1}{A}\frac{x_2}{A}\cdots\frac{x_n}{A}\le\exp\left\{\frac{x_1}{A}-1+\frac{x_2}{A}-1\dots\frac{x_n}{A}-1\right\}$.

But $\frac{x_1}{A}-1+\frac{x_2}{A}-1\dots\frac{x_n}{A}-1=(x_1+x_2+\dots+x_n)\cdot\frac{1}{A}-n=0$.

Hence $\frac{x_1}{A}\frac{x_2}{A}\cdots\frac{x_n}{A}\le 1\Rightarrow x_1x_2\dots x_n\le A^n$. So $(x_1x_2\dots x_n)^{1/n}\le A$.

Problem HAR3_13.

If $a_1,a_2,\dots,a_n$ are positive numbers, show that $(a_1+a_2+\dots+a_n)\cdot\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)\ge n^2$.

Solution: We have $(a_1a_2\dots a_n)^{1/n}\le\frac{a_1+a_2+\dots+a_n}{n}$. The substitution $a_1\Rightarrow\frac{1}{a_1},\dots,a_n\Rightarrow\frac{1}{a_n}$

gives $\left(\frac{1}{a_1}\frac{1}{a_2}\dots\frac{1}{a_n}\right)^{1/n}\le\frac{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}{n}$.

By multiplication $(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)\cdot\frac{1}{n^2}\ge 1$.

Hence $(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)\ge n^2$.

Problem HAR3_14.

If $a>0, b>0, c>0$ and $a+b+c=1$, show that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 27$.

Solution: Clearly $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=(a+b+c)^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$; $a+b+c=1$.

Use the inequality between the arithmetic mean and the geometric mean for a,b,c and then for $\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2}$. We have $\frac{a+b+c}{3}\ge\sqrt[3]{abc}\Rightarrow(a+b+c)^2\ge 9\left(\sqrt[3]{abc}\right)^2$,

$\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{3}\ge\sqrt[3]{\left(\frac{1}{a^2}\cdot\frac{1}{b^2}\cdot\frac{1}{c^2}\right)}\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 3\frac{1}{\sqrt[3]{a^2b^2c^2}}$.

By multiplication, $(a+b+c)^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 9\left(\sqrt[3]{abc}\right)^2 3\frac{1}{\sqrt[3]{a^2b^2c^2}}=27$.

Problem HAR3_15.

If $a_1,a_2,\dots,a_n$ are positive numbers, show that

(i) $a_1^n+a_2^n+\dots+a_n^n\ge na_1a_2\dots a_n$

(ii) $\sqrt[n]{a_1a_2\dots a_n}\ge\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}$.

Solution: Use the inequality between the arithmetic mean and the geometric mean $(x_1x_2\dots x_n)^{1/n}\le\frac{x_1+x_2+\dots+x_n}{n}$, $x_1>0,\dots,x_n>0$.

(i) The substitution $x_1=a_1^n,\dots,x_n=a_n^n$ gives

$(a_1^na_2^n\dots a_n^n)^{1/n}\le\frac{a_1^n+a_2^n+\dots+a_n^n}{n}\Rightarrow a_1^n+a_2^n+\dots+a_n^n\ge na_1a_2\dots a_n$.

(ii) The substitution $x_1=a_1^{-1},\dots,x_n=a_n^{-1}$ gives

$\left(\frac{1}{a_1}\frac{1}{a_2}\dots\frac{1}{a_n}\right)^{1/n}\le\frac{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}{n}\Rightarrow(a_1a_2\dots a_n)^{1/n}\ge\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}$.

Problem HAR3_16.

Show that $x-\frac{1}{3}x^3<\tan^{-1}x<x-\frac{1}{3}x^3+\frac{1}{5}x^5$ for $x>0$.

Solution: Let us prove that for $t>0$ $\quad 1-t^2<\frac{1}{1+t^2}<1-t^2+t^4$.

First, it is easily seen that $(1-t^2)(1+t^2)=1-t^4<1$ for $t>0$, and hence $1-t^2<\frac{1}{1+t^2}$ for $t>0$. Further, it is clear that $(1-t^2+t^4)(1+t^2)-1=t^6>0$ for $t>0$.

Thus, we have $\frac{1}{1+t^2}<1-t^2+t^4$ for $t>0$.

Hence, we arrive to the desired result $1-t^2<\frac{1}{1+t^2}<1-t^2+t^4$ for $t>0$.

By integrating the last inequality between 0 and x, we derive

$\int_0^x(1-t^2)dt<\int_0^x\frac{dt}{1+t^2}<\int_0^x(1-t^2+t^4)dt$, $x-\frac{1}{3}x^3<\tan^{-1}x<x-\frac{1}{3}x^3+\frac{1}{5}x^5$ for $x>0$.

Problem HAR3_17.

Show that $xy\le e^{x-1}+y\ln y$ for all real x and all positive y. When does equality hold?

Solution: Let $f(x)=e^{x-1}+y\ln y-xy$ be the function with parameter $y>0$. It is easily to get $f'(x)=e^{x-1}-y$, $f'(x)=0\Leftrightarrow e^{x-1}=y$ or $x=1+\ln y$.

Furthermore, $f'(x)>0$ for $x>1+\ln y$ and $f'(x)<0$ for $x<1+\ln y$, as we can see that, if $x=\Delta x+1+\ln y$, then $f'(x)=y(e^{\Delta x}-1)$, and $f'(x)>0$ if $\Delta x>0$, and $f'(x)<0$ if $\Delta x<0$.

Hence $f(x)$ has an absolute minimum of 0 when $x=1+\ln y$. As a result, we get for all x $f(x)\ge 0$, and $e^{x-1}+y\ln y\ge xy$ with equality iff $x=1+\ln y$.

Problem HAR3_18.

Show that $\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx < \int_0^1 x^4(1-x)^4\,dx$ and hence deduce that

$\frac{22}{7}-\frac{1}{630} < \pi < \frac{22}{7}-\frac{1}{1260}$.

Solution:

$$\int_0^1 x^4(1-x)^4\,dx = \int_0^1 (x^4-4x^5+6x^6-4x^7+x^8)dx = \left[\frac{x^5}{5}\right]_0^1 - 4\left[\frac{x^6}{6}\right]_0^1 + 6\left[\frac{x^7}{7}\right]_0^1 - 4\left[\frac{x^8}{8}\right]_0^1 + \left[\frac{x^9}{9}\right]_0^1$$

$$=\frac{1}{5}-\frac{2}{3}+\frac{6}{7}-\frac{1}{2}+\frac{1}{9}=\frac{1}{630}.$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \frac{x^4-4x^5+6x^6-4x^7+x^8}{1+x^2}dx$$

By using the representation $x^8-4x^7+6x^6-4x^5+x^4=(1+x^2)(x^6-4x^5+5x^4-4x^2+4)-4$, we get

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 (x^6-4x^5+5x^4-4x^2+4)dx - 4\int_0^1 \frac{dx}{1+x^2}$$

$$=\left[\frac{x^7}{7}\right]_0^1 - 4\left[\frac{x^6}{6}\right]_0^1 + 5\left[\frac{x^5}{5}\right]_0^1 - 4\left[\frac{x^3}{3}\right]_0^1 + 4[x]_0^1 - 4\left[\tan^{-1}x\right]_0^1 = \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi = \frac{22}{7}-\pi.$$

It is easily seen that for $0 < x < 1$

$$\frac{1}{2} < \frac{1}{1+x^2} < 1,$$

because of $\frac{1}{1+x^2}-\frac{1}{2}=\frac{1-x^2}{2(1+x^2)}>0,\quad 1-\frac{1}{1+x^2}=\frac{x^2}{1+x^2}>0.$

Since $x^4(1-x)^4>0$, we get $\frac{1}{2}x^4(1-x)^4 < \frac{x^4(1-x)^4}{1+x^2} < x^4(1-x)^4$.

By integrating this inequality with respect to x between 0 and 1, we deduce that

$$\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx < \int_0^1 x^4(1-x)^4\,dx.$$

In view of $\int_0^1 x^4(1-x)^4\,dx = \frac{1}{630}$ and $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7}-\pi$, we obtain

$$\frac{1}{1260} < \frac{22}{7}-\pi < \frac{1}{630} \Rightarrow -\frac{1}{630} < \pi-\frac{22}{7} < -\frac{1}{1260} \Rightarrow \frac{22}{7}-\frac{1}{630} < \pi < \frac{22}{7}-\frac{1}{1260}.$$

Problem HAR3_19.

Show that $1-e^{-\pi/2} \le \int_0^{\pi/2} e^{-\sin x}dx \le \frac{\pi}{2e}(e-1)$. Hint: if $\frac{\pi}{2} \ge x \ge 0$, show that $x \ge \sin x \ge \frac{2}{\pi}x$.

Solution: Let us show that $\sin x \le x$ for $0 \le x \le \frac{\pi}{2}$. It is easily to deduce that for $f(x)=x-\sin x$, we get $f'(x)=1-\cos x \ge 0$. Hence for $x \ge 0$ $f(x)$ is a non-decreasing function with absolute minimum 0 when $x=0$.

Thus $f(x)\ge 0$ for $x \ge 0$, and $\sin x \le x$ for $0 \le x \le \frac{\pi}{2}$.

Let us show that $\sin x \ge \frac{2}{\pi}x$ for $0 \le x \le \frac{\pi}{2}$. It is not difficult to establish for $g(x)=\sin x-\frac{2}{\pi}x$ that $g'(x)=\cos x-\frac{2}{\pi}$ and $g'(x)=0$ when $x=\arccos\frac{2}{\pi}$.

Furthermore, for $0 \le x \le \frac{\pi}{2}$ function $g(x)$ has the only absolute maximum of $\sin(\arccos 2/\pi)-2/\pi \arccos 2/\pi \ge 0$ when $x=\arccos 2/\pi$, since

$g'(x)=\cos x-\frac{2}{\pi}\ge 0$ for $x<\arccos\frac{2}{\pi}$,

$g'(x)=\cos x-\frac{2}{\pi}\le 0$ for $x>\arccos\frac{2}{\pi}$.

Function $g(x)$ reaches absolute minimum of 0 when $x=0,\frac{\pi}{2}$.

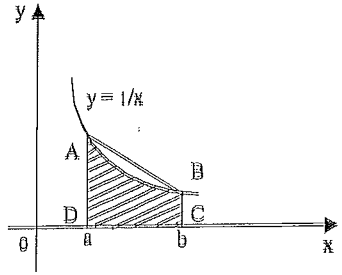
Thus $g(x)\ge 0$ for $0\le x\le\frac{\pi}{2}$, that is, $\frac{2}{\pi}x \le \sin x$, and, finally,

$\frac{2}{\pi}x \le \sin x \le x,\quad -\frac{2}{\pi}x \ge -\sin x \ge -x,\quad e^{-x2/\pi} \ge e^{-\sin x} \ge e^{-x}.$

By integrating the last inequality with respect to x between 0 and $\frac{\pi}{2}$, we come to

$$\int_0^{\pi/2} e^{-x}dx \le \int_0^{\pi/2} e^{-\sin x}dx \le \int_0^{\pi/2} e^{-x2/\pi}dx,\quad -\left[e^{-x}\right]_0^{\pi/2} \le \int_0^{\pi/2} e^{-\sin x}dx \le -\frac{\pi}{2}\left[e^{-x2/\pi}\right]_0^{\pi/2},$$

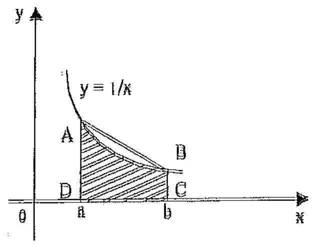
$$1-e^{-\pi/2} \le \int_0^{\pi/2} e^{-\sin x}dx \le \frac{\pi}{2e}(e-1).$$



Problem HAR3_20.

If $b > a > 0$, show that $\dfrac{\ln b - \ln a}{b - a} < \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

Solution: Shaded area < area $ABCD$,



$\int_a^b \frac{1}{x}dx < \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)(b-a)$, $\dfrac{\ln b - \ln a}{b - a} < \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

Problem HAR3_21.

If $b > a > 0$, show that $\dfrac{b-a}{\ln b - \ln a} < \dfrac{a+b}{2}$.

Solution: Rewrite the inequality to be proved in the equivalent form

$2(b-a) < (\ln b - \ln a)(a-b)$ or $\ln\left(\dfrac{b}{a}\right)\cdot(b+a) - 2(b-a) > 0$.

Let $f(t) = \ln\left(\dfrac{t}{a}\right)(t+a) - 2(t-a) \Rightarrow f(a) = 0$, $f'(t) = \ln\left(\dfrac{t}{a}\right) + \dfrac{a}{t} - 1 \Rightarrow f'(a) = 0$,

$f''(t) = \dfrac{1}{t} - \dfrac{a}{t^2} = \dfrac{1}{t}\left(1 - \dfrac{a}{t}\right) > 0$ for $t > a$.

So $f'(a) = 0$ and $f'(t)$ is an increasing function for $t > a \Rightarrow f'(t) > 0$ for all $t > a$.
But $f(a) = 0 \Rightarrow f(t) > 0$ for all $t > a$. Consequently,

$f(b) > 0$ as $b > a \Rightarrow \ln\left(\dfrac{b}{a}\right)(b+a) - 2(b-a) > 0$.

Problem HAR3_22.

Show that for $n \geq 1$, $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Solution: Let $u_n = 1^3 + 2^3 + \ldots + n^3 \Rightarrow u_{n+1} = u_n + (n+1)^3$, $u_1 = 1$, $n = 1,2,\ldots$ Hence it suffices to show that $u_n = \dfrac{n^2(n+1)^2}{4}$. Define the statement $S(n): u_n = \dfrac{n^2(n+1)}{4}$, $n = 1,2,\ldots$

Consider $S(1): u_1 = 1 = \dfrac{1^2(1+1)^2}{4} \Rightarrow S(1)$ is true. Let k be a positive integer. If $S(k)$ is true, then

$u_k = \dfrac{k^2(k+1)^2}{4}$.

Consider $S(k+1): u_{k+1} = u_k + (k+1)^3 = \dfrac{k^2(k+1)^2}{4} + (k+1)^3$, if $S(k)$ is true

$= \dfrac{(k+1)^2(k^2+4k+4)}{4} = \dfrac{(k+1)^2(k+2)^2}{4} \Rightarrow S(k+1)$ is true.

Hence for all $k \geq 1$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, hence by induction $S(n)$ is true for all $n \geq 1$, $1^3 + 2^3 + 3^3 + \ldots + n^3 = u_n = \dfrac{n^2(n+1)^2}{4}$.

Problem HAR3_23.

Show that for $n \geq 1$, $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$.

Solution: Define the statement $S(n)$: $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$, $n = 1,2,\ldots$

Consider $S(1)$: $n = 1$, $\dfrac{1}{2!} = 1 - \dfrac{1}{2!} = \dfrac{1}{2}$. Hence $S(1)$ is true.

Let k be a positive integer. If $S(k)$ is true, then $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{k}{(k+1)!} = 1 - \dfrac{1}{(k+1)!}$.

Consider $S(k+1)$. As $S(k)$ is true, we get

$\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{k}{(k+1)!} + \dfrac{k+1}{(k+2)!} = 1 - \dfrac{1}{(k+1)!} + \dfrac{k+1}{(k+2)!} =$

$= 1 - \dfrac{k+2-(k+1)}{(k+2)!} = 1 - \dfrac{1}{(k+2)!}$.

Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers n:

$\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$ for $n \geq 1$.

Problem HAR3_24.

Show that for $n \geq 1$, $1+\frac{x}{1!}+\frac{x(x+1)}{2!}+...+\frac{x(x+1)...(x+n-1)}{n!}=\frac{(x+1)(x+2)...(x+n)}{n!}$.

Solution: Define the statement $S(n)$:

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+...+\frac{x(x+1)...(x+n-1)}{n!}=\frac{(x+1)(x+2)...(x+n)}{n!},\ n=1,2,...;$$

Consider $S(1)$: $n=1$, $1+\frac{x}{1!}=1+x=\frac{x+1}{1}$. Hence $S(1)$ is true.

Let k be a positive integer. If $S(k)$ is true, then

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+...+\frac{x(x+1)...(x+k-1)}{k!}=\frac{(x+1)(x+2)...(x+k)}{k!}.$$

Consider $S(k+1)$. As $S(k)$ is true, we get

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+...+\frac{x(x+1)...(x+k-1)}{k!}+\frac{x(x+1)...(x+k)}{(k+1)!}$$

$$=\frac{(x+1)(x+2)...(x+k)}{k!}+\frac{x(x+1)...(x+k)}{(k+1)!}=\frac{(x+1)(x+2)...(x+k)(k+1+x)}{(k+1)!}.$$

Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers n:

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+...+\frac{x(x+1)...(x+n-1)}{n!}=\frac{(x+1)(x+2)...(x+n)}{n!}.$$

Problem HAR3_25.

Show that for $n \geq 1$, $2\cdot 1!+5\cdot 2!+10\cdot 3!+...+(n^2+1)n!=n(n+1)!$.

Solution: Define the statement $S(n)$: $2\cdot 1!+5\cdot 2!+10\cdot 3!+...+(n^2+1)n!=n\cdot(n+1)!$, $n=1,2,...$.

Consider $S(1)$: $n=1$, $2\cdot 1!=1\cdot 2!$. Hence $S(1)$ is true.

Let k be a positive integer. If $S(k)$ is true, then

$2\cdot 1!+5\cdot 2!+10\cdot 3!+...+(k^2+1)k!=k\cdot(k+1)!$.

Consider $S(k+1)$. As $S(k)$ is true, we get

$2\cdot 1!+5\cdot 2!+10\cdot 3!+...+(k^2+1)k!+((k+1)^2+1)(k+1)!=k(k+1)!+((k+1)^2+1)(k+1)!=$

$=(k+1)!(k+(k+1)^2+1)=(k+1)!(k+1)(k+2)=(k+2)!(k+1)$.

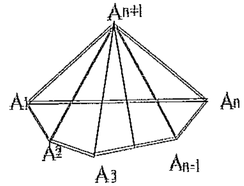
Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers n:

$2\cdot 1!+5\cdot 2!+10\cdot 3!+...+(n^2+1)n!=n\cdot(n+1)!$ for $n\geq 1$.

Problem HAR3_26.

Show that a convex polygon with $n\geq 4$ sides has $\frac{n(n-3)}{2}$ diagonals.

Solution: Let the function $f(n)$ define the quantity of diagonals for a convex polygon with $n\geq 4$ sides. It is easily seen that $f(n+1)=f(n)+n-1$ (see figure 15), since including an additional point A_{n+1} for a polygon with n sides leads to new $n-2$ diagonals with respect to the points $A_1, A_3, ..., A_{n-1}$, besides the side A_1A_n becomes a new diagonal.



Define the statement $S(n): f(n)=\frac{n(n-3)}{2}$, $n=4,5,...$.

Consider $S(4)$: $f(4)=\frac{4\cdot 1}{2}=2\Rightarrow S(1)$ true.

Let k be a positive integer, $k\geq 4$. If $S(k)$ is true, then

$f(k)=\frac{k(k-3)}{2}$.

Consider $S(k+1)$. As $S(k)$ is true, we get

$$f(k+1)=f(k)+k-1=\frac{k(k-3)}{2}+k-1=\frac{k^2-k-2}{2}=\frac{(k+1)(k-2)}{2}$$

We see that for all $k\geq 4$, $S(k)$ true implies $S(k+1)$ true. But $S(4)$ is true. Hence by Induction, $S(n)$ is true for all integers $n\geq 4$.

Problem HAR3_27.

Show that $(1+x)^n-nx-1$ is divisible by x^2 for $n\geq 2$.

Solution: Define the statement $S(n)$: $(1+x)^n-nx-1$ is divisible by x^2, $n=2,3,...$

Consider $S(2)$: $(1+x)^2-2x-1=x^2\Rightarrow S(2)$ is true.

Let k be a positive integer more that 1. If $S(k)$ is true, then $(1+x)^k-kx-1=x^2M$, where M is a polynomial in $x\Rightarrow(1+x)^k=1+kx+x^2M$.

Consider $S(k+1)$: $(1+x)^{k+1}-(k+1)^{k+1}-(k+1)x-1=(1+x)(1+x)^k-(k+1)x-1$

$=(1+x)(1+kx+x^2M)-(k+1)x-1$, if $S(k)$ is true

$=x^2M+kx^2+x^3M=x^2(m+k+xm)\Rightarrow$ $(1+x)^{k+1}-(k+1)x-1$ is divisible by x^2.

Hence for all positive integers $k\geq 2$, $S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true, hence by induction $S(n)$ is true for all positive integers $n\geq 2$: $(1+x)^n-nx-1$ is divisible by x^2.

Problem HAR3_28.

Show that for $n\geq 10, 2^n>n^3$.

Solution: Define the statement $S(n)$: $2^n>n^3, n=1,2,...$

Consider $S(10)$: $2^{10}=1024>10^3\Rightarrow S(10)$ is true.

Let k be a positive integer, $k\geq 10$. If $S(k)$ is true, then $2^k>k^3$.

Consider $S(k+1)$: $2^{k+1}=2\cdot 2^k>2\cdot k^3$, if $S(k)$ is true. But

$2k^3-(k+1)^3=k^3-3k^2-3k-1>k^3-3k^2-3k^2-k^2=k^2(k-7)>0\Rightarrow 2k^3>(k+1)^3$.

Hence $2^{k+1}>(k+1)^3\Rightarrow S(k+1)$ is true.

Hence for all positive integers $k\geq 10$, $S(k)$ true implies $S(k+1)$ true. But $S(10)$ is true, hence $S(n)$ is true for all $n\geq 10, 2^n>n^3$.

Problem HAR3_29.

Show that for $n \geq 2$, $n^n > (n+1)^{n-1}$.

Solution: Define the statement $S(n)$: $n^n > (n+1)^{n-1}$, $n = 2,3,\ldots$

Consider $S(2)$: $2^2 > (2+1) \Rightarrow S(2)$ is true. Let k be a positive integer, $k \geq 2$. If $S(k)$ is true, then $k^k > (k+1)^{k-1}$.

Consider $S(k+1)$: $(k+1)^{k+1} = \dfrac{(k+1)^{k+1}}{k^k} \cdot k^k > \dfrac{(k+1)^{k+1}}{k^k} \cdot (k+1)^{k-1}$, if $S(k)$ is true

$= \dfrac{(k+1)^{2k}}{k^k}$.

Let us show that $\dfrac{(k+1)^{2k}}{k^k} > (k+2)^k \Leftrightarrow (k+1)^{2k} > (k(k+2))^k \Leftrightarrow (k+1)^2 > k(k+2) \Leftrightarrow$ $k^2 + 2k + 1 > k^2 + 2k$.

So we have $(k+1)^{k+1} > (k+2)^k \Rightarrow S(k+1)$ is true. Hence for all $k \geq 2$, $S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true, hence by induction $S(n)$ is true for all $n \geq 2$, $n^n > (n+1)^{n-1}$.

Problem HAR3_30.

Show that for $n \geq 1$, $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$.

Solution: Let $u_n = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}} \Rightarrow u_{n+1} = u_n + \dfrac{1}{\sqrt{n+1}}$, $u_1 = 1$, $n = 1,2,\ldots$

Hence it suffices to show that $u_n > 2(\sqrt{n+1} - 1)$.

Define the statement $S(n)$: $u_n > 2(\sqrt{n+1} - 1)$, $n = 1,2,\ldots$

Consider $S(1)$: $1 > 2(\sqrt{2} - 1) \Leftrightarrow S(1)$ is true.

Let k be a positive integer. If $S(k)$ is true, then $u_k > 2(\sqrt{k+1} - 1)$.

Consider $S(k+1)$: $u_{k+1} = u_k + \dfrac{1}{\sqrt{k+1}} > 2(\sqrt{k+1} - 1) + \dfrac{1}{\sqrt{k+1}}$, if $S(k)$ is true.

But $2(\sqrt{k+1} - 1) + \dfrac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1) \Leftrightarrow 2(k+1) + 1 > 2\sqrt{(k+1)(k+2)} \Leftrightarrow$

$(2k+3)^2 > 4(k+1)(k+2) \Leftrightarrow 4k^2 + 12k + 9 > 4k^2 + 12k + 8$. So we have that $u_{k+1} > 2(\sqrt{k+2} - 1) \Rightarrow S(k+1)$ is true. Hence for all $k \geq 1$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, hence $S(n)$ is true for all $n \geq 1$, $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$.

Problem HAR3_31.

Show that for $n \geq 1$, $\dfrac{1}{(1!)^2} + \dfrac{1}{(2!)^2} + \ldots + \dfrac{1}{(n!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4^n}\right)$.

Solution: Show that for $n \geq 1$, $n! \geq 2^{n-1}$.

Define the statement $S(n)$: $n! \geq 2^{n-1}$, $n = 1,2,\ldots$.

Consider $S(1)$: $1! \geq 2^0 = 1$. Hence $S(1)$ is true.

Let k be a positive integer. If $S(k)$ is true, then $k! \geq 2^{k-1}$.

Consider $S(k+1)$. As $S(k)$ is true, we get

$(k+1)! = k!(k+1) \geq 2^{k-1}(k+1) = 2^k \cdot \dfrac{k+1}{2} \geq 2^k$, since $\dfrac{(k+1)}{2} \geq 1$ for $k \geq 1$.

Hence $(k+1)! \geq 2^k$, $k \geq 1$.

We see that for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n \geq 1$: $n! \geq 2^{n-1}$.

Define the statement $S(n)$: $\dfrac{1}{(1!)^2} + \dfrac{1}{(2!)^2} + \ldots + \dfrac{1}{(n!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4^n}\right)$, $n = 1,2,\ldots$.

Consider $S(1)$: $\dfrac{1}{(1!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4}\right) = 1$. Hence $S(1)$ is true.

Let k be a positive integer. If $S(k)$ is true, then

$\dfrac{1}{(1!)^2} + \dfrac{1}{(2!)^2} + \ldots + \dfrac{1}{(k!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4^k}\right)$.

Consider $S(k+1)$. As $S(k)$ is true, we get

$\dfrac{1}{(1!)^2} + \dfrac{1}{(2!)^2} + \ldots + \dfrac{1}{(k!)^2} + \dfrac{1}{((k+1)!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4^k}\right) + \dfrac{1}{((k+1)!)^2}$.

Since $k! \geq 2^{k-1}$ for $k \geq 1$ we get $\dfrac{1}{((k+1)!)^2} \leq \dfrac{1}{4^k}$. By using this inequality, we come to

$\dfrac{1}{(1!)^2} + \dfrac{1}{(2!)^2} + \ldots + \dfrac{1}{(k!)^2} + \dfrac{1}{((k+1)!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4^k}\right) + \dfrac{1}{4^k} = \dfrac{4}{3}\left(1 - \dfrac{1}{4^{k+1}}\right)$.

Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers n:

$\dfrac{1}{(1!)^2} + \dfrac{1}{(2!)^2} + \ldots + \dfrac{1}{(n!)^2} \leq \dfrac{4}{3}\left(1 - \dfrac{1}{4^n}\right)$.

Problem HAR3_32.

Show that $35^n + 3 \cdot 7^n + 2 \cdot 5^n + 6$ is divisible by 12 for $n \geq 0$.

Solution: Let $u_n = 35^n + 3 \cdot 7^n + 2 \cdot 5^n + 6$, then

$u_{n+1} = 35 \cdot 35^n + 21 \cdot 7^n + 10 \cdot 5^n + 6 = (36 - 1) \cdot 35^n + (24 - 3) \cdot 7^n + (12 - 2) \cdot 5^n + (12 - 6)$

$= 36 \cdot 35^n + 24 \cdot 7^n + 12 \cdot 5^n + 12 - u_n = 12(3 \cdot 35^n + 2 \cdot 7^n + 5^n + 1) - u_n$

Define the statement $S(n)$: u_n is divisible by 12, $n = 1,2,\ldots$

Consider $S(0)$: $u_0 = 12$ is divisible by $12 \Rightarrow S(0)$ is true.

Let k be an integer, $k \geq 0$. If $S(k)$ is true, then $u_k = 12 \cdot M$ for some integer M.

Consider $S(k+1)$: $u_{k+1} = 12(3 \cdot 35^k + 2 \cdot 7^k + 5^k + 1) - u_k =$ $12(3 \cdot 35^k + 2 \cdot 7^k + 5^k + 1) - 12M$, if $S(k)$ is true.

u_{k+1} is divisible by $12 \Rightarrow S(k+1)$ is true. Hence for all integer $k \geq 0$, $S(k)$ true implies $S(k+1)$ true. But $S(0)$ is true, hence by induction $S(n)$ is true for all integers $n \geq 0$: $35^n + 3 \cdot 7^n + 2 \cdot 5^n + 6$ is divisible by 12.

Problem HAR3_33.

Show that $7^n + 15^n$ is divisible by 11 for odd $n \geq 1$.

Solution: Let us introduce $f(n)=7^n+15^n$. It is easily seen that

$f(n+2)=7^{n+2}+15^{n+2}=7^2(7^n+15^n)-49\cdot 15^n+225\cdot 15^n=49f(n)+176\cdot 15^n$

Define the statement $S(n)$: $f(n)$ is divisible by 11, $n=1,3,5,\ldots$.

Consider $S(1)$: $f(1)=7+15=22=11\cdot 2\Rightarrow S(1)$ is true, since $f(1)$ is divisible by 11.

Let k be a positive odd integer. If $S(k)$ is true, then $f(k)=11\cdot M$ for some integer M.

Consider $S(k+2)$ ($k+2$ is the next odd integer). As $S(k)$ is true, we get

$f(k+2)=49f(k)+15^k\cdot 176=49\cdot 11M+15^k\cdot 16\cdot 11=11(49\cdot M+15^k\cdot 16)$.

Since $49M+16\cdot 15^k$ is integer, we see that $f(k+2)$ is divisible by 11.

Hence for all odd positive integers k, $S(k)$ true implies $S(k+2)$ is true. But $S(1)$ is true.

Therefore by induction, $S(n)$ is true for all odd positive integers n: 7^n+15^n is divisible by 11 for odd $n\ge 1$.

Problem HAR3_34.

The equation $x^2-x+1=0$ has roots α and β, and $A_n=\alpha^n+\beta^n$ for $n\ge 1$.

(i) Without solving the equation, show that $A_1=1$, $A_2=-1$ and $A_n=A_{n-1}-A_{n-2}$ for $n\ge 3$.

(ii) Hence show by induction that for $n\ge 1$, $A_n=2\cos\frac{n\pi}{3}$.

Solution: (i) If α,β are roots of $x^2-x+1=0$, then $\alpha+\beta=1$ and $\alpha\beta=1$ by *Vieta's* theorem.

Hence $(\alpha+\beta)^2=1\Rightarrow\ \alpha^2+\beta^2=1-2\alpha\beta\Rightarrow\alpha^2+\beta^2=-1$, $A_n=\alpha^n+\beta^n=\alpha^{n-1}\cdot\alpha+\beta^{n-1}\cdot\beta$,

$\alpha+\beta=1\Rightarrow\alpha=1-\beta$ and $\beta=1-\alpha$. Hence $A_n=\alpha^{n-1}(1-\beta)+\beta^{n-1}(1-\alpha)\Rightarrow$

$A_n=\alpha^{n-1}+\beta^{n-1}-\alpha^{n-1}\beta-\beta^{n-1}\alpha\Rightarrow\ A_n=A_{n-1}-\alpha\beta(\alpha^{n-2}+\beta^{n-2})$; $\alpha\beta=1\Rightarrow$

$A_n=A_{n-1}-A_{n-2}$ for $n\ge 3$.

(ii) Define the statement $S(n)$: $A_n=2\cos\frac{n\pi}{3}$, $n=1,2,\ldots$.

Consider $S(1)$: $n=1$, $A_1=2\cos\frac{\pi}{3}=1\Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2$, $A_2=2\cos\frac{2\pi}{3}=-1\Rightarrow S(2)$ is true.

Let k be a positive integer, $k\ge 2$. If $S(n)$ is true for all integer $n\le k$, then

$A_n=2\cos\frac{n\pi}{3}$, $n=1,2,3,\ldots k$.

Consider $S(k+1)$. As $S(n)$ is true for $n=1,2,3,\ldots,k$, we get

$$A_{k+1}=A_k-A_{k-1}=\ 2\cos\frac{k\pi}{3}-2\cos\frac{(k-1)\pi}{3}=2\left(\cos\frac{k\pi}{3}-\cos\frac{(k-1)\pi}{3}\right).$$

But $\cos a-\cos b=2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$, $a,b\in\mathbb{R}$.

Hence $A_{k+1}=4\sin\left(\frac{(2k-1)}{2}\cdot\frac{\pi}{3}\right)\sin\left(-\frac{1}{2}\cdot\frac{\pi}{3}\right)\Rightarrow\ A_{k+1}=-4\sin\frac{\pi}{6}\sin\left(\left(k-\frac{1}{2}\right)\frac{\pi}{3}\right)\Rightarrow$

$A_{k+1}=-2\sin\left((k+1)\frac{\pi}{3}-\frac{\pi}{2}\right)\Rightarrow\ A_{k+1}=2\cos(k+1)\frac{\pi}{3}$.

Hence for $k\ge 2$, $S(n)$ true for all positive integers $n\le k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers n:

$A_n=2\cos\frac{n\pi}{3}$ for $n\ge 1$.

Problem HAR3_35.

If $u_1=1$, $u_2=1$ and $u_n=u_{n-1}+u_{n-2}$ for $n\ge 3$, show that for

$$n\ge 1\quad u_n=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right\}.$$

Solution: Define the statement $S(n)$: $u_n=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right\}$, $n=1,2,\ldots$.

Consider $S(1)$: $u_1=\frac{1}{\sqrt5}\left\{\frac{1+\sqrt5}{2}-\frac{1-\sqrt5}{2}\right\}=1\Rightarrow S(1)$ is true.

Consider $S(2)$: $u_2=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^2-\left(\frac{1-\sqrt5}{2}\right)^2\right\}=\frac{1}{\sqrt5}\left\{\frac{4\sqrt5}{4}\right\}=1\Rightarrow S(2)$ is true.

Let k be a positive integer, $k\ge 2$. If $S(n)$ is true for all integers $n\le k$, then

$u_n=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right\}$ for $n=1,2,\ldots,k$.

Consider $S(k+1)$:

$$u_{k+1}=u_k+u_{k-1}=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^k-\left(\frac{1-\sqrt5}{2}\right)^k\right\}+\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^{k-1}-\left(\frac{1-\sqrt5}{2}\right)^{k-1}\right\}$$

as $S(k-1)$, $S(k)$ are true

$$=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^{k-1}\cdot\left(\frac{1+\sqrt5}{2}+1\right)-\left(\frac{1-\sqrt5}{2}\right)^{k-1}\cdot\left(\frac{1-\sqrt5}{2}+1\right)\right\}$$

$$=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^{k-1}\cdot\left(\frac{1+\sqrt5}{2}\right)^2-\left(\frac{1-\sqrt5}{2}\right)^{k-1}\cdot\left(\frac{1-\sqrt5}{2}\right)^2\right\}=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^{k+1}-\left(\frac{1-\sqrt5}{2}\right)^{k+1}\right\},$$

Hence for all integers $k\ge 1$ $S(n)$ true for all positive integers $n\le k$ For $k=2,3,\ldots$, $S(n)$ is true for all positive integers $n\le k$ implies $S(k+1)$ true. But $S(1)$, $S(2)$ are true. Hence by induction, $S(n)$ is true for all positive integers n:

$$u_n=\frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right\}.$$

Problem HAR3_36.

If $u_1 = 1$ and $u_n = \dfrac{2u_{n-1}^3 + 27}{3u_{n-1}^2}$ for $n \geq 2$, show that

(a) $u_n \geqslant 3$ for $n \geq 2$;

(b) $u_{n+1} \leqslant u_n$ for $n \geq 2$.

Solution: (a) Define the statement $S(n): u_n \geqslant 3,\ n = 2,3,\ldots$.

Consider $S(2)$: $u_2 = \dfrac{2+27}{3\cdot 1} = \dfrac{29}{3} \geqslant 3 \Rightarrow S(2)$ is true.

Let k be a positive integer $k \geq 2$. If $S(k)$ is true then $u_k \geqslant 3$.

Consider $S(k+1)$. Show that $u_{k+1} \geqslant 3$. To this end, let the function $f(x)$ be given by

$f(x) = \dfrac{2}{3}x + \dfrac{9}{x^2},\ x \geq 3$.

It is easily seen that

$f'(x) = \dfrac{2}{3} - \dfrac{18}{x^3}$.

We obtain that $f'(x) = 0$ when $x = 3$ and $f'(x) \geqslant 0$ for $x \geqslant 3$. Thus the function $f(x)$ has an absolute minimum of $f(3) = 3$ on the set $x \geq 3$. Hence

$f(x) = \dfrac{2}{3}x + \dfrac{9}{x^2} \geqslant 3$

Thus, as $S(k)$ is true ($u_k \geqslant 3,\quad k \geq 2$), using this inequality, we get

$S(k+1)$: $u_{k+1} = \dfrac{2u_k^3 + 27}{3u_k^2} = \dfrac{2}{3}u_k + \dfrac{9}{u_k^2} \geqslant 3$.

Hence for all positive integers $k \geq 2$, $S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n \geq 2$: $u_n \geqslant 3$.

(b) Show that $u_{n+1} \leqslant u_n$ for $n \geq 2$.

One can deduce that $u_n - u_{n+1} = \dfrac{1}{3}u_n - \dfrac{9}{u_n^2}$, since $u_{n+1} = \dfrac{2}{3}u_n + \dfrac{9}{u_n^2}$.

Let the function $g(x)$ be given by $g(x) = \dfrac{1}{3}x - \dfrac{9}{x^2},\quad x \geq 3$.

It is easy to see that $g'(x) = \dfrac{1}{3} + \dfrac{18}{x^3} \geqslant 0$ for $x \geq 3$. Hence $g(x)$ is a monotonically increasing function for $x \geq 3$ and $g(3) = 0$. Thus $g(x) \geqslant 0$ for $x \geqslant 3$, and $\dfrac{1}{3}x - \dfrac{9}{x^2} \geqslant 0,\quad x \geqslant 3$.

By using this inequality and the fact proved in (a) that $u_n \geqslant 3, n \geq 2$, we get $\dfrac{1}{3}u_n - \dfrac{9}{u_n^2} \geqslant 0$.

Hence $u_{n+1} \leqslant u_n, n \geq 2$.

Problem HAR3_37.

If $u_1 = 1$ and $u_n = \dfrac{1}{2}\left(u_{n-1} + \dfrac{3}{u_{n-1}}\right)$ for $n \geq 2$, show that

(a) $u_n^2 \geqslant 3$ for $n \geq 2$;

(b) $u_{n+1} \leqslant u_n$ for $n \geq 2$.

Solution: (a) Define the statement $S(n): u_n^2 = \dfrac{1}{4}\left(u_{n-1}^2 + 6 + \dfrac{9}{u_{n-1}^2}\right) \geqslant 3,\ n = 2,3,\ldots$.

Consider $S(2)$: $u_2^2 = \dfrac{1}{4}(1+6+9) = 4 \geqslant 3 \Rightarrow S(2)$ is true.

Let k be a positive integer $k \geq 2$. If $S(k)$ is true, then $u_k^2 \geqslant 3$ i.e.,

$u_k^2 = \dfrac{1}{4}\left(u_{k-1}^2 + 6 + \dfrac{9}{u_{k-1}^2}\right) \geqslant 3$.

Consider $S(k+1)$. Show that $u_{k+1}^2 \geqslant 3$, that is,

$u_{k+1}^2 = \dfrac{1}{4}\left(u_k^2 + 6 + \dfrac{9}{u_k^2}\right) \geqslant 3$.

Show that $S(k+1)$ is true. Let the function $f(x)$ be given by $f(x) = \dfrac{1}{4}\left(x + 6 + \dfrac{9}{x}\right),\quad x \geq 3$.

It is easy to see that $f'(x) = \dfrac{1}{4}\left(1 - \dfrac{9}{x^2}\right)$

$\Rightarrow f'(x) = 0$ when $x = 3$ and $f'(x) \geqslant 0$ for $x \geqslant 3$.

Hence the function $f(x)$ has an absolute minimum of 3 when $x = 3$, and thus

$\dfrac{1}{4}\left(x + 6 + \dfrac{9}{x}\right) \geqslant 3$ for $x \geqslant 3$.

By using this inequality, in view of $u_k^2 \geqslant 3$, we get $u_{k+1}^2 = \dfrac{1}{4}\left(u_k^2 + 6 + \dfrac{9}{u_k^2}\right) \geqslant 3$.

Hence for all positive integers true implies $S(k+1)$ true. But $S(2)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n \geq 2$: $u_n^2 \geqslant 3$.

(b) Show that $u_{n+1} \leqslant u_n$ for $n \geq 2$.

As $u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{3}{u_n}\right)$, we have that $u_n - u_{n+1} = \dfrac{1}{2}u_n - \dfrac{3}{2u_n}$ and $u_n \geqslant 0$ for $n \geq 2$.

Let the function $g(x)$ be given by $g(x) = \dfrac{1}{2}x - \dfrac{3}{2x},\quad x \geqslant 0$.

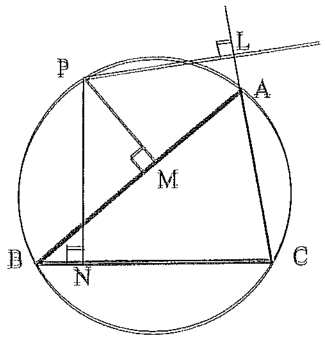
It is easy to see that $g'(x) = \dfrac{1}{2} + \dfrac{3}{2x^2} \geqslant 0$ for all $x \geqslant 0$.

Hence $g(x)$ is a monotonically increasing function and $g(x) = 0$ when $x = \sqrt{3}$. Thus $g(x) \geqslant 0$ for $x \geqslant \sqrt{3}$, and $\dfrac{1}{2}x - \dfrac{3}{2x} \geqslant 0$ for $x \geqslant \sqrt{3}$.

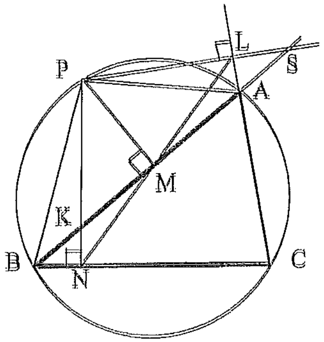
By using this inequality and the fact that $u_n^2 \geqslant 3$ or $u_n \geqslant \sqrt{3}$, proved in (a), we get $\dfrac{1}{2}u_n - \dfrac{3}{2u_n} \geqslant 0$ for $n \geq 2$. Hence $u_{n+1} \leqslant u_n$ for $n \geq 2$.

Problem HAR3_38.

ABC is a triangle inscribed in the a circle. *P* is a point on the minor *arc AB*. The points *L*, *M*, and *N* are the feet of the perpendiculars from *P* to *CA* produced, *AB*, and *BC* respectively. Show that *L*, *M* and *N* are collinear. (The line *NL* is called the Simpson line.)



Solution:

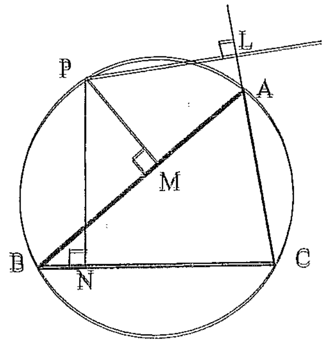


In order to prove that *L*, *M* and *N* are collinear, it is sufficient to show that $\angle LMA = \angle NMB$.
For this purpose we show, that $\angle NMB = \angle BPN = \angle SPA = \angle LMA$.
The first step: $\angle NMB = \angle BPN$. The triangles *PKM* and *BKN* are rectangular and $\angle PKM = \angle BKN \Rightarrow \Delta PKM$ are similar $\Delta BKN \Rightarrow \frac{BK}{PK} = \frac{NK}{MK}$. But $\angle PKB = \angle MKN \Rightarrow \Delta PKB$ are similar $\Delta MKN \Rightarrow \angle NMB = \angle BPN$.

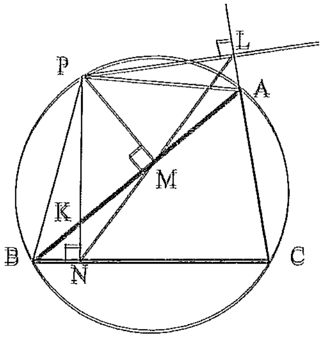
The second step: $\angle BPN = \angle SPA$. The point *P* lies on the circle $\Rightarrow$ *PACB* is a cyclic quadrilateral $\Rightarrow \angle PAC + \angle PBC = 180°$. But $\angle PAC + \angle PAL = 180°$. Hence $\angle PBC = \angle PAL$.
From here, as the triangles *PNB* and *PLA* are rectangular, we have ΔPNB are similar $\Delta PLA \Rightarrow \angle BPN = \angle APL$.
The third step: $\angle SPA = \angle LMA$. It is obvious that ΔALS is similar ΔPMS, as these rectangular triangles have the common angle $\angle PSM$. Hence $\frac{PS}{AS} = \frac{MS}{LS} \Rightarrow \Delta MLS$ is similar $\Delta PAS \Rightarrow \angle SPA = \angle LMA$.

Problem HAR3_39.

ΔABC is inscribed in a circle. *P* is a point on a minor *arc AB*. The points *M*, *L* and *N* are the feet of the perpendiculars from *P* to *AB* produced, *AC* and *BC* respectively.
Show that $\Delta PNL \parallel \Delta PBA$. Hint: use the fact that the points *N*, *M*, and *L* are collinear.



Solution:



We use the known fact that the points N, M and L are collinear (the line NL is the Simpson line, see problem 39). It suffices to show that $\angle PNL = \angle PBA$ and $\angle NPL = \angle BPA$.

The first step: $\angle PNL = \angle PBA$. The rectangular triangles KMP and KNB are similar as

$\angle PKM = \angle BKN \Rightarrow \dfrac{KN}{KM} = \dfrac{KB}{KP}$. But $\angle NKM = \angle BKP \Rightarrow \Delta NKM \; \Delta BKP \Rightarrow \angle PNL = \angle PBA$.

The second step: $\angle NPL = \angle BPA$. The sum of any quadrilateral is 360^0. But the quadrilateral $PNCA$ has two right angles $\Rightarrow \angle NPL + \angle BCA = 180^0$. The quadrilateral $PBCA$ is a cyclic one $\Rightarrow \angle BPA + \angle BCA = 180^0$ (as opposite angles). Hence $\angle NPL + \angle BCA = \angle BPA + \angle BCA \Rightarrow$ $\angle NPL = \angle BPA$.

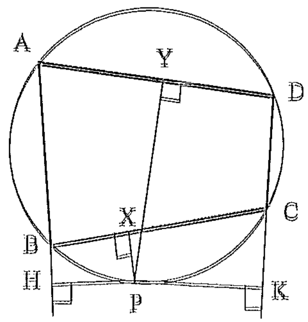
Problem HAR3_40.

$ABCD$ is a cyclic quadrilateral. P is a point on the circle through ABC and D. PH, PX, PK and PY are the perpendiculars from P to AB produced, BC, DC produced and DA, respectively.

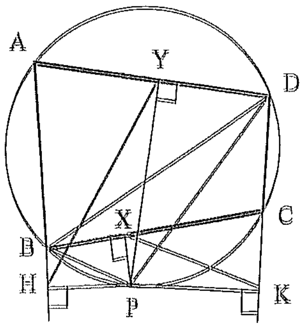
(i) Show that $\Delta XPK \; ||| \; \Delta HPY$;

(ii) Hence show that $PX \cdot PY = PH \cdot PK$ and $\dfrac{PX \cdot PK}{PH \cdot PY} = \dfrac{(XK)^2}{(HY)^2}$.

Hint: Use the result of the problem 39.



Solution:



(i) The triangle ABD and CBD are inscribed in a circle. Hence according to the result of the problem 39 ΔHPY is similar to ΔBPD and ΔBPD is similar to ΔXPK.

Therefore $\Delta XPK \; ||| \; \Delta HPY$.

(ii) $\Delta\, XPK$ is similar to $\Delta\, HPY$, hence $\dfrac{PX}{PH} = \dfrac{PK}{PY} \Rightarrow PX \cdot PY = PH \cdot PK$. Also $\dfrac{PX}{PH} = \dfrac{XK}{HY}$ and $\dfrac{PK}{PY} = \dfrac{XK}{HY}$, multiplying these equalities, $\dfrac{PX}{PH} \cdot \dfrac{PK}{PY} = \dfrac{(XK)^2}{(HY)^2}$.