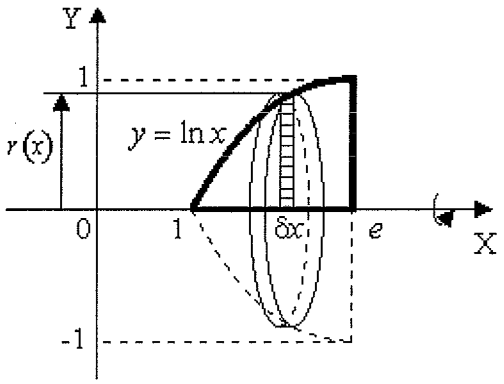
Topic 8. Volumes.
Level 3.
Problem VOL3_01.
By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x, y) : 0 \le x \le e,\ 0 \le y \le \ln x\}$ about the x-axis.

Answer: $\pi(e-2)$.

Solution:



A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x) = \ln x$. The slice has volume

$\delta V = \pi r^2(x)\delta x = \pi \ln^2 x\,\delta x$.

$$\therefore\quad V = \lim_{\delta x \to 0} \sum_{x=1}^{e} \pi \ln^2 x\,\delta x = \int_1^e \pi \ln^2 x\,dx$$

$$= \pi x \ln^2 x\Big|_1^e - \pi \int_1^e \left(2\ln x \cdot \frac{1}{x}\right) x\,dx$$

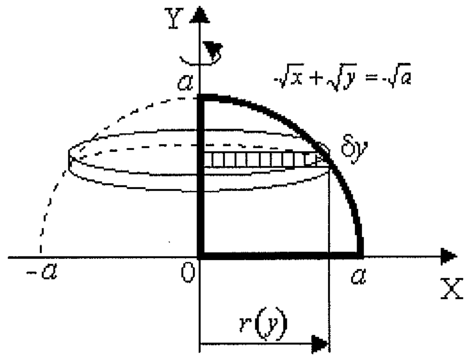
$$= \pi e - 2\pi \int_1^e \ln x\,dx = \pi e - 2\pi\left(x\ln x\Big|_1^e - \int_1^e x \cdot \frac{1}{x}\,dx \right) = -\pi e + 2\pi x\Big|_1^e = \pi(e-2).$$

$\therefore$ the volume of the solid is $\pi(e-2)$ cubic units.

Problem VOL3_02.
By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x, y) : 0 \le x \le 2,\ 0 \le y \le x^3\}$ about the line $y = 8$.

Answer: $\frac{320}{7}\pi$.

Solution:



A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y)$.

Deduce the equation of $r(y)$: $\sqrt{r}+\sqrt{y}=\sqrt{a} \quad \Rightarrow \quad r=\left(\sqrt{a}-\sqrt{y}\right)^2$.

The slice has volume $\delta V=\pi r^2(y)\delta y=\pi\left(\sqrt{a}-\sqrt{y}\right)^4\delta y$.

$\therefore \quad V=\lim\limits_{\delta y\to 0}\sum\limits_{y=0}^{a}\pi\left(\sqrt{a}-\sqrt{y}\right)^4\delta y \ =\pi\int\limits_0^a\left(\sqrt{a}-\sqrt{y}\right)^4 dy$.

Substitution $y=a(z+1)^2$, $dy=2a(z+1)\,dz$ yields

$$V=2\pi a\int\limits_{-1}^{0}\left[\sqrt{a}-\sqrt{a}(z+1)\right]^4(z+1)\,dz=2\pi a^3\int\limits_{-1}^{0}z^4(z+1)\,dz=2\pi a^3\left(\frac{z^6}{6}+\frac{z^5}{5}\right)\Bigg|_{-1}^{0}$$

$=\dfrac{\pi a^3}{15}$.

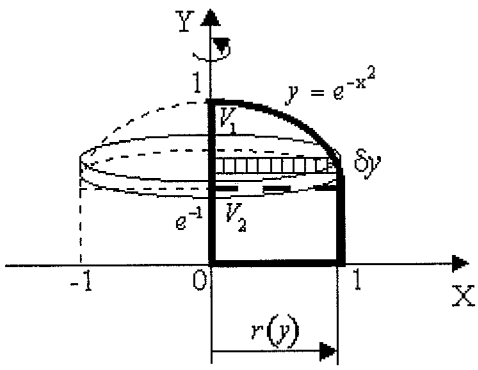
$\therefore$ the volume of the solid is $\dfrac{\pi a^3}{15}$ cubic units.

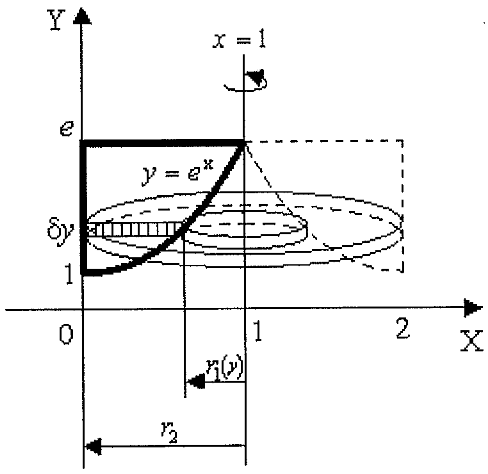
Problem VOL3_08.

By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x,y):0\le x\le 1,\ 0\le y\le e^{-x^2}\}$ about the y-axis.

Answer: $\pi\left(1-e^{-1}\right)$.

Solution:





A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)=1-\ln y$ and $r_2=1$. The slice has volume $\delta V=\pi\left(r_2^2-r_1^2\right)\delta y=\pi\left[1-(1-\ln y)^2\right]\delta y$.

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=1}^{e}\pi\left[1-(1-\ln y)^2\right]\delta y=\pi\int_1^e\left[1-(1-\ln y)^2\right]dy=\pi\int_1^e\left(2\ln y-\ln^2 y\right)dy$$

$$=\pi\left[\left(2\ln y-\ln^2 y\right)y\Big|_1^e-\int_1^e\left(\frac{2}{y}-\frac{2\ln y}{y}\right)y\,dy\right]$$

$$=\pi\left[e-2\int_1^e(1-\ln y)\,dy\right]=\pi\left[e-2y\Big|_1^e+2\int_1^e\ln y\,dy\right]=\left[2-e+2\left(y\ln y\Big|_1^e-\int_1^e\frac{1}{y}\cdot y\,dy\right)\right]$$

$$=\pi\left[2+e-2\int_1^e dy\right]=\pi\left(2+e-2y\Big|_1^e\right)=\pi(4-e).$$

$\therefore$ the volume of the solid is $\pi(4-e)$ cubic units.

Problem VOL3_07.

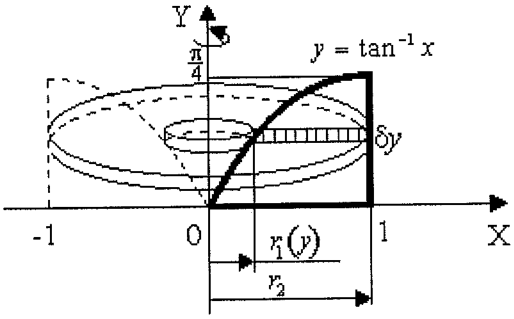
By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region enclosed by the curve $x^{1/2}+y^{1/2}=a^{1/2}$ and the coordinate axes about the y-axis.

Answer: $\dfrac{\pi a^3}{15}$.

Solution:

Answer: $\frac{\pi}{2}(\pi-2)$.

Solution:



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y) = \tan y$ and $r_2 = 1$. The slice has volume $\delta V = \pi\left(r_2^2 - r_1^2\right)\delta y = \pi\left[1-\tan^2 y\right]\delta y$.

$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{\pi/4} \pi\left(1-\tan^2 y\right)\delta y = \pi \int_0^{\pi/4} \left(1-\tan^2 y\right) dy$$

$$= \pi \int_0^{\pi/4} dy - \pi \int_0^{\pi/4} \frac{\sin^2 y}{\cos^2 y}\, dy = \pi\, y\Big|_0^{\pi/4} - \pi \int_0^{\pi/4} \sin y\, d\left(\frac{1}{\cos y}\right)$$

$$= \frac{\pi^2}{4} - \pi\left(\frac{\sin y}{\cos y}\Bigg|_0^{\pi/4} - \int_0^{\pi/4} \frac{1}{\cos y}\, d\sin y\right) = \frac{\pi^2}{4} - \pi + \pi \int_0^{\pi/4} dy = \frac{\pi}{2}(\pi-2).$$

$\therefore$ the volume of the solid is $\frac{\pi}{2}(\pi-2)$ cubic units.

Problem VOL3_06.

By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x,y): 0 \le x \le 1,\ e^x \le y \le e\}$ about the line $x = 1$.

Answer: $\pi(4-e)$.

Solution:

A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)=\sqrt{4-y}$ and $r_2=2$. The slice has volume

$$\delta V=\pi\left(r_2^2-r_1^2\right)\delta y=\left[2^2-\left(\sqrt{4-y}\right)^2\right]\delta y=\pi y\delta y.$$

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{4}\pi y\delta y=\int_0^4\pi y dy=\pi\frac{y^2}{2}\bigg|_0^4=8\pi.$$

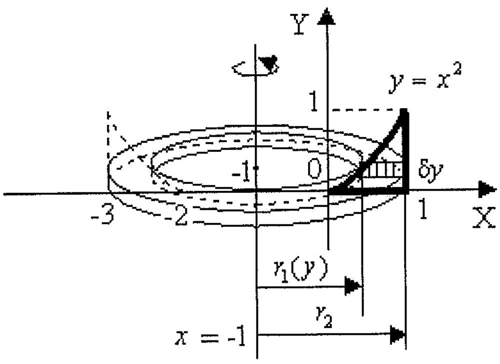
$\therefore$ the volume of the solid is 8π cubic units.

Problem VOL3_04.

By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x,y):0\le x\le 1,\ 0\le y\le x^2\}$ about the $x=-1$.

Answer: $\frac{13}{6}\pi$.

Solution:



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy and radii $r_2=2$ and $r_1=x+1$. But $y=x^2$, hence $r_1(y)=\sqrt{y}+1$. The slice has volume

$$\delta V=\pi(r_2^2-r_1^2)\delta y=\pi[4-(\sqrt{y}+1)^2]\delta y=\pi(3-2\sqrt{y}+y)\delta y.$$

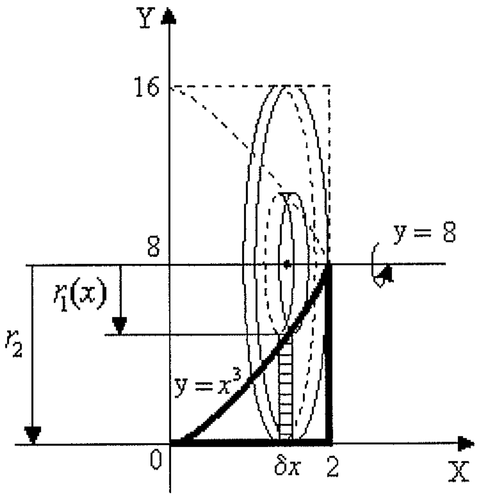
$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{1}\pi\left(3-2\sqrt{y}+y\right)\delta y=\int_0^1\pi\left(3-2\sqrt{y}+y\right)dy$$

$$=\pi\left(3y-\frac{4}{3}y^{3/2}+\frac{y^2}{2}\right)\bigg|_0^1=\pi\left(3-\frac{4}{3}+\frac{1}{2}\right)=\frac{13}{6}\pi.$$

$\therefore$ the volume of the solid is $\frac{13}{6}\pi$ cubic units.

Problem VOL3_05.

By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x,y):0\le x\le 1,\ 0\le y\le \tan^{-1}x\}$ about the y-axis.



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy and radii $r_1(x) = 8 - x^3$ and $r_2 = 8$. The slice has volume $\delta V = \pi(r_2^2 - r_1^2)\delta x = \pi[64 - (8 - x^3)^2]\delta x$ $= \pi\left(16x^3 - x^6\right)\delta x$.

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{2} \pi\left(16x^3 - x^6\right)^2 \delta x = \int_0^2 \pi\left(16x^3 - 6x^6\right)dx$$

$$= \pi\left(4x^4 - \frac{x^7}{7}\right)\Bigg|_0^2 = \pi\left(2^6 - \frac{2}{7}\cdot 2^6\right) = \frac{320}{7}\pi .$$

$\therefore$ the volume of the solid is $\dfrac{320}{7}\pi$ cubic units.

Problem VOL3_03.
By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x, y) : 0 \le x \le 2,\ 4 - x^2 \le y \le 4\}$ about the y-axis.

Answer: 8π.

Solution: