Ext.2 – Topic 13 – Integration III

28. Find $\int x^2 \cos x . dx$

29. Find $\int x \sec x \tan x . dx$

30. Find $\int \frac{1}{\cos^3 x} . dx$

31. Find $\int \ln(x^2 - 1).dx$

32. Find $\int \frac{\ln x}{x^n}.dx$

33. Find $\int_0^{\frac{\pi}{2}} x.\sin x \cos x.dx$

34. If $I_n = \int \sin^n x.dx$ for $n \geq 0$, show that:

$$I_n = -\frac{1}{n}\cos x.\sin^{n-1} x + \frac{n-1}{n}.I_{n-2} \text{ where } n \geq 2$$

Hence show that: $I_n = \int_0^{\frac{\pi}{2}} \sin^n x.dx = \frac{n-1}{n} I_{n-2}$

and that: $I_5 \times I_6 = \frac{\pi}{12}$

35. If $I_n = \int \sec^n x.dx$ for $n \geq 0$, show that: $I_n = \frac{1}{n-1}\tan x.\sec^{n-2} x + \frac{n-2}{n-1}.I_{n-2}$ for $n \geq 2$	36. If $I_n = \int_0^1 x(1-x^3)^n.dx$ for $n \geq 0$, show that: $I_n = \frac{3n}{2+3n}.I_{n-1}$ *for* $n \geq 1$ hence show: $I_n = \frac{3^n n!}{(3n+2)(3n-1).....8.5.2}$

37. If $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\theta}{\sin\theta}.dx$ show that:

$I_n - I_{n-1} = 0 \quad for \ n \geq 1$

– hence find the value of $I_n \quad for \ n \geq 1$:

38. Show that $\int_0^a f(x).dx = \int_0^a f(a-x).dx$ & hence

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}.dx = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}.dx$$

& hence evaluate the integral.

39. Show that $\int_{-a}^{a} f(x).dx = \int_{0}^{a} \{f(x) + f(-x)\}.dx$

Hence show: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}.dx = \int_{0}^{\frac{\pi}{4}} \frac{2}{\cos^2 x}.dx$

& hence evaluate the integral.

40. Show that $\int_{0}^{a} f(x).dx = \int_{0}^{a} f(a-x).dx$

& hence show that: $\int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}.dx = \frac{\pi}{4}$

[SOLUTIONS]

Ext.2 – Topic 13 – Integration III

28. Find $\int \overset{D}{x^2} \overset{I}{\cos x}.dx$

$$I = (\sin x).x^2 - \int \overset{I}{(\sin x)}.\overset{D}{2x}.dx$$

$$= x^2 \sin x - \left\{ (-\cos x)2x - \int (-\cos x).2\,dx \right\}$$

$$= x^2 \sin x + 2x \cos x - 2\int \cos x\,dx$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + c$$

29. Find $\int \overset{D}{x} \underbrace{\sec x \tan x}_{}^{I}.dx$

$$I = (\sec x).x - \int (\sec x).1\,dx$$

$$= x\sec x - \ln|\sec x + \tan x| + c$$

30. Find $\int \frac{1}{\cos^3 x}.dx$

$$I = \int \overset{I}{\sec^2 x}.\overset{D}{\sec x}\,dx$$

$$= (\tan x)\sec x - \int (\tan x).\sec x.\tan x.dx$$

$$= \tan x \sec x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \tan x \sec x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore I = \tan x \sec x - I + \ln|\sec x + \tan x| + c$$

$$\therefore 2I = \tan x \sec x + \ln|\sec x + \tan x| + c$$

$$\therefore I = \frac{1}{2}\tan x.\sec x + \frac{1}{2}\ln|\sec x + \tan x| + c$$

31. Find $\int \ln(x^2-1).dx$

$$I = \int 1.(x^2-1)\,dx$$

$$= x\ln(x^2-1) - \int x.\frac{2x}{x^2-1}\,dx$$

$$= x\ln(x^2-1) - \int \frac{2x^2-2}{x^2-1} + \frac{2}{x^2-1}\,dx$$

$$= x\ln(x^2-1) - \int 2\,dx - \int \frac{2}{x^2-1}\,dx$$

$$= x\ln(x^2-1) - 2x - \ln\left|\frac{x-1}{x+1}\right| + c$$

$$\frac{2}{x^2-1} \equiv \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 \equiv Ax + A + Bx - B$$

$$\therefore \left.\begin{aligned} A+B &= 0 \\ A-B &= 2 \end{aligned}\right\} \quad \begin{aligned} A &= 1 \\ B &= -1 \end{aligned}$$

$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx$$

$$= \ln|x-1| - \ln|x+1| + c$$

$$= \ln\left|\frac{x-1}{x+1}\right| + c$$

32. Find $\int \frac{\ln x}{x^n}.dx$

$$I = \int \overset{I}{x^{-n}} \overset{D}{\ln x}$$

$$= \frac{x^{-n+1}}{1-n}.\ln x - \int \frac{x^{-n+1}}{1-n}.\frac{1}{x}\,dx$$

$$= \frac{x^{1-n}}{1-n}\ln x - \int \frac{x^{-n}}{1-n}\,dx$$

$$I = \frac{\ln x}{(1-n)x^{n-1}} - \frac{1}{1-n}.\frac{x^{-n+1}}{(1-n)} + c$$

$$= \frac{1}{(1-n)x^{n-1}}.\left(\ln x - \frac{1}{1-n}\right) + c$$

33. Find $\int_0^{\frac{\pi}{2}} x.\sin x\cos x.dx$

D I

$I = \int_0^{\pi/2} \frac{x}{2}.\sin 2x\,dx$

$= \left[\left(-\frac{1}{2}\cos 2x\right).\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\left(-\frac{1}{2}\cos 2x\right).\frac{1}{2}\,dx$

$= \left(\frac{1}{2}.\pi/4 - 0\right) + \frac{1}{4}\int_0^{\pi/2}\cos 2x\,dx$

$= \frac{\pi}{8} + \frac{1}{4}\left[\frac{1}{2}\sin 2x\right]_0^{\pi/2}$

$= \pi/8 + \frac{1}{4}(0-0)$

$= \pi/8$

34. If $I_n = \int \sin^n x.dx$ for $n \geq 0$, show that:

$$I_n = -\frac{1}{n}\cos x.\sin^{n-1}x + \frac{n-1}{n}I_{n-2} \text{ where } n \geq 2$$

Hence show that: $I_n = \int_0^{\frac{\pi}{2}} \sin^n x.dx = \frac{n-1}{n}I_{n-2}$

and that: $I_5 \times I_6 = \frac{\pi}{12}$

D I

$I_n = \int \sin^{n-1}x.\sin x\,dx$

$= (-\cos x).\sin^{n-1}x - \int(-\cos x).(n-1)\sin^{n-2}x\cos x\,dx$

$= -\cos x.\sin^{n-1}x + (n-1)\int\cos^2 x.\sin^{n-2}x$

$= -\cos x\sin^{n-1}x + (n-1)\int(1-\sin^2 x).\sin^{n-2}x\,dx$

$= -\cos x\sin^{n-1}x + (n-1)\int\sin^{n-2}x - (n-1)I_n$

$I_n + (n-1)I_n = -\cos x\sin^{n-1}x + (n-1)I_{n-2}$

$nI_n = $ " "

$\therefore I_n = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}I_{n-2}$

$\therefore \int_0^{\pi/2}\sin^n x\,dx$

$= \left[\left(-\frac{1}{n}(0)(1)^{n-1} - 0\right)\right] + \frac{n-1}{n}I_{n-2}$ —— (*)

$I_0 = [x]_0^{\pi/2} = \pi/2$

$I_1 = [-\cos x]_0^{\pi/2} = (0) - (-1) = 1$

$\therefore I_2 = \frac{2-1}{2}I_0 = \pi/4$

$I_4 = \frac{4-1}{4}.I_2 = 3/4.\pi/4 = \frac{3\pi}{16}$

$I_6 = \frac{6-1}{6}.I_4 = 5/6.3\pi/16 = \frac{5\pi}{32}$

$I_3 = \frac{3-1}{3}.I_1 = \frac{2}{3}$

$I_5 = \frac{5-1}{5}I_3 = \frac{4}{5}.\frac{2}{3} = \frac{8}{15}$

$\therefore I_5 \times I_6 = \frac{5\pi}{32}\times\frac{8}{15} = \pi/12$

35. If $I_n = \int\sec^n x.dx$ for $n \geq 0$, show that:

$$I_n = \frac{1}{n-1}\tan x.\sec^{n-2}x + \frac{n-2}{n-1}.I_{n-2} \text{ for } n \geq 2$$

D I

$I_n = \int\sec^{n-2}x.\sec^2 x\,dx$

$= (\tan x).\sec^{n-2}x - \int(\tan x).(n-2)\sec^{n-3}x.\sec x.\tan x\,dx$

$= \tan x\sec^{n-2}x - (n-2)\int\tan^2 x.\sec^{n-2}x\,dx$

$I_n = \tan x\sec^{n-2}x - (n-2)\int\sec^n x\,dx - (n-2)\int\sec^{n-2}x\,dx$

$I_n = \tan x\sec^{n-2}x - (n-2)I_n - (n-2)I_{n-2}$

$\therefore (n-2)I_n + I_n = \tan x\sec^{n-2}x - (n-2)I_{n-2}$

$(n-1)I_n = \tan x\sec^{n-2}x - (n-2)I_{n-2}$

$I_n = \frac{1}{n-1}\tan x\sec^{n-2}x - \frac{n-2}{n-1}I_{n-2}$

36. If $I_n = \int_0^1 x(1-x^3)^n.dx$ for $n \geq 0$, show that:

$$I_n = \frac{3n}{2+3n}.I_{n-1} \quad \textit{for } n \geq 1 \text{ hence show:}$$

$$I_n = \frac{3^n n!}{(3n+2)(3n-1).....8.5.2}$$

$I_n = \left[\frac{x^2}{2}.(1-x^3)^n\right]_0^1 - \int_0^1\frac{x^2}{2}.n(1-x^3)^{n-1}.(-3x^2)\,dx$

$= 0 + \frac{3n}{2}\int_0^1 x.(1-x^3)^{n-1}.x^3\,dx$

$= -\frac{3}{2}n\int_0^1 x(1-x^3)^{n-1}.(-x^3)\,dx$ $\quad = (1-x^3)-1$

$= -\frac{3}{2}n\int_0^1 x(1-x^3)^{n-1}(1-x^3) - x(1-x^3)^{n-1}\,dx$

$= -\frac{3}{2}n\int_0^1 x(1-x^3)^n\,dx + \frac{3}{2}n\int x(1-x^3)^{n-1}\,dx$

$= -\frac{3}{2}n\,I_n + \frac{3}{2}n\,I_{n-1}$

$\therefore \left(1+\frac{3n}{2}\right)I_n = \frac{3n}{2}I_{n-1}$

$\left(\frac{2+3n}{2}\right)I_n = \frac{3n}{2}.I_{n-1}$

$I_n = \frac{3n}{2+3n}.I_{n-1}$

$\therefore I_n = \frac{3n}{2+3n}.\frac{3(n-1)}{3n-1}.I_{n-2}$

$= \frac{3n}{3n+2}.\frac{3(n-1)}{3n-1}.\frac{3(n-2)}{3n-4}.I_{n-3}$

$= \frac{3^n.n!}{(3n+2)(3n-1)\cdots 8.5.2}$

37. If $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\theta}{\sin\theta}.dx$ show that:

$I_n - I_{n-1} = 0 \;\; for\ n \geq 1$

– hence find the value of I_n $for\ n \geq 1$:

$$I_n - I_{n-1} = \int_0^{\pi/2} \frac{\sin(2n+1)\theta - \sin(2n-1)\theta}{\sin\theta} d\theta$$

Since $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$\Rightarrow I_n - I_{n-1} = 2\int_0^{\pi/2} \frac{\cos(2n\theta)\cancel{\sin\theta}}{\cancel{\sin\theta}} d\theta$$

$$= 2\int_0^{\pi/2} \cos 2n\theta \, d\theta$$

$$= 2\left[\frac{1}{2n}\sin 2n\theta\right]_0^{\pi/2}$$

$$= 2\left[(0) - (0)\right] = 0$$

$$\therefore I_n = I_{n-1} = I_{n-2} = \cdots = I_0$$

$$I_0 = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta} d\theta$$

$$= \left[\theta\right]_0^{\pi/2}$$

$$= \pi/2 .$$

38. Show that $\int_0^a f(x).dx = \int_0^a f(a-x).dx$ & hence

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}.dx = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}.dx$$

& hence evaluate the integral.

(a)

$$\int_0^a f(a-x)dx$$

$$= \int_a^0 f(u).(-1).du$$

$$= \int_0^a f(u)\,du.$$

$$\left[\begin{array}{ll} \text{Let } u = a-x & \\ du = -1dx & \\ x=0 & u=a \\ x=a & u=0 \end{array}\right.$$

(b) LHS: from (a)...

$$I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx = \int_0^{\pi/2} \frac{\cos^3(\pi/2 - x)\,.\,dx}{\cos^3(\pi/2 - x) + \sin^3(\pi/2 - x)}$$

$$= \int_0^{\pi/2} \frac{\sin^3 x \, dx}{\sin^3 x + \cos^3 x}$$

(c)

$$I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x \, dx}{\sin^3 x + \cos^3 x} - \int \frac{\cos^3 x\,.\,dx}{\sin^3 x + \cos^3 x}$$

$$= \left[x\right]_0^{\pi/2} - I.$$

$$2I = (\pi/2 - 0) = \pi/2$$

$$\therefore I = \pi/4$$

39. Show that $\int_{-a}^a f(x).dx = \int_0^a \{f(x)+f(-x)\}.dx$

Hence show: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}.dx = \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2 x}.dx$

& hence evaluate the integral.

$$I = \int_0^a f(x)dx + \int_{-a}^0 f(x)dx$$

$$= \int_0^a f(x)dx + \int_a^0 f(-u).-du$$

$$= \int_0^a f(x)dx + \int_0^a f(-u).du$$

$$\left[\begin{array}{ll} \text{Let } x = -u & \\ dx = -du & \\ x=0 & u=0 \\ x=-a & u=a \end{array}\right.$$

$$\text{or} = \int_0^a f(x)dx + \int_0^a f(-x)dx$$

$$\therefore \int = \int_0^{\pi/4} \frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)} dx$$

$$= \int_0^{\pi/4} \frac{1}{1+\sin x} + \frac{1}{1-\sin x} dx$$

$$= \int_0^{\pi/4} \frac{2 \; dx}{1-\sin^2 x} = \int_0^{\pi/4} \frac{2 \; dx}{\cos^2 x}$$

$$= 2\int_0^{\pi/4} \sec^2 x \, dx$$

$$= 2\left[\tan x\right]_0^{\pi/4}$$

$$= 2(1-0)$$

$$= 2.$$

40. Show that $\int_0^a f(x).dx = \int_0^a f(a-x).dx$

& hence show that: $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}.dx = \frac{\pi}{4}$

a) See qu (30)

b)

$$I = \int_0^{\pi/2} \frac{\cos x \, dx}{\cos x + \sin x}$$

$$= \int_0^{\pi/2} \frac{\cos(\pi/2 - x)\,.\,dx}{\cos(\pi/2 - x) + \sin(\pi/2 - x)} \quad \text{from (a)}$$

$$= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx - \int_0^{\pi/2} \frac{\cos x \,.\, dx}{\cos x + \sin x}$$

$$= \left[x\right]_0^{\pi/2} - I$$

$$\therefore 2I = \pi/2 - 0$$

$$I = \pi/4$$