BOARD OF STUDIES
NEW SOUTH WALES

2006

HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx$. **2**

(b) By completing the square, find $\displaystyle\int \frac{dx}{x^2-6x+13}$. **2**

(c) (i) Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as **3**

$$\frac{16x-43}{(x-3)^2(x+2)} = \frac{a}{(x-3)^2} + \frac{b}{x-3} + \frac{c}{x+2},$$

where a, b and c are real numbers, find a, b and c.

(ii) Hence find $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$. **2**

(d) Evaluate $\displaystyle\int_0^2 te^{-t}\,dt$. **3**

(e) Use the substitution $t=\tan\dfrac{\theta}{2}$ to show that **3**

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta} = \frac{1}{2}\log 3.$$

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z=3+i$ and $w=2-5i$. Find, in the form $x+iy$,

(i) z^2 **1**

(ii) $\overline{z}w$ **1**

(iii) $\dfrac{w}{z}$. **1**

(b) (i) Express $\sqrt{3}-i$ in modulus-argument form. **2**

(ii) Express $\left(\sqrt{3}-i\right)^7$ in modulus-argument form. **2**

(iii) Hence express $\left(\sqrt{3}-i\right)^7$ in the form $x+iy$. **1**

(c) Find, in modulus-argument form, all solutions of $z^3=-1$. **2**

(d) The equation $|z-1-3i| + |z-9-3i| = 10$ corresponds to an ellipse in the Argand diagram.

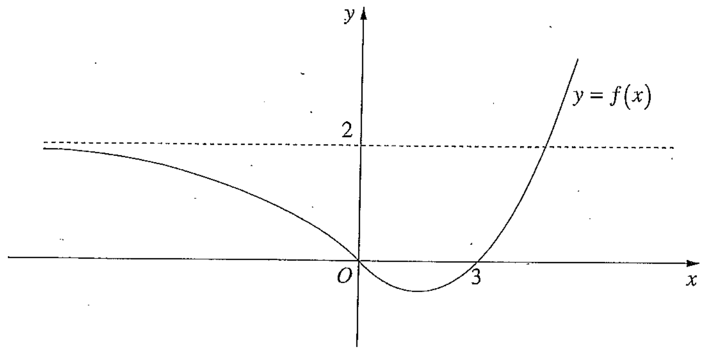
(i) Write down the complex number corresponding to the centre of the ellipse. **1**

(ii) Sketch the ellipse, and state the lengths of the major and minor axes. **3**

(iii) Write down the range of values of $\arg(z)$ for complex numbers z corresponding to points on the ellipse. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \left(f(x)\right)^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

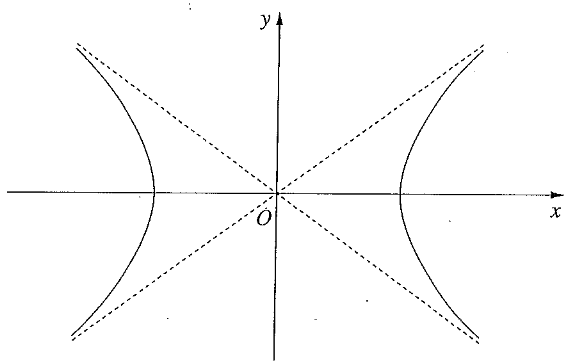
(iii) $y = xf(x)$. **2**

Question 3 continues on page 5

 Marks

(b) The diagram shows the graph of the hyperbola

$$\frac{x^2}{144} - \frac{y^2}{25} = 1.$$



(i) Find the coordinates of the points where the hyperbola intersects the x-axis. **1**

(ii) Find the coordinates of the foci of the hyperbola. **2**

(iii) Find the equations of the directrices and the asymptotes of the hyperbola. **2**

(c) Two of the zeros of $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ are $a + ib$ and $a + 2ib$, where a and b are real and $b > 0$.

(i) Find the values of a and b. **3**

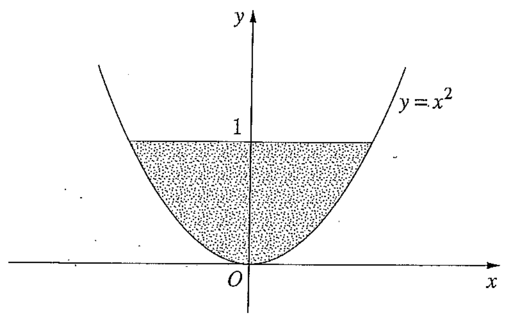
(ii) Hence, or otherwise, express $P(x)$ as the product of quadratic factors with real coefficients. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The polynomial $p(x) = ax^3 + bx + c$ has a multiple zero at 1 and has remainder 4 when divided by $x + 1$. Find a, b and c. **3**

(b) The base of a solid is the parabolic region $x^2 \le y \le 1$ shaded in the diagram. **3**



Vertical cross-sections of the solid perpendicular to the y-axis are squares.

Find the volume of the solid.

(c) Let $P\left(p, \frac{1}{p}\right)$, $Q\left(q, \frac{1}{q}\right)$ and $R\left(r, \frac{1}{r}\right)$ be three distinct points on the hyperbola $xy = 1$.

(i) Show that the equation of the line, ℓ, through R, perpendicular to PQ, is $y = pqx - pqr + \frac{1}{r}$. **2**

(ii) Write down the equation of the line, m, through P, perpendicular to QR. **1**

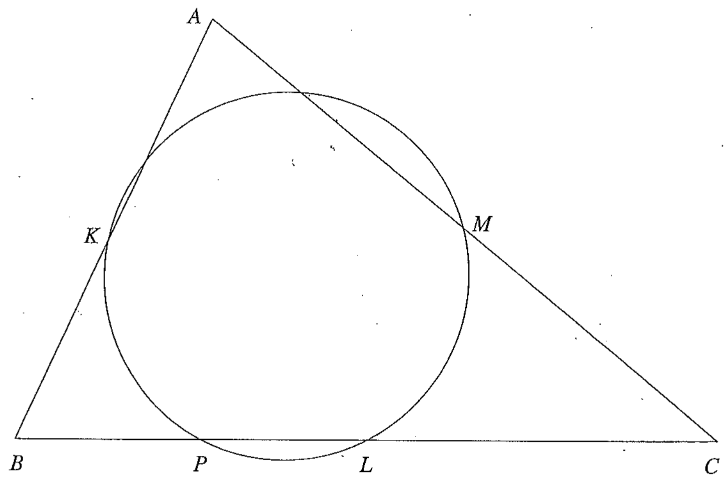
(iii) The lines ℓ and m intersect at T. **2**

Show that T lies on the hyperbola.

Question 4 continues on page 7

Question 4 (continued) **Marks**

(d)



In the acute-angled triangle ABC, K is the midpoint of AB, L is the midpoint of BC and M is the midpoint of CA. The circle through K, L and M also cuts BC at P as shown in the diagram.

Copy or trace the diagram into your writing booklet.

(i) Prove that $KMLB$ is a parallelogram. **1**

(ii) Prove that $\angle KPB = \angle KML$. **1**

(iii) Prove that $AP \perp BC$. **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A solid is formed by rotating the region bounded by the curve $y=x(x-1)^2$ and the line $y=0$ about the y-axis. Use the method of cylindrical shells to find the volume of this solid. **3**

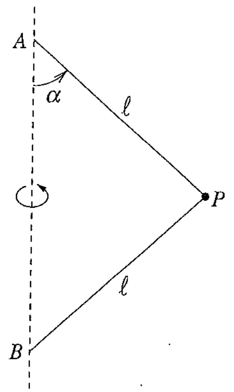
(b) (i) Show that $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$. **1**

(ii) Hence, or otherwise, solve the equation **3**

$$\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 0$$

for $0 \leq \theta \leq 2\pi$.

(c) A particle, P, of mass m is attached by two strings, each of length ℓ, to two fixed points, A and B, which lie on a vertical line as shown in the diagram.



The system revolves with constant angular velocity ω about AB. The string AP makes an angle α with the vertical. The tension in the string AP is T_1 and the tension in the string BP is T_2 where $T_1 \geq 0$ and $T_2 \geq 0$. The particle is also subject to a downward force, mg, due to gravity.

(i) Resolve the forces on P in the horizontal and vertical directions. **2**

(ii) If $T_2 = 0$, find the value of ω in terms of ℓ, g and α. **1**

Question 5 continues on page 9

Question 5 (continued) **Marks**

(d) In a chess match between the Home team and the Away team, a game is played on each of board 1, board 2, board 3 and board 4.

On each board, the probability that the Home team wins is 0.2, the probability of a draw is 0.6 and the probability that the Home team loses is 0.2.

The results are recorded by listing the outcomes of the games for the Home team in board order. For example, if the Home team wins on board 1, draws on board 2, loses on board 3 and draws on board 4, the result is recorded as WDLD.

(i) How many different recordings are possible? **1**

(ii) Calculate the probability of the result which is recorded as WDLD. **1**

(iii) Teams score 1 point for each game won, $\frac{1}{2}$ a point for each game drawn and 0 points for each game lost. **3**

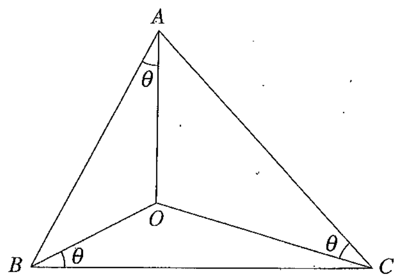
What is the probability that the Home team scores more points than the Away team?

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) In ΔABC, $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. The point O is chosen inside ΔABC so that $\angle OAB = \angle OBC = \angle OCA = \theta$, as shown in the diagram.



(i) Show that $\dfrac{OA}{OB} = \dfrac{\sin(\beta-\theta)}{\sin\theta}$. **1**

(ii) Hence show that $\sin^3\theta = \sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)$. **2**

(iii) Prove the identity $\cot x - \cot y = \dfrac{\sin(y-x)}{\sin x \sin y}$. **1**

(iv) Hence show that **1**

$$(\cot\theta - \cot\alpha)(\cot\theta - \cot\beta)(\cot\theta - \cot\gamma) = \operatorname{cosec}\alpha \operatorname{cosec}\beta \operatorname{cosec}\gamma.$$

(v) Hence find the value of θ when ΔABC is an isosceles right triangle. **2**

Question 6 continues on page 11

Marks

Question 6 (continued)

(b) In an alien universe, the gravitational attraction between two bodies is proportional to x^{-3}, where x is the distance between their centres.

A particle is projected upward from the surface of a planet with velocity u at time $t = 0$. Its distance x from the centre of the planet satisfies the equation $\ddot{x} = -\dfrac{k}{x^3}$.

(i) Show that $k = gR^3$, where g is the magnitude of the acceleration due to gravity at the surface of the planet and R is the radius of the planet. **1**

(ii) Show that v, the velocity of the particle, is given by **3**

$$v^2 = \frac{gR^3}{x^2} - \left(gR - u^2\right).$$

(iii) It can be shown that $x = \sqrt{R^2 + 2uRt - \left(gR - u^2\right)t^2}$. **2**

(Do NOT prove this.)

Show that if $u \geq \sqrt{gR}$ the particle will not return to the planet.

(iv) If $u < \sqrt{gR}$ the particle reaches a point whose distance from the centre of the planet is D, and then falls back.

(1) Use the formula in part (ii) to find D in terms of u, R and g. **1**

(2) Use the formula in part (iii) to find the time taken for the particle to return to the surface of the planet in terms of u, R and g. **1**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The curves $y = \cos x$ and $y = \tan x$ intersect at a point P whose x-coordinate is α.

(i) Show that the curves intersect at right angles at P. **3**

(ii) Show that $\sec^2 \alpha = \dfrac{1+\sqrt{5}}{2}$. **2**

(b) (i) Let $I_n = \displaystyle\int_0^x \sec^n t\, dt$, where $0 \le x < \dfrac{\pi}{2}$. Show that **3**

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

(ii) Hence find the exact value of **2**

$$\int_0^{\frac{\pi}{3}} \sec^4 t\, dt.$$

Question 7 continues on page 13

Question 7 (continued) **Marks**

(c) The sequence $\{x_n\}$ is given by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{4 + x_n}{1 + x_n} \text{ for } n \ge 1.$$

(i) Prove by induction that for $n \ge 1$ **4**

$$x_n = 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right),$$

where $\alpha = -\dfrac{1}{3}$.

(ii) Hence find the limiting value of x_n as $n \to \infty$. **1**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Suppose $0 \le t \le \frac{1}{\sqrt{2}}$.

(i) Show that $0 \le \frac{2t^2}{1-t^2} \le 4t^2$. **2**

(ii) Hence show that $0 \le \frac{1}{1+t} + \frac{1}{1-t} - 2 \le 4t^2$. **1**

(iii) By integrating the expressions in the inequality in part (ii) with respect to t from $t=0$ to $t=x$ (where $0 \le x \le \frac{1}{\sqrt{2}}$), show that **2**

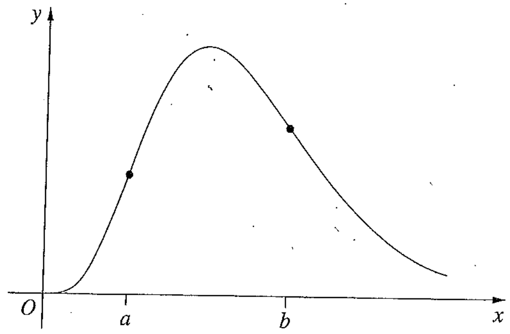
$$0 \le \log_e\left(\frac{1+x}{1-x}\right) - 2x \le \frac{4x^3}{3}.$$

(iv) Hence show that for $0 \le x \le \frac{1}{\sqrt{2}}$ **1**

$$1 \le \left(\frac{1+x}{1-x}\right)e^{-2x} \le e^{\frac{4x^3}{3}}.$$

Question 8 continues on page 15

(b) For $x > 0$, let $f(x) = x^n e^{-x}$, where n is an integer and $n \ge 2$.



(i) The two points of inflexion of $f(x)$ occur at $x = a$ and $x = b$, where $0 < a < b$. Find a and b in terms of n. **4**

(ii) Show that **2**

$$\frac{f(b)}{f(a)} = \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}}.$$

(iii) Using the result of part (a) (iv), show that **2**

$$1 \le \frac{f(b)}{f(a)} \le e^{\frac{4}{3\sqrt{n}}}.$$

(iv) What can be said about the ratio $\frac{f(b)}{f(a)}$ as $n \to \infty$? **1**

End of paper

2006 Higher School Certificate Solutions
Mathematics Extension 2

QUESTION 1

(a) $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx = -\frac{1}{8}\int \frac{-8x}{\sqrt{9-4x^2}}\,dx$

$\displaystyle = -\frac{1}{4}\sqrt{9-4x^2}+c.$

(b) $\displaystyle\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+4}$

$\displaystyle = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c.$

(c) (i) $\displaystyle\frac{16x-43}{(x-3)^2(x+2)}$

$\displaystyle = \frac{a}{(x-3)^2}+\frac{b}{x-3}+\frac{c}{x+2}$

$\displaystyle = \frac{a(x+2)+b(x-3)(x+2)+c(x-3)^2}{(x-3)^2(x+2)}$

Equating numerators:

$16x-43$

$= a(x+2)+b(x-3)(x+2)+c(x-3)^2$

Let $x=-2$: $25c = -75$

$c = -3.$

Let $x = 3$: $5a = 5$

$a = 1.$

Coefficient of x^2: $b+c=0$

$b = 3.$

$\therefore\ a = 1,\ b = 3,\ c = -3.$

(ii) $\displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx$

$\displaystyle = \int \frac{1}{(x-3)^2}\,dx + \int \frac{3}{x-3}\,dx - \int \frac{3}{x+2}\,dx$

$\displaystyle = \frac{-1}{x-3}+3\ln|x-3|-3\ln|x+2|+c.$

$\displaystyle = \frac{-1}{x-3}+3\ln\left|\frac{x-3}{x+2}\right|+c.$

(d) $u = t \qquad \dfrac{du}{dt} = 1$

$\dfrac{dv}{dt} = e^{-t} \qquad v = -e^{-t}$

$\displaystyle\int_0^2 te^{-t}\,dt = \left[-te^{-t}\right]_0^2+\int_0^2 e^{-t}\,dt$

$= -2e^{-2}+\left[-e^{-t}\right]_0^2$

$= -2e^{-2}-e^{-2}+1$

$= 1-3e^{-2}.$

(e) $\sin\theta = \dfrac{2t}{1+t^2}$ and $d\theta = \dfrac{2}{1+t^2}\,dt$

When $\theta = \dfrac{\pi}{2}$, $\quad t = \tan\dfrac{\pi}{4} = 1.$

When $\theta = \dfrac{2\pi}{3}$, $\quad t = \tan\dfrac{\pi}{3} = \sqrt{3}.$

$\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta} = \int_1^{\sqrt{3}} \frac{t^2+1}{2t}\times\frac{2}{1+t^2}\,dt$

$\displaystyle = \int_1^{\sqrt{3}} \frac{dt}{t}$

$= \left[\log_e t\right]_1^{\sqrt{3}}$

$= \log_e\sqrt{3}-\log_e 1$

$= \log_e 3^{\frac{1}{2}}-0$

$= \dfrac{1}{2}\log_e 3.$

QUESTION 2

(a) (i) $z = 3+i,\ w = 2-5i$

$z^2 = (3+i)^2$

$= 9+6i-1$

$= 8+6i.$

(ii) $\bar{z}w = (3-i)(2-5i)$

$= 6-15i-2i-5$

$= 1-17i.$

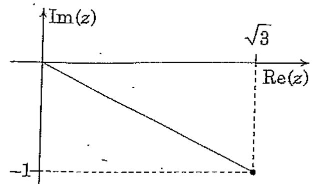
(iii) $\dfrac{w}{z} = \dfrac{2-5i}{3+i}$

$= \dfrac{(2-5i)(3-i)}{(3+i)(3-i)}$

$= \dfrac{1-17i}{10}$

$= \dfrac{1}{10}-\dfrac{17}{10}i.$

(b) (i) $\sqrt{3}-i = r\operatorname{cis}\theta$



$r = \sqrt{\sqrt{3}^2+1^2}$

$= 2.$

$\theta = \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}.$

$\therefore\ \sqrt{3}-i = 2\operatorname{cis}\left(-\dfrac{\pi}{6}\right).$

(ii) $(\sqrt{3}-i)^7 = \left[2\operatorname{cis}\left(-\dfrac{\pi}{6}\right)\right]^7$

$= 2^7\operatorname{cis}\left(-\dfrac{7\pi}{6}\right),$

using De Moivre's theorem

$= 128\operatorname{cis}\left(\dfrac{5\pi}{6}\right).$

(iii) $(\sqrt{3}-i)^7 = 128\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$

$= 128\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)$

$= -64\sqrt{3}+64i.$

(c) $z^3 = -1$

$= 1\operatorname{cis}(\pi+2k\pi) \qquad k = 0,\ \pm 1$

$\therefore\ z = 1\operatorname{cis}\left(\dfrac{\pi}{3}+\dfrac{2k\pi}{3}\right)$, using De Moivre's theorem.

When $k=0$, $\quad z = \operatorname{cis}\left(\dfrac{\pi}{3}\right).$

When $k=1$, $\quad z = \operatorname{cis}\left(\dfrac{\pi}{3}+\dfrac{2\pi}{3}\right)$

$= \operatorname{cis}(\pi)$

$= 1.$

When $k=-1$, $\quad z = \operatorname{cis}\left(\dfrac{\pi}{3}-\dfrac{2\pi}{3}\right)$

$= \operatorname{cis}\left(-\dfrac{\pi}{3}\right).$

$\therefore$ The three solutions are $\operatorname{cis}\left(\dfrac{\pi}{3}\right)$, -1 and $\operatorname{cis}\left(-\dfrac{\pi}{3}\right).$

(d) $|z-1-3i|+|z-9-3i| = 10.$

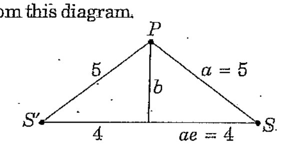
(i) $|z-(1+3i)|+|z-(9+3i)| = 10.$

This corresponds to $PS'+PS = 2a$, where P represents z, the foci S' and S are represented by $1+3i$ and $9+3i$ and the major axis has a length of $2a = 10$.

The centre of the ellipse is the midpoint of S' and S which is represented by $5+3i$.

(ii) The major axis has length $2a = 10$.

The minor axis has length $2b$ which can be determined using Pythagoras' theorem from this diagram.



So $b = 3$ and the minor axis has length $2b = 6$.

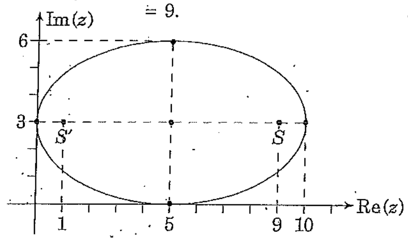
The minor axis can also be determined algebraically using

$b^2 = a^2(1-e^2)$

$= a^2-(ae)^2$

$= 5^2-4^2$

$= 9.$



(iii) From the diagram we note that

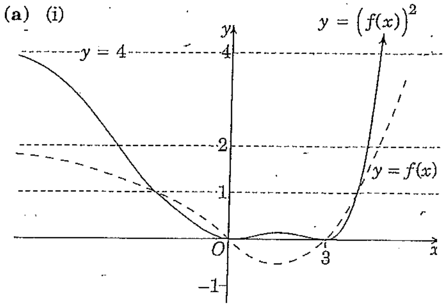
$\arg(5+0i) = 0$

and $\arg(0+3i) = \dfrac{\pi}{2}.$

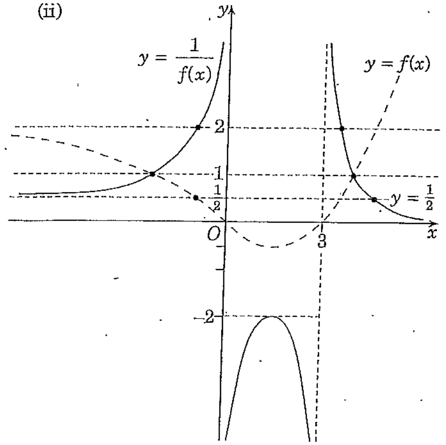
Hence $0 \le \arg(z) \le \dfrac{\pi}{2}.$

QUESTION 3

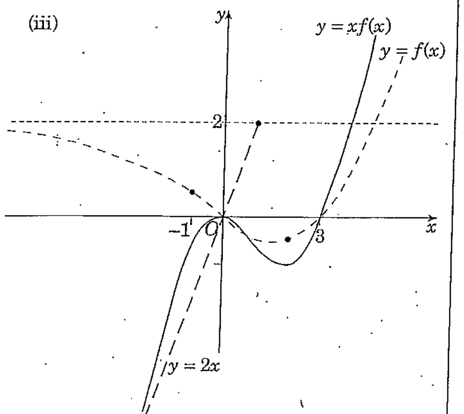
(a) (i)



(ii)



(iii)



(b) $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1.$

(i) $y = 0$: $\dfrac{x^2}{144} = 1$

$x = \pm 12.$

∴ The hyperbola intersects the x-axis at $(12, 0)$ and $(-12, 0)$.

(ii) The foci have coordinates $(\pm ae, 0)$.

$$a = 12$$
$$b^2 = a^2(e^2 - 1)$$
$$25 = 144(e^2 - 1)$$
$$e^2 - 1 = \frac{25}{144}$$
$$e^2 = \frac{169}{144}$$
$$e = \frac{13}{12}, \quad \text{as } e > 0.$$

∴ The foci have coordinates $(13, 0)$ and $(-13, 0)$.

(iii) Directrices: $x = \pm\dfrac{a}{e}$

$$x = \frac{144}{13} \text{ and } x = -\frac{144}{13}.$$

Asymptotes: $y = \pm\dfrac{b}{a}x$

$$y = \frac{5}{12}x \text{ and } y = -\frac{5}{12}x.$$

(c) (i) Since $P(x)$ has real coefficients, the zeros occur in complex conjugate pairs. Hence the zeros are $a \pm ib$ and $a \pm 2ib$.

Sum of the zeros:

$$(a+ib)+(a-ib)+(a+2ib)+(a-2ib) = 12$$
$$4a = 12$$
$$a = 3.$$

Product of zeros:

$$(a+ib)(a-ib)(a+2ib)(a-2ib) = 130$$
$$(a^2+b^2)(a^2+4b^2) = 130$$
$$(9+b^2)(9+4b^2) = 130, \text{ using } a = 3$$
$$4b^4 + 45b^2 - 49 = 0$$
$$(b^2-1)(4b^2+49) = 0$$

∴ $b^2 = 1$ since b is real

∴ $b = 1$ since $b > 0$

∴ $a = 3$ and $b = 1$.

(ii)
$$P(x) = (x-3+i)(x-3-i) \times (x-3+2i)(x-3-2i)$$
$$= \left[(x-3)^2+1\right]\left[(x-3)^2+4\right]$$
$$= (x^2-6x+10)(x^2-6x+13).$$

QUESTION 4

(a) $p(x) = ax^3 + bx + c$ and $p'(x) = 3ax^2 + b$.

$p(x)$ has a multiple root at $x = 1$, so $p(1) = 0$ and $p'(1) = 0$.

$$a + b + c = 0 \qquad \text{— ①}$$
$$3a + b = 0 \qquad \text{— ②}$$

$p(x)$ has a remainder 4 when divided by $(x+1)$, so $p(-1) = 4$.

$$-a - b + c = 4 \qquad \text{— ③}$$

① + ③: $2c = 4$

∴ $c = 2.$

Substitute $c = 2$ into ①:

$$a + b + 2 = 0 \qquad \text{— ④}$$

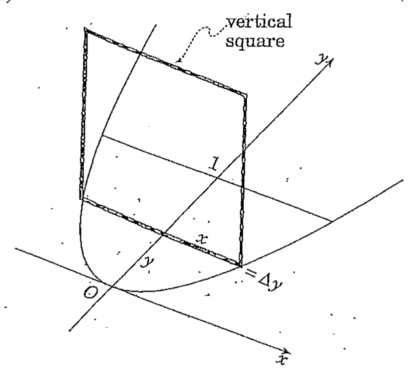
④ − ②: $-2a + 2 = 0$

∴ $a = 1$

$b = -3.$

∴ $a = 1$, $b = -3$ and $c = 2$.

(b)



Length of square $= 2x$.

∴ Area $= 4x^2$

$= 4y.$

Thickness of square $= \Delta y$

∴ $\Delta V = 4y\Delta y.$

$$V = \lim_{\Delta y \to 0} \sum_0^1 4y\Delta y$$
$$= \int_0^1 4y\, dy$$
$$= \left[2y^2\right]_0^1$$
$$= 2 \text{ units}^3.$$

(c) $P\left(p, \dfrac{1}{p}\right)$, $Q\left(q, \dfrac{1}{q}\right)$, $R\left(r, \dfrac{1}{r}\right)$.

(i) Gradient of PQ
$$= \frac{\frac{1}{p} - \frac{1}{q}}{p - q}$$
$$= \frac{p - q}{pq(q-p)}$$
$$= -\frac{1}{pq}.$$

Gradient of the perpendicular $= pq$.

The equation of the line ℓ through R is

$$y - \frac{1}{r} = pq(x - r)$$
$$y = pqx - pqr + \frac{1}{r}. \qquad \text{— ①}$$

(ii) Similarly the equation of the line m through P, perpendicular to QR, is found by exchanging p and r in ①.

$$y = qrx - pqr + \frac{1}{p}. \qquad \text{— ②}$$

(iii) Solve ① and ② simultaneously.

① − ②: $(pq - qr)x + \left(\dfrac{1}{r} - \dfrac{1}{p}\right) = 0$

$$q(p-r)x = \left(\frac{r-p}{pr}\right)$$

∴ $x = -\dfrac{1}{pqr}.$

$$y = pq\left(-\frac{1}{pqr}\right) - pqr + \frac{1}{r}$$
$$= -pqr.$$

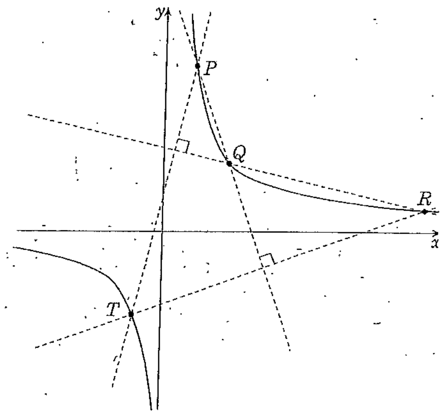
∴ T has coordinates $x = -\dfrac{1}{pqr}$ and $y = -pqr$.

Substitute these coordinates into the equation of the hyperbola: $xy = 1$

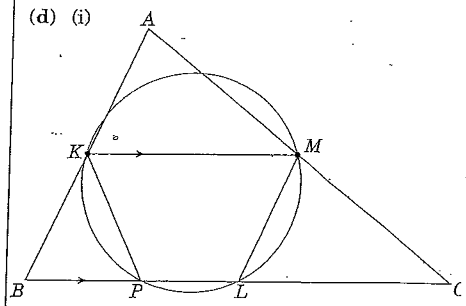
LHS $= -\dfrac{1}{pqr} \times -pqr = 1 =$ RHS.

∴ T lies on the hyperbola $xy = 1$.

This is illustrated in the following diagram (not required for the exam).



(d) (i)



In triangles ABC and AKM,

$$\frac{AB}{AC} = \frac{2AK}{2AM} = \frac{AK}{AM}$$

and the included angle $\angle A$ is common.

$\therefore\ \triangle ABC \,|||\, \triangle AKM$ (corresponding sides in proportion and included $\angle$ equal).

$\angle AKM = \angle ABC$ (corresponding $\angle$s in similar $\triangle$s)

$\therefore\ KM \parallel BL$ (corresponding $\angle$s are equal).

The scale ratio $\dfrac{AK}{AB} = \dfrac{1}{2}$ (K is the midpoint of AB).

$\therefore\ KM = \dfrac{1}{2}BC.$

But $BL = \dfrac{1}{2}BC$ (L is the midpoint of BC).

$\therefore\ KM = BL.$

$\therefore$ $KMLB$ is a parallelogram (one pair of opposite sides are equal and parallel).

(ii) $KMLP$ is a cyclic quadrilateral.

$\angle KPB = \angle KML$ (exterior $\angle$ of a cyclic quadrilateral is equal to the opposite interior $\angle$).

(iii) $\angle KPB = \angle KML$, from (ii)

$\angle KML = \angle KBP$, from (i) (opposite $\angle$s in a parallelogram are equal).

Hence $\angle KPB = \angle KBP$.

$\therefore\ \triangle KBP$ is isosceles with $KB = KP$.

$KB = KA$ (K is the midpoint of AB).

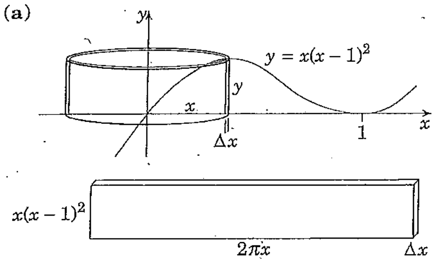
Therefore $KB = KA = KP$, so that K is the centre of a circle with diameter AB and passing through P.

$\therefore\ \angle APB = 90°$ ($\angle$ in a semi-circle is a right angle).

$\therefore\ AP \perp BC.$

QUESTION 5

(a)



$$\Delta V = 2\pi x \times x(x-1)^2 \times \Delta x$$
$$= 2\pi x^2(x-1)^2\,\Delta x.$$
$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi x^2 (x-1)^2\,\Delta x$$
$$= 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right)\,dx$$
$$= 2\pi\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_0^1$$
$$= 2\pi\left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right)$$
$$= \frac{\pi}{15}\ \text{units}^3.$$

(b) (i) $\cos(\alpha+\beta) + \cos(\alpha-\beta)$
$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$= 2\cos\alpha\cos\beta.$

(ii)
$$\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 0$$
$$\cos(2\theta-\theta) + \cos(3\theta-\theta) + \cos(2\theta+\theta) + \cos(3\theta+\theta) = 0$$
$$2\cos\theta\cos 2\theta + 2\cos\theta\cos 3\theta = 0$$
$$2\cos\theta(\cos 2\theta + \cos 3\theta) = 0$$
$$2\cos\theta\left[\cos\left(\frac{5\theta}{2} - \frac{\theta}{2}\right) + \cos\left(\frac{5\theta}{2} + \frac{\theta}{2}\right)\right] = 0$$
$$4\cos\theta\cos\frac{5\theta}{2}\cos\frac{\theta}{2} = 0$$

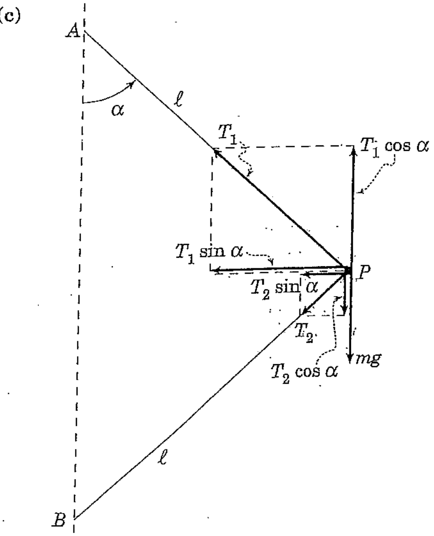
$\cos\theta = 0$
$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\cos\dfrac{\theta}{2} = 0$
$\dfrac{\theta}{2} = \dfrac{\pi}{2}$
$\theta = \pi$

$\cos\dfrac{5\theta}{2} = 0$
$\dfrac{5\theta}{2} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}$
$\theta = \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \pi, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}.$

Hence the solutions are
$$\theta = \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}; \pi, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{9\pi}{5}.$$

(c)



(i) Sum of horizontal components $= mw^2 r$.
$T_1\sin\alpha + T_2\sin\alpha = mw^2 r.$ — ①
Sum of vertical components $= 0$.
$T_1\cos\alpha - T_2\cos\alpha - mg = 0.$ — ②

(ii) If $T_2 = 0$, then
$T_1 \sin\alpha = mw^2 \ell\sin\alpha$, from ① using $r = \ell\sin\alpha$
$T_1 = mw^2\ell$
$T_1\cos\alpha = mg$, from ②.
$T_1 = \dfrac{mg}{\cos\alpha}$
$\therefore\ mw^2\ell = \dfrac{mg}{\cos\alpha}$
$w^2 = \dfrac{g}{\ell\cos\alpha}$
$w = \sqrt{\dfrac{g}{\ell\cos\alpha}},\quad w > 0.$

(d) (i) There are 3 possible outcomes on each board. Therefore, by the multiplication principle, there are $3^4 = 81$ possible recordings.

(ii) $P(WDLD) = 0{\cdot}2 \times 0{\cdot}6 \times 0{\cdot}2 \times 0{\cdot}6$
$= 0{\cdot}0144.$

(iii) The probabilities are symmetric, so we determine the probability that one side wins by first finding the probability of equal points. There are three possible cases.

$P(4\text{ draws}) = 0{\cdot}6^4 = 0{\cdot}1296.$

$P(2\text{ wins, }2\text{ losses}) = \dfrac{4!}{2!\,2!} \times 0{\cdot}2^4$
$= 0{\cdot}0096.$

$P(1\text{ win, }1\text{ loss, }2\text{ draws})$
$= \dfrac{4!}{2!} \times 0{\cdot}2^2 \times 0{\cdot}6^2$
$= 0{\cdot}1728.$

$P(\text{equal points})$
$= 0{\cdot}1296 + 0{\cdot}0096 + 0{\cdot}1728$
$= 0{\cdot}312.$

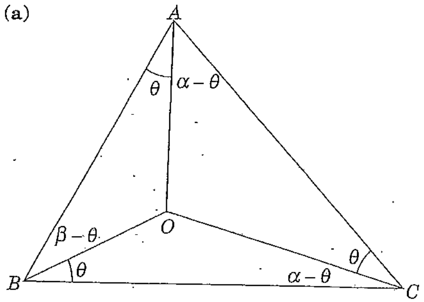
$P(\text{unequal points}) = 0{\cdot}688.$

$P(\text{home team scores more points}) = 0{\cdot}344.$

Note that to calculate this result directly requires considering 6 possible cases:
1 win, 3 draws; 2 wins, 2 draws;
2 wins, 1 draw, 1 loss; 3 wins, 1 draw;
3 wins, 1 loss; 4 wins.

QUESTION 6

(a)



(i) In $\triangle OAB$, $\angle ABO = \beta - \theta$
$\dfrac{OA}{\sin(\beta-\theta)} = \dfrac{OB}{\sin\theta}$ (sine rule)
$\therefore\ \dfrac{OA}{OB} = \dfrac{\sin(\beta-\theta)}{\sin\theta}.$

(ii) Similarly, $\dfrac{OC}{OA} = \dfrac{\sin(\alpha-\theta)}{\sin\theta}$
$\dfrac{OB}{OC} = \dfrac{\sin(\gamma-\theta)}{\sin\theta}.$
$\therefore\ \dfrac{OA}{OB} \times \dfrac{OC}{OA} \times \dfrac{OB}{OC} = 1$
$1 = \dfrac{\sin(\beta-\theta)}{\sin\theta} \times \dfrac{\sin(\alpha-\theta)}{\sin\theta} \times \dfrac{\sin(\gamma-\theta)}{\sin\theta}$
$1 = \dfrac{\sin(\beta-\theta)\sin(\alpha-\theta)\sin(\gamma-\theta)}{\sin^3\theta}$
$\therefore\ \sin^3\theta = \sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta).$

(iii) $\cot x - \cot y = \dfrac{\cos x}{\sin x} - \dfrac{\cos y}{\sin y}$
$= \dfrac{\cos x\sin y - \cos y\sin x}{\sin x\sin y}$
$= \dfrac{\sin(y-x)}{\sin x\sin y}$
$\therefore\ \cot x - \cot y = \dfrac{\sin(y-x)}{\sin x\sin y}.$

Note that this is an identity. It applies generally and is not specific to this problem.

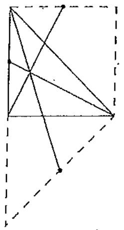
(iv) $(\cot\theta - \cot\alpha)(\cot\theta - \cot\beta)(\cot\theta - \cot\gamma)$
$= \dfrac{\sin(\alpha-\theta)}{\sin\alpha\sin\theta} \times \dfrac{\sin(\beta-\theta)}{\sin\beta\sin\theta} \times \dfrac{\sin(\gamma-\theta)}{\sin\gamma\sin\theta}$, using (iii)
$= \dfrac{\sin^3\theta}{\sin\alpha\sin\beta\sin\gamma\sin^3\theta}$, using (ii)
$= \dfrac{1}{\sin\alpha\sin\beta\sin\gamma}$
$= \operatorname{cosec}\alpha\,\operatorname{cosec}\beta\,\operatorname{cosec}\gamma.$

(v) Assume, say, that $\alpha = \beta = \frac{\pi}{4}$ and $\gamma = \frac{\pi}{2}$.

Substituting these in (iv) gives

$$\left(\cot\theta - \cot\frac{\pi}{4}\right)^2\left(\cot\theta - \cot\frac{\pi}{2}\right)$$
$$= \operatorname{cosec}^2\frac{\pi}{4} \times \operatorname{cosec}\frac{\pi}{2},$$

$$\therefore \quad (\cot\theta - 1)^2\cot\theta = \sqrt{2}^2 \times 1$$
$$\cot\theta(\cot^2\theta - 2\cot\theta + 1) = 2$$
$$\cot^3\theta - 2\cot^2\theta + \cot\theta - 2 = 0$$
$$\cot^2\theta(\cot\theta - 2) + (\cot\theta - 2) = 0$$
$$(\cot^2\theta + 1)(\cot\theta - 2) = 0.$$
$$\cot\theta = 2, \text{ since } \cot^2\theta + 1 \neq 0$$
$$\tan\theta = \frac{1}{2}$$
$$\theta = \tan^{-1}\frac{1}{2}.$$



Here is an illustration of this case (not required for the exam).

(b) (i) $\ddot{x} = -\dfrac{k}{x^3}$.

The acceleration due to gravity is towards the centre of the planet, in the opposite direction to x. Therefore, substitute $\ddot{x} = -g$ when $x = R$.

$$-g = -\frac{k}{R^3}$$
$$k = gR^3.$$

(ii) $\ddot{x} = -\dfrac{gR^3}{x^3}$

$$\frac{1}{2}v^2 = \int -\frac{gR^3}{x^3}\,dx, \text{ using } \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{gR^3}{2x^2} + c$$
$$v^2 = \frac{gR^3}{x^2} + c', \text{ where } c' = 2c.$$

When $t = 0$, $x = R$ and $v = u$,

$$u^2 = \frac{gR^3}{R^2} + c'$$
$$c' = u^2 - gR$$
$$v^2 = \frac{gR^3}{x^2} + u^2 - gR$$
$$= \frac{gR^3}{x^2} - (gR - u^2).$$

(iii) If $u \geq \sqrt{gR}$, then $gR - u^2 \leq 0$.

METHOD 1

$$x = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$$
$$\geq \sqrt{R^2 + 2\sqrt{gR}Rt - (gR - gR)t^2}$$
$$> \sqrt{R^2} \quad \text{if } t > 0$$
$$= R.$$

Thus if $t > 0$, then $x > R$ and the particle will never return to the planet.

METHOD 2

$$v^2 = \frac{gR^3}{x^2} - (gR - u^2), \quad \text{from (ii)}$$
$$> \frac{gR^3}{x^2}.$$

$\therefore$ v is never zero and the particle continues to move away from the planet forever and hence never returns to it.

(iv) (1) The maximum displacement $x = D$ occurs when $v = 0$.

Substituting in (ii) gives

$$\frac{gR^3}{D^2} - (gR - u^2) = 0$$
$$\frac{gR^3}{D^2} = gR - u^2$$
$$D^2 = \frac{gR^3}{gR - u^2}$$
$$D = \sqrt{\frac{gR^3}{gR - u^2}}$$
$$= R\sqrt{\frac{gR}{gR - u^2}}$$

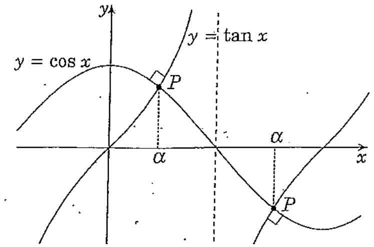
The square root exists since $gR - u^2 > 0$ when $u < \sqrt{gR}$.

(2) To find the time to return to the surface, substitute $x = R$ into (iii) and solve for $t > 0$.

$$R = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$$
$$R^2 = R^2 + 2uRt - (gR - u^2)t^2$$
$$\therefore \quad 2uR - (gR - u^2)t = 0, \text{ since } t > 0$$
$$\therefore \quad t = \frac{2uR}{gR - u^2}.$$

QUESTION 7

(a) The diagram shows two possible values of α.



Note that at P, $\cos\alpha = \tan\alpha$.

(i) For $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x.$$

At $x = \alpha$, $m_1 = \sec^2\alpha$.

For $y = \cos x$

$$\frac{dy}{dx} = -\sin x.$$

At $x = \alpha$, $m_2 = -\sin\alpha$.

$$\therefore \quad m_1 m_2 = -\sec^2\alpha\sin\alpha.$$
$$= -\frac{\tan\alpha}{\cos\alpha}$$
$$= -1 \text{ since } \tan\alpha = \cos\alpha.$$

Hence the curves intersect at right angles at P.

(ii) *METHOD 1*

$$\cos\alpha = \tan\alpha$$
$$\cos^2\alpha = \tan^2\alpha$$
$$\cos^2\alpha = \sec^2\alpha - 1$$
$$1 = \sec^4\alpha - \sec^2\alpha$$
$$\sec^4\alpha - \sec^2\alpha - 1 = 0$$
$$\sec^2\alpha = \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \frac{1 + \sqrt{5}}{2},$$
since $\sec^2\alpha > 0$.

METHOD 2

$$\tan\alpha = \cos\alpha$$
$$\sin\alpha = \cos^2\alpha$$
$$\sin\alpha = 1 - \sin^2\alpha.$$
$$\sin^2\alpha + \sin\alpha - 1 = 0$$
$$\sin\alpha = \frac{-1 \pm \sqrt{5}}{2}$$
$$\cos^2\alpha = \frac{-1 \pm \sqrt{5}}{2},$$
since $\cos^2\alpha = \sin\alpha$ and $\cos^2\alpha > 0$.

$$\sec^2\alpha = \frac{2}{-1 + \sqrt{5}} \times \frac{1 + \sqrt{5}}{1 + \sqrt{5}}$$
$$= \frac{2 + 2\sqrt{5}}{4}$$
$$= \frac{1 + \sqrt{5}}{2}.$$

(b) (i) $I_n = \int_0^x \sec^n t \qquad 0 \leq x < \frac{\pi}{2}.$

$$= \int_0^x \sec^{n-2} t \sec^2 t \,dt$$
$$= \int_0^x \sec^{n-2} t\left(\frac{d}{dt}\tan t\right) dt$$
$$= \left[\sec^{n-2} t \tan t\right]_0^x - \int_0^x \tan t(n-2)\sec^{n-3} t \sec t \tan t \,dt$$
$$= \sec^{n-2} x \tan x - (n-2)\int_0^x \sec^{n-2} t \tan^2 t \,dt$$
$$= \sec^{n-2} x \tan x - (n-2)\int_0^x \sec^{n-2} t(\sec^2 t - 1)\,dt$$
$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$
$$(n-1)I_n = \sec^{n-2} x \tan x - (n-2)I_{n-2}$$
$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}.$$

(ii) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^4 t \,dt = I_4$

$$= \frac{\tan\frac{\pi}{3}\sec^2\frac{\pi}{3}}{3} + \frac{2}{3}I_2$$
$$= \frac{\sqrt{3}(4)}{3} + \frac{2}{3}\int_0^{\frac{\pi}{3}} \sec^2 t \,dt$$
$$= \frac{4\sqrt{3}}{3} + \frac{2}{3}\left[\tan t\right]_0^{\frac{\pi}{3}}$$
$$= \frac{4\sqrt{3}}{3} + \frac{2}{3}(\sqrt{3})$$
$$= \frac{6\sqrt{3}}{3}$$
$$= 2\sqrt{3}.$$

(c) (i) Check the formula when $n = 1$.

$$x_1 = 2\left[\frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}\right]$$
$$= 2 \times \frac{\frac{2}{3}}{\frac{4}{3}}$$
$$= 1 \quad \text{as required.}$$

Assume the formula is true for $n = k$,

that is, assume $x_k = 2\left[\dfrac{1 + \alpha^k}{1 - \alpha^k}\right]$.

Prove the formula is true for $n = k + 1$,

that is, prove $x_{k+1} = 2\left[\dfrac{1 + \alpha^{k+1}}{1 - \alpha^{k+1}}\right]$.

Now $x_{k+1} = \dfrac{4+x_k}{1+x_k}$ from the definition

$$= \frac{4+2\left[\dfrac{1+\alpha^k}{1-\alpha^k}\right]}{1+2\left[\dfrac{1+\alpha^k}{1-\alpha^k}\right]} \text{ by assumption}$$

$$= \frac{4(1-\alpha^k)+2(1+\alpha^k)}{1-\alpha^k+2(1+\alpha^k)}$$

$$= \frac{6-2\alpha^k}{3+\alpha^k}$$

$$= 2\left[\frac{3-\alpha^k}{3+\alpha^k}\right]$$

$$= 2\left[\frac{1-\frac{1}{3}\alpha^k}{1+\frac{1}{3}\alpha^k}\right]$$

$$= 2\left[\frac{1+\left(-\frac{1}{3}\right)\alpha^k}{1-\left(-\frac{1}{3}\right)\alpha^k}\right]$$

$$= 2\left[\frac{1+\alpha^{k+1}}{1-\alpha^{k+1}}\right], \text{ using } \alpha = -\frac{1}{3}.$$

This is the required formula. Hence by the principle of mathematical induction, the result is true for all $n > 1$.

(ii) $\lim_{n\to\infty}\left(-\dfrac{1}{3}\right)^n = 0$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2\left[\frac{1+\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)^n}\right]$$

$$= 2\left(\frac{1+0}{1-0}\right)$$

$$= 2.$$

QUESTION 8

(a) Suppose $0 \le t \le \dfrac{1}{\sqrt{2}}$.

(i)

$$0 \le t^2 \le \frac{1}{2}$$

$$0 \ge -t^2 \ge -\frac{1}{2}$$

$$1 \ge 1-t^2 \ge \frac{1}{2}$$

$$\therefore\quad 1 \le \frac{1}{1-t^2} \le 2$$

$$2t^2 \le \frac{2t^2}{1-t^2} \le 4t^2$$

$$\therefore\quad 0 \le \frac{2t^2}{1-t^2} \le 4t^2, \text{ since } 2t^2 \ge 0.$$

(ii)

$$\frac{1}{1+t}+\frac{1}{1-t}-2 = \frac{(1-t)+(1+t)-2(1-t^2)}{(1+t)(1-t)}$$

$$= \frac{2-2+2t^2}{1-t^2}$$

$$= \frac{2t}{1-t^2}.$$

Hence from (i),

$$0 \le \frac{1}{1+t}+\frac{1}{1-t}-2 \le 4t^2 \text{ for } 0 \le t \le \frac{1}{\sqrt{2}}.$$

(iii) For $0 \le x \le \dfrac{1}{\sqrt{2}}$, we have

$$\int_0^x 0\,dt \le \int_0^x\left(\frac{1}{1+t}+\frac{1}{1-t}-2\right)dt \le \int_0^x 4t^2\,dt$$

$$0 \le \left[\log_e(1+t)-\log_e(1-t)-2t\right]_0^x \le \left[\frac{4t^3}{3}\right]_0^x$$

$$0 \le \left[\log_e\left(\frac{1+x}{1-x}\right)-2x\right]-0 \le \frac{4x^3}{3}$$

$$\therefore\quad 0 \le \log_e\left(\frac{1+x}{1-x}\right)-2x \le \frac{4x^3}{3}.$$

(iv) Raise all the expressions in (iii) to the power of e: Note that e^x is monotonic increasing for all x. That is, if $a \le b$, then $e^a \le e^b$.

So (iii) becomes

$$e^0 \le e^{\ln\left(\frac{1+x}{1-x}\right)-2x} \le e^{\frac{4x^3}{3}}$$

$$1 \le e^{\ln\left(\frac{1+x}{1-x}\right)}\times e^{-2x} \le e^{\frac{4x^3}{3}}$$

$$1 \le \left(\frac{1+x}{1-x}\right)e^{-2x} \le e^{\frac{4x^3}{3}},$$

provided $0 \le x \le \dfrac{1}{\sqrt{2}}$.

(b) $f(x) = x^n e^{-x}$, where $n \ge 2$ is an integer.

(i)

$$f'(x) = nx^{n-1}e^{-x} - x^n e^{-x}$$

$$= (n-x)x^{n-1}e^{-x}.$$

Note that the stationary point between the two points of inflexion is at $x = n$.

$$f''(x) = n(n-1)x^{n-2}e^{-x} - 2nx^{n-1}e^{-x} + x^n e^{-x}$$

$$= e^{-x}x^{n-2}\left[x^2-2nx+n(n-1)\right].$$

The points of inflexion satisfy $f''(x) = 0$ and since a and b are both positive, they are the solutions of

$$x^2-2nx+n(n-1) = 0$$

$$(x-n)^2 = n$$

$$x = n \pm \sqrt{n}.$$

Hence $a = n-\sqrt{n}$ and $b = n+\sqrt{n}$.

(ii)

$$\frac{f(b)}{f(a)} = \frac{(n+\sqrt{n})^n e^{-(n+\sqrt{n})}}{(n-\sqrt{n})^n e^{-(n-\sqrt{n})}}$$

$$= \left(\frac{n+\sqrt{n}}{n-\sqrt{n}}\right)^n e^{-2\sqrt{n}}$$

$$= \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}}.$$

(iii) Substitute $x = \dfrac{1}{\sqrt{n}}$ into (a)(iv): Note that since $n \ge 2$, therefore $\dfrac{1}{\sqrt{n}} \le \dfrac{1}{\sqrt{2}}$ and so $x = \dfrac{1}{\sqrt{n}}$ satisfies the condition for (a)(iv) to be true.

$$\therefore\quad 1 \le \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)e^{-\frac{2}{\sqrt{n}}} \le e^{\frac{4}{3n\sqrt{n}}}.$$

Raise all these expressions to the power of n. Note that for $n \ge 2$ and $x > 0$ the function x^n is monotonic increasing. That is, if $0 < p \le q$, then $p^n \le q^n$.

So these inequalities become

$$1 \le \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)^n\left(e^{-\frac{2}{\sqrt{n}}}\right)^n \le \left(e^{\frac{4}{3n\sqrt{n}}}\right)^n$$

$$1 \le \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}} \le e^{\frac{4}{3\sqrt{n}}}$$

$$\therefore\quad 1 \le \frac{f(b)}{f(a)} \le e^{\frac{4}{3\sqrt{n}}}, \text{ using (ii).}$$

(iv)

$$\lim_{n\to\infty} e^{\frac{4}{3\sqrt{n}}} = e^0 = 1.$$

$$\therefore\quad \frac{f(b)}{f(a)} \to 1 \text{ as } n \to \infty.$$

END OF EXTENSION 2 SOLUTIONS