BOARD OF STUDIES

NEW SOUTH WALES

2007

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Write $\left(1+\sqrt{5}\right)^3$ in the form $a+b\sqrt{5}$, where a and b are integers. **2**

(b) The interval AB, where A is $(4, 5)$ and B is $(19, -5)$, is divided internally in the ratio 2 : 3 by the point $P(x, y)$. Find the values of x and y. **2**

(c) Differentiate $\tan^{-1}(x^4)$ with respect to x. **2**

(d) The graphs of the line $x-2y+3=0$ and the curve $y=x^3+1$ intersect at $(1, 2)$. Find the exact value, in radians, of the acute angle between the line and the tangent to the curve at the point of intersection. **3**

(e) Use the substitution $u=25-x^2$ to evaluate $\displaystyle\int_3^4 \frac{2x}{\sqrt{25-x^2}}\,dx$. **3**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) By using the substitution $t=\tan\frac{\theta}{2}$, or otherwise, show that $\dfrac{1-\cos\theta}{\sin\theta}=\tan\frac{\theta}{2}$. **2**

(b) Let $f(x)=2\cos^{-1}x$.

(i) Sketch the graph of $y=f(x)$, indicating clearly the coordinates of the endpoints of the graph. **2**

(ii) State the range of $f(x)$. **1**

(c) The polynomial $P(x)=x^2+ax+b$ has a zero at $x=2$. When $P(x)$ is divided by $x+1$, the remainder is 18. **3**

Find the values of a and b.

(d) A skydiver jumps from a hot air balloon which is 2000 metres above the ground. The velocity, v metres per second, at which she is falling t seconds after jumping is given by $v=50(1-e^{-0.2t})$.

(i) Find her acceleration ten seconds after she jumps. Give your answer correct to one decimal place. **2**

(ii) Find the distance that she has fallen in the first ten seconds. Give your answer correct to the nearest metre. **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Find the volume of the solid of revolution formed when the region bounded by the curve $y = \dfrac{1}{\sqrt{9+x^2}}$, the x-axis, the y-axis and the line $x = 3$, is rotated about the x-axis. **3**

(b) (i) Find the vertical and horizontal asymptotes of the hyperbola $y = \dfrac{x-2}{x-4}$ and hence sketch the graph of $y = \dfrac{x-2}{x-4}$. **3**

(ii) Hence, or otherwise, find the values of x for which $\dfrac{x-2}{x-4} \le 3$. **2**

(c) A particle is moving in a straight line with its acceleration as a function of x given by $\ddot{x} = -e^{-2x}$. It is initially at the origin and is travelling with a velocity of 1 metre per second.

(i) Show that $\dot{x} = e^{-x}$. **2**

(ii) Hence show that $x = \log_e(t+1)$. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) In a large city, 10% of the population has green eyes.

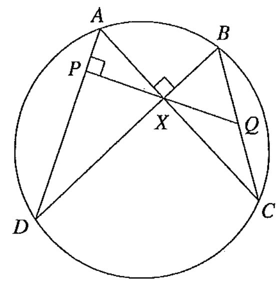
(i) What is the probability that two randomly chosen people both have green eyes? **1**

(ii) What is the probability that exactly two of a group of 20 randomly chosen people have green eyes? Give your answer correct to three decimal places. **1**

(iii) What is the probability that more than two of a group of 20 randomly chosen people have green eyes? Give your answer correct to two decimal places. **2**

(b) Use mathematical induction to prove that $7^{2n-1} + 5$ is divisible by 12, for all integers $n \ge 1$. **3**

(c)



NOT TO SCALE

The diagram shows points A, B, C and D on a circle. The lines AC and BD are perpendicular and intersect at X. The perpendicular to AD through X meets AD at P and BC at Q.

Copy or trace this diagram into your writing booklet.

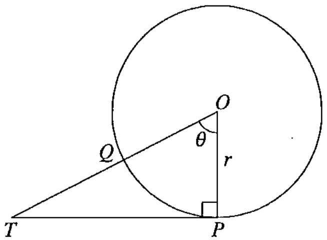
(i) Prove that $\angle QXB = \angle QBX$. **3**

(ii) Prove that Q bisects BC. **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



The points P and Q lie on the circle with centre O and radius r. The arc PQ subtends an angle θ at O. The tangent at P and the line OQ intersect at T, as shown in the diagram.

(i) The arc PQ divides triangle TPO into two regions of equal area. **2**

Show that $\tan\theta = 2\theta$.

(ii) A first approximation to the solution of the equation $2\theta - \tan\theta = 0$ is $\theta = 1.15$ radians. Use one application of Newton's method to find a better approximation. Give your answer correct to four decimal places. **2**

(b) Mr and Mrs Roberts and their four children go to the theatre. They are randomly allocated six adjacent seats in a single row. **2**

What is the probability that the four children are allocated seats next to each other?

(c) Find the exact values of x and y which satisfy the simultaneous equations **3**

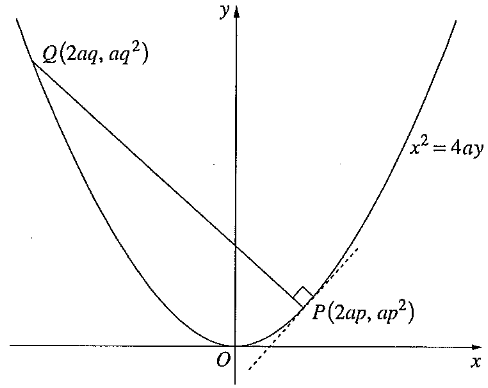
$$\sin^{-1}x + \frac{1}{2}\cos^{-1}y = \frac{\pi}{3} \quad \text{and}$$

$$3\sin^{-1}x - \frac{1}{2}\cos^{-1}y = \frac{2\pi}{3}.$$

Question 5 continues on page 7

Marks

(d)



The diagram shows a point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$. The normal to the parabola at P intersects the parabola again at $Q(2aq, aq^2)$.

The equation of PQ is $x + py - 2ap - ap^3 = 0$. (Do NOT prove this.)

(i) Prove that $p^2 + pq + 2 = 0$. **1**

(ii) If the chords OP and OQ are perpendicular, show that $p^2 = 2$. **2**

End of Question 5

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Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) A particle moves in a straight line. Its displacement, x metres, after t seconds is given by

$$x = \sqrt{3}\sin 2t - \cos 2t + 3.$$

(i) Prove that the particle is moving in simple harmonic motion about $x = 3$ by showing that $\ddot{x} = -4(x-3)$. **2**

(ii) What is the period of the motion? **1**

(iii) Express the velocity of the particle in the form $\dot{x} = A\cos(2t - \alpha)$, where α is in radians. **2**

(iv) Hence, or otherwise, find all times within the first π seconds when the particle is moving at 2 metres per second in either direction. **2**

(b) Consider the function $f(x) = e^x - e^{-x}$.

(i) Show that $f(x)$ is increasing for all values of x. **1**

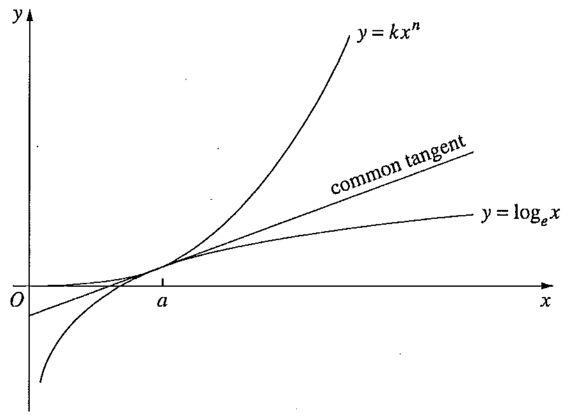
(ii) Show that the inverse function is given by **3**

$$f^{-1}(x) = \log_e\left(\frac{x + \sqrt{x^2 + 4}}{2}\right).$$

(iii) Hence, or otherwise, solve $e^x - e^{-x} = 5$. Give your answer correct to two decimal places. **1**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graphs of the functions $y = kx^n$ and $y = \log_e x$ have a common tangent at $x = a$, as shown in the diagram.

(i) By considering gradients, show that $a^n = \dfrac{1}{nk}$. **1**

(ii) Express k as a function of n by eliminating a. **2**

Question 7 continues on page 11

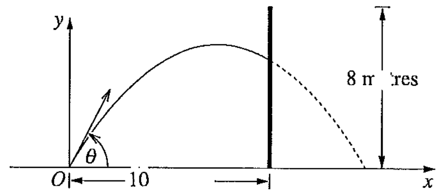
(b) A small paintball is fired from the origin with initial velocity 14 metres per second towards an eight-metre high barrier. The origin is at ground level, 10 metres from the base of the barrier.

The equations of motion are

$$x = 14t\cos\theta$$

$$y = 14t\sin\theta - 4.9t^2$$

where θ is the angle to the horizontal at which the paintball is fired and t is the time in seconds. (Do NOT prove these equations of motion.)



(i) Show that the equation of trajectory of the paintball is **2**

$$y = mx - \left(\frac{1+m^2}{40}\right)x^2, \quad \text{where } m = \tan\theta.$$

(ii) Show that the paintball hits the barrier at height h metres when **2**

$$m = 2 \pm \sqrt{3 - 0.4h}\,.$$

Hence determine the maximum value of h.

(iii) There is a large hole in the barrier. The bottom of the hole is 3.9 metres above the ground and the top of the hole is 5.9 metres above the ground. The paintball passes through the hole if m is in one of two intervals. One interval is $2.8 \le m \le 3.2$. **2**

Find the other interval.

(iv) Show that, if the paintball passes through the hole, the range is **3**

$$\frac{40m}{1+m^2} \text{ metres.}$$

Hence find the widths of the two intervals in which the paintball can land at ground level on the other side of the barrier.

End of paper

2007 Higher School Certificate Solutions

Mathematics Extension 1

Question 1

(a) $\left(1+\sqrt{5}\right)^3 = 1+3\sqrt{5}+3\left(\sqrt{5}\right)^2+\left(\sqrt{5}\right)^3$

$= 1+3\sqrt{5}+15+5\sqrt{5}$

$= 16+8\sqrt{5}.$

(b) $A(4, 5)$ $B(19, -5)$ $P(x, y)$

$x = \dfrac{mx_2+nx_1}{m+n}, \quad y = \dfrac{my_2+ny_1}{m+n}$

When $m:n = 2:3$,

$x = \dfrac{2\times 19+3\times 4}{2+3}, \quad y = \dfrac{2\times -5+3\times 5}{2+3}$

$= \dfrac{50}{5} \qquad\qquad = \dfrac{5}{5}$

$\therefore\ x = 10$ and $y = 1.$

(c) $\dfrac{d}{dx}\left(\tan^{-1}\left(x^4\right)\right)$

Let $y = \tan^{-1} u$, where $u = x^4$.

Then $\dfrac{dy}{du} = \dfrac{1}{1+u^2} \qquad \dfrac{du}{dx} = 4x^3$

$\therefore\ \dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx}$

$= \dfrac{1}{1+u^2}\times 4x^3$

$= \dfrac{4x^3}{1+\left(x^4\right)^2}$

$= \dfrac{4x^3}{1+x^8}.$

(d) For $x-2y+3=0$, $m_1 = \dfrac{1}{2}$

For $y = x^3+1$,

$\dfrac{dy}{dx} = 3x^2$

When $x=1$, $m_2 = 3(1)^2 = 3$

$\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$ (where θ is the acute angle between line and tangent)

$= \left|\dfrac{\frac{1}{2}-3}{1+\frac{1}{2}\times 3}\right|$

$= 1$

$\therefore\ \theta = \tan^{-1} 1$

$= \dfrac{\pi}{4}$ radians.

(e) Let $u = 25-x^2$

$\dfrac{du}{dx} = -2x$

When $x=3$, $u = 25-(3)^2 = 16$

When $x=4$, $u = 25-(4)^2 = 9$

$\therefore\ \displaystyle\int_3^4 \frac{2x}{\sqrt{25-x^2}}\,dx = \int_{16}^{9} \frac{-du}{\sqrt{u}}$

$= \displaystyle\int_9^{16} u^{-\frac{1}{2}}\,du$

$= \left[\dfrac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_9^{16}$

$= 2\left[\sqrt{16}-\sqrt{9}\right]$

$= 2.$

Question 2

(a) Let $t = \tan\dfrac{\theta}{2}$

Then $\cos\theta = \dfrac{1-t^2}{1+t^2}, \quad \sin\theta = \dfrac{2t}{1+t^2}$

$\therefore$ LHS $= \dfrac{1-\cos\theta}{\sin\theta}$

$= \dfrac{1-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$

$= \dfrac{\left(1+t^2\right)-\left(1-t^2\right)}{\left(1+t^2\right)}\times\dfrac{\left(1+t^2\right)}{2t}$

$= \dfrac{2t^2}{2t}$

$= t$

$= \tan\dfrac{\theta}{2}$

= RHS.

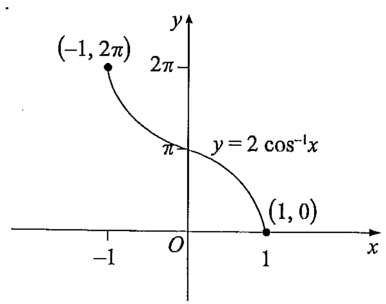
(b) (i) Let $y = f(x) = 2\cos^{-1} x$

$\therefore\ \dfrac{y}{2} = \cos^{-1} x$

Domain: $-1 \le x \le 1$

Range: $0 \le \dfrac{y}{2} \le \pi$

$\therefore\ 0 \le y \le 2\pi.$



(ii) Range: $0 \le f(x) \le 2\pi.$

(c) $P(x) = x^2+ax+b$

Zero at $x=2$, $\therefore\ P(2)=0$

$(2)^2+a(2)+b=0$

$4+2a+b=0$ —①

$P(-1)=18$, by remainder theorem.

$(-1)^2+a(-1)+b=18$

$1-a+b=18$ —②

From ①: $2a+b=-4$ —③

From ②: $-a+b=17$ —④

③ − ④: $3a=-21$

$a=-7$

In ①: $4+2\times -7+b=0$

$\therefore\ b=10$

$\therefore\ a=-7, b=10$

(d) (i) $v = 50\left(1-e^{-0.2t}\right)$

$a = \dfrac{dv}{dt}$

$= 50\times 0.2\,e^{-0.2t}$

$= 10\,e^{-0.2t}$

When $t=10$,

$a = 10\,e^{-0.2\times 10}$

$= 10\,e^{-2}$

$= 1.3533\ldots$

$\therefore\ a = 1.4$ m/s² (to 1 d.p.).

(ii) $\dfrac{dx}{dt} = 50\left(1-e^{-0.2t}\right)$

$x = \displaystyle\int_0^{10} 50\left(1-e^{-0.2t}\right)dt$

$= 50\left[t+\dfrac{e^{-0.2t}}{0.2}\right]_0^{10}$

$= 50\left[\left(10+\dfrac{e^{-2}}{0.2}\right)-\dfrac{1}{0.2}\right]$

$= 283.8338\ldots$

$\therefore$ Distance $\doteqdot 284$ m (to the nearest m).

Question 3

(a) $V=\int_0^3 \pi y^2\,dx,$

where $y=\dfrac{1}{\sqrt{9+x^2}}$

$=\pi\int_0^3 \dfrac{1}{9+x^2}\,dx$

$=\pi\left[\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}\right]_0^3$

$=\dfrac{\pi}{3}\left[\tan^{-1}1-\tan^{-1}0\right]$

$=\dfrac{\pi}{3}\left[\dfrac{\pi}{4}-0\right]$

$\therefore$ Volume $=\dfrac{\pi}{12}$ units^3.

(b) (i) $y=\dfrac{x-2}{x-4},\ x\neq 4$

$\therefore$ Vertical asymptote is $x=4$.

Horizontal asymptote as $x\to\pm\infty$.

METHOD 1:

$$\lim_{x\to\infty}\left(\frac{x-2}{x-4}\right)=\lim_{x\to\infty}\left(\frac{1-\frac{2}{x}}{1-\frac{4}{x}}\right)=1$$

$\therefore$ Horizontal asymptote is $y=1$.

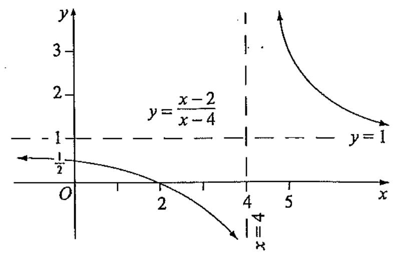
METHOD 2: $y=\dfrac{x-2}{x-4}=\dfrac{x-4+2}{x-4}$

$=1+\dfrac{2}{x-4}$

As $x\to\infty,\ \dfrac{2}{x-4}\to 0, y\to 1$

$\therefore$ Horizontal asymptote is $y=1$.

When $x=0$, $y=\dfrac{1}{2}$ and when $y=0$, $x=2$.



(ii) When $y=3$, $\dfrac{x-2}{x-4}=3$

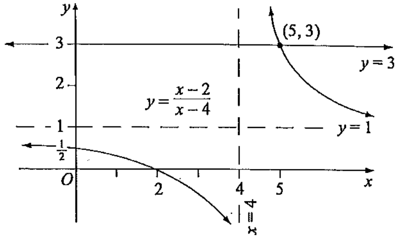
$x-2=3x-12$

$10=2x$

$x=5$

$\therefore$ When $\dfrac{x-2}{x-4}\le 3$,

the graph of $y=\dfrac{x-2}{x-4}$ intersects with or is below the graph of $y=3$.



This occurs when $x<4$ or $x\ge 5$.

(c) (i) $\ddot{x}=-e^{-2x}$

$\dfrac{d}{dx}\dot{x}=-e^{-2x}$

$\dfrac{d}{dx}\left(\dfrac{1}{2}\dot{x}^2\right)=-e^{-2x}$

$\dfrac{1}{2}\dot{x}^2=\int -e^{-2x}\,dx$

$\dfrac{1}{2}\dot{x}^2=\dfrac{e^{-2x}}{2}+c$

When $t=0$, $x=0$, $\dot{x}=1$.

$\dfrac{1}{2}(1)^2=\dfrac{e^0}{2}+c$

$\dfrac{1}{2}=\dfrac{1}{2}+c$

$\therefore\quad c=0$

$\therefore\ \dfrac{1}{2}\dot{x}^2=\dfrac{e^{-2x}}{2}$

$\dot{x}^2=e^{-2x}$

$\dot{x}=\pm e^{-x}$

Initially, $\dot{x}>0$, $\therefore\ \dot{x}=e^{-x}$.

(ii) $\dot{x}=\dfrac{dx}{dt}=e^{-x}$

$\dfrac{dt}{dx}=e^{x}$

$t=\int e^x\,dx$

$\therefore\ t=e^x+c$

When $t=0$, $x=0$.

$0=e^0+c$

$\therefore\ c=-1$

$\therefore\ t=e^x-1$

$e^x=t+1$

$x=\log_e(t+1)$.

Question 4

(a) (i) $P(\text{green})=0.1$, $P(\text{not green})=0.9$

$P(2\text{ green})=0.1\times 0.1$

$=0.01$.

(ii) $P(\text{exactly 2 green})$

$={}^{20}C_2(0.1)^2(0.9)^{18}$

$=0.2851\ldots$

$\doteqdot 0.285$ (to 3 d.p.).

(iii) $P(\text{more than 2 green})$

$=1-P(0,1\text{ or }2)$

$=1-\left({}^{20}C_0(0.1)^0(0.9)^{20}+{}^{20}C_1(0.1)^1(0.9)^{19}+{}^{20}C_2(0.1)^2(0.9)^{18}\right)$

$=1-(0.6769\ldots)$

$=0.3230\ldots$

$\doteqdot 0.32$ (to 2 d.p.).

(b) For $n\ge 1$.

For $n=1$, $7^{2-1}+5=12$, which is divisible by 12.

$\therefore$ Statement is true for $n=1$.

Assume true for $n=k$,

i.e. $7^{2k-1}+5=12M$

where M is an integer.

$\therefore\ 7^{2k-1}=12M-5$

Required to prove true for $n=k+1$,

i.e. $7^{2(k+1)-1}+5=7^{2k+1}+5$

is divisible by 12.

When $n=k+1$,

$7^{2k+1}+5=7^2\times 7^{2k-1}+5$

$=49(12M-5)+5$, by assumpti

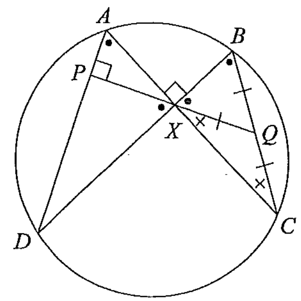
$=12\times 49M-48\times 5$

$=12(49M-20)$, which is divisi by 12 as M is a integer.

$\therefore$ The statement is true for $n=k+1$.

$\therefore$ By the principle of mathematical induction, the result is true for all integers $n\ge 1$.

(c)



(i)

$\angle QXB=\angle PXD$ (vertically opposite $\angle$s)

$\angle AXP=90°-\angle PXD$ ($AC\perp BD$)

$\angle PAX=90°-\angle AXD$ (angle sum of ΔAPX)

$=90°-(90°-\angle PXD)$

$=\angle PXD$

$\angle PAX=\angle QBX$ ($\angle$s in same segment on

$\therefore\ \angle PXD=\angle QBX$ (both equal to $\angle PAX$)

$\therefore\ \angle QXB=\angle QBX$ (both equal to $\angle PXD$).

(c) (ii) $\angle QXB = \angle QBX$ (from (i))

$\therefore \Delta BXQ$ is isosceles.

$\therefore QB = QX$ (sides opposite equal $\angle$s of ΔBXQ)

$\angle QXC = 90° - \angle QXB$ ($AC \perp BD$)

Also, $\angle BCX = 90° - \angle QBX$ (angle sum of ΔXBC)

$= 90° - \angle QXB$ (from (i))

$\therefore QXC = BCX$

$\therefore \Delta XQC$ is isosceles.

$\therefore QX = QC$ (sides opposite equal $\angle$'s of ΔQXC)

$\therefore QB = QC$ (both equal to QX)

$\therefore Q$ bisects BC.

Question 5

(a) (i) Area of sector $POQ = \frac{1}{2}r^2\theta$

Area of $\Delta TPO = \frac{1}{2} \times PT \times r$

$2 \times$ area sector $OPQ =$ area ΔTPO

$\therefore r^2\theta = \frac{1}{2} \times PT \times r$

Now, $\tan\theta = \frac{PT}{r}$, $\therefore PT = r\tan\theta$

$\therefore r^2\theta = \frac{1}{2} \times r\tan\theta \times r$

$r^2\theta = \frac{1}{2}r^2\tan\theta$

$\therefore \tan\theta = 2\theta.$

(ii) Let $f(x) = 2\theta - \tan\theta.$

Where $\theta = 1.15$ radians (1st approx.)

Then $f'(x) = 2 - \sec^2\theta$

Newton's method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = 1\cdot15 - \frac{2\times1\cdot15 - \tan1\cdot15}{2-\sec^2 1\cdot15}$$

$$= 1\cdot15 - \frac{0\cdot0655\ldots}{-3\cdot9929\ldots}$$

$$\doteqdot 1\cdot1664 \text{ (to 4 d.p.)}$$

$\therefore \theta = 1.1664$ radians is a better approximation.

(b) Number of arrangements of children = 4!

Considering the children as one group, number of arrangements of 2 adults and the children as one group = 3!

Total arrangements = 6!

$$P(\text{children sit together}) = \frac{3!4!}{6!} = \frac{1}{5}.$$

(c)

$$\sin^{-1}x + \frac{1}{2}\cos^{-1}y = \frac{\pi}{3} \quad \text{—①}$$

$$3\sin^{-1}x - \frac{1}{2}\cos^{-1}y = \frac{2\pi}{3} \quad \text{—②}$$

①+②:

$$4\sin^{-1}x = \pi$$

$$\sin^{-1}x = \frac{\pi}{4}$$

$$x = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Substituting in ①,

$$\frac{\pi}{4} + \frac{1}{2}\cos^{-1}y = \frac{\pi}{3}$$

$$\frac{1}{2}\cos^{-1}y = \frac{\pi}{3} - \frac{\pi}{4}$$

$$\frac{1}{2}\cos^{-1}y = \frac{\pi}{12}$$

$$\cos^{-1}y = \frac{\pi}{6}$$

$$y = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

(d) (i) *METHOD 1*

$$x + py - 2ap - ap^3 = 0$$

Substituting $Q(2aq, aq^2)$,

$$2aq + apq^2 - 2ap - ap^3 = 0$$

$$2aq - 2ap + apq^2 - ap^3 = 0$$

$$2a(q-p) + ap(q^2 - p^2) = 0$$

$$2a(q-p) + ap(q-p)(q+p) = 0$$

$$2a + ap(q+p) = 0$$

$$2 + p(q+p) = 0$$

$\therefore 2 + pq + p^2 = 0$ since $p \neq q$.

METHOD 2

$P(2ap, ap^2), Q(2ap, ap^2)$

Gradient of PQ:

$$\frac{aq^2 - ap^2}{2aq - 2ap} = \frac{a(q-p)(q+p)}{2a(q-p)} = \frac{p+q}{2}$$

Gradient of PQ from equation $= \frac{-1}{p}$

Equating gradients,

$$\frac{p+q}{2} = \frac{-1}{p}$$

$$p^2 + pq = -2$$

$\therefore p^2 + pq + 2 = 0.$

(ii) $m_{OP} = \frac{ap^2}{2ap} = \frac{p}{2}$ $\quad m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$

$OP \perp OQ.$

$\therefore m_{OP} \times m_{OQ} = -1$

$$\frac{p}{2} \times \frac{q}{2} = -1$$

$$pq = -4$$

From (i): $p^2 + pq + 2 = 0$

$\therefore p^2 - 4 + 2 = 0$

$\therefore p^2 = 2.$

Question 6

(a) (i) $x = \sqrt{3}\sin 2t - \cos 2t + 3$

i.e. $x - 3 = \sqrt{3}\sin 2t - \cos 2t$

$$\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$$

$$\ddot{x} = -4\sqrt{3}\sin 2t + 4\cos 2t$$

$$= -4\left(\sqrt{3}\sin 2t - \cos 2t\right)$$

$$= -4(x-3)$$

$\therefore$ The particle is moving in simple harmonic motion about $x = 3$.

(ii) $\ddot{x} = -n^2(x - x_0)$

$-4(x-3) = -n2(x-3)$ from (i)

$n^2 = 4$

$\therefore n = 2$

$\therefore$ Period $\frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ seconds.

(iii) $\dot{x} = A\cos(2t - \alpha)$

$= A\cos 2t\cos\alpha + A\sin 2t\sin\alpha$

$= (A\cos\alpha)\cos 2t + (A\sin\alpha)\sin 2t$

$= 2\sqrt{3}\cos 2t + 2\sin 2t$ from (i)

By equating coefficients,

$A\cos\alpha = 2\sqrt{3}$ ....①

$A\sin\alpha = 2$ ....②

②÷①: $\frac{\sin\alpha}{\cos\alpha} = \frac{1}{\sqrt{3}}$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \frac{\pi}{6}$

①² + ②²:

$A^2\cos^2\alpha + A^2\sin^2\alpha = \left(2\sqrt{3}\right)^2 + (2)^2$

$A^2\left(\sin^2\alpha + \cos^2\alpha\right) = 16$

$A^2 = 16$

$\therefore A = 4$

$$\therefore \dot{x} = 4\cos\left(2t - \frac{\pi}{6}\right).$$

(iv) $\dot{x} = 4\cos\left(2t - \frac{\pi}{6}\right)$

$\dot{x} = \pm 2$ m/s and $0 \le t \le \pi$

$\therefore\ 0 \le 2t \ge 2\pi$

$-\frac{\pi}{6} \le \left(2t - \frac{\pi}{6}\right) \le \frac{11\pi}{6}$

(1st, 4th quadrants)

When $\dot{x} = 2$,

$$2 = 4\cos\left(2t - \frac{\pi}{6}\right)$$

$$\cos\left(2t - \frac{\pi}{6}\right) = \frac{1}{2}$$

$$2t - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$2t = \frac{\pi}{2}, \frac{11\pi}{6}$$

$$\therefore\ t = \frac{\pi}{4}, \frac{11\pi}{12}$$

When $\dot{x} = -2$,

$$-2 = 4\cos\left(2t - \frac{\pi}{6}\right)$$

$$\therefore \cos\left(2t - \frac{\pi}{6}\right) = -\frac{1}{2}$$

$$2t - \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$$

(2nd, 3rd quadrants)

$$2t = \frac{5\pi}{6}, \frac{3\pi}{2}$$

$$\therefore\ t = \frac{5\pi}{12}, \frac{3\pi}{4}$$

$\therefore$ The times when the particle is moving at 2 m/s in either direction are $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$ and $\frac{11\pi}{12}$ seconds.

(b) $f(x) = e^x - e^{-x}$

(i) $f'(x) = e^x + e^{-x}$

$f'(x) > 0$ for all real x since $e^x > 0$ and $e^{-x} > 0$ for all real x.

$\therefore f(x)$ is increasing for all values of x.

(ii) Let $y = f^{-1}(x)$

Interchanging x and y gives

$$x = e^y - e^{-y}$$

$$= e^y - \frac{1}{e^y}$$

$$= \frac{e^{2y} - 1}{e^y}$$

$$x\,e^y = e^{2y} - 1$$

$$e^{2y} - x\,e^y - 1 = 0$$

Solving as a quadratic equation in terms of e^y gives:

$$e^y = \frac{x \pm \sqrt{x^2 + 4}}{2}$$

Since $e^y > 0$,

$$e^y = \frac{x + \sqrt{x^2 + 4}}{2}$$

$$\therefore\ y = \log_e\left(\frac{x + \sqrt{x^2 + 4}}{2}\right)$$

i.e. $f^{-1}(x) = \log_e\left(\frac{x + \sqrt{x^2 + 4}}{2}\right)$.

(iii) $e^x - e^{-x} = 5$

$f(x) = 5$

$\therefore\ x = f^{-1}(5)$

Using the result in (ii),

$$x = \log_e\left(\frac{5 + \sqrt{5^2 + 4}}{2}\right)$$

$$= 1.6472\ldots$$

$\therefore\ x \doteqdot 1.65$ (to 2 d.p.).

Question 7

(a) (i) For $y = kx^n$, $\frac{dy}{dx} = nkx^{n-1}$

For $y = \log_e x$, $\frac{dy}{dx} = \frac{1}{x}$

At $x = a$, $m_1 = nka^{n-1}$

$m_2 = \frac{1}{a}$

Since the graphs have a common tangent, the gradients are equal.

$$nka^{n-1} = \frac{1}{a}$$

i.e. $a^n = \frac{1}{nk}$.

(ii) At $x = a$, $y = ka^n$

and $y = \log_e a$

$\therefore \log_e a = ka^n$

$= k\left(\frac{1}{nk}\right)$ from (i)

$= \frac{1}{n}$

$\therefore\ a = e^{\frac{1}{n}}$

Now, $a^n = \left(e^{\frac{1}{n}}\right)^n = e$

$\therefore\ \frac{1}{nk} = e$ from (i)

$\therefore\ k = \frac{1}{ne}$.

(b) (i) $x = 14t\cos\theta$ ......①

$y = 14t\sin\theta - 4.9t^2$ ......②

From ①, $t = \frac{x}{14\cos\theta}$

Substituting into ②,

$$y = 14\left(\frac{x}{14\cos\theta}\right)\sin\theta - 4.9\left(\frac{x}{14\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{x^2}{40\cos^2\theta}$$

$$= x\tan\theta - \frac{x^2\sec^2\theta}{40}$$

$$= x\tan\theta - \frac{x^2\left(1 + \tan^2\theta\right)}{40}$$

$\therefore y = mx - \left(\frac{1 + m^2}{40}\right)x^2$, where $m = \tan\theta$.

(ii) The paintball hits the barrier at $(10, h)$.

$\therefore\ h = 10m - \left(\frac{1 + m^2}{40}\right)100$ from (i)

$$h = 10m - \left(\frac{1 + m^2}{2}\right)5$$

$$2h = 20m - 5 - 5m^2$$

$$5m^2 - 20m + 5 + 2h = 0$$

$$\therefore m = \frac{20 \pm \sqrt{(20)^2 - 4(5)(5 + 2h)}}{10}$$

$$= \frac{20 \pm \sqrt{300 - 40h}}{10}$$

$\therefore\ m = 2 \pm \sqrt{3 - 0.4h}$

For real value of m,

$3 - 0.4h \ge 0$

$-0.4h \ge -3$

$\therefore\ h \le 7.5$

$\therefore$ Maximum height is 7.5 m.

(iii) $m = 2 \pm \sqrt{3 - 0.4h}$

When $h = 3.9$,

$$m = 2 \pm \sqrt{3 - 0.4 \times 3.9}$$
$$= 2 \pm \sqrt{1.44}$$

$\therefore m = 0.8$ or 3.2

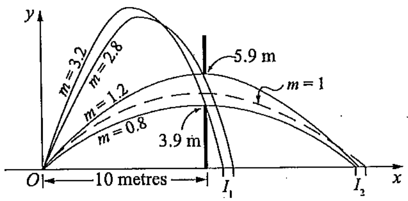
When $h = 5.9$,

$$m = 2 \pm \sqrt{3 - 0.4 \times 5.9}$$
$$= 2 \pm \sqrt{0.64}$$

$\therefore m = 1.2$ or 2.8

So the paintball passes through the hole if $0.8 \le m \le 1.2$ or $2.8 \le m \le 3.2$.

(iv)



If the paintball passes through the hole, the range occurs when $y = 0$.

When $y = 0$,

$$0 = mx - \left(\frac{1+m^2}{40}\right)x^2 \quad \text{from (i)}$$
$$= x\left[m - \left(\frac{1+m^2}{40}\right)x\right]$$
$$\therefore\ m - \left(\frac{1+m^2}{40}\right)x = 0$$
$$\left(\frac{1+m^2}{40}\right)x = m$$
$$\therefore\ x = \frac{40m}{1+m^2}$$

For interval I_1, $2.8 \le m \le 3.2$, range is

$$f(2.8) = \frac{40 \times 2.8}{1 + 2.8^2} = 12.6696\ldots$$
$$f(3.2) = \frac{40 \times 3.2}{1 + 3.2^2} = 11.3879\ldots$$

$\therefore$ Width of this interval

$= 12.6696\ldots - 11.3879\ldots$

$\doteqdot 1.28$ m (to 2 d.p.)

For interval I_2, $0.8 \le m \le 1.2$, maximum range occurs when $m = 1$ (i.e. $\tan 45^\circ$).

$$f(1) = \frac{40 \times 1}{1 + 1^2} = 20$$
$$f(0.8) = \frac{40 \times 0.8}{1 + 0.8^2} = 19.5121\ldots$$
$$f(1.2) = \frac{40 \times 1.2}{1 + 1.2^2} = 19.6721\ldots$$

$\therefore$ Width of this interval

$= 20 - 19.5121\ldots$

$\doteqdot 0.49$ m (to 2 d.p.).

End of Mathematics Extension 1 solutions