BOARD OF STUDIES
NEW SOUTH WALES

2007

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$. **2**

(b) Find $\displaystyle\int \tan^2 x \sec^2 x\,dx$. **2**

(c) Evaluate $\displaystyle\int_0^{\pi} x\cos x\,dx$. **3**

(d) Evaluate $\displaystyle\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx$. **4**

(e) It can be shown that **4**

$$\frac{2}{x^3+x^2+x+1} = \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}. \quad \text{(Do NOT prove this.)}$$

Use this result to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx$.

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 4 + i$ and $w = \bar{z}$. Find, in the form $x + iy$,

(i) w **1**

(ii) $w - z$ **1**

(iii) $\dfrac{z}{w}$. **1**

(b) (i) Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$. **2**

(ii) Hence, or otherwise, find $(1+i)^{17}$ in the form $a + ib$, where a and b are integers. **3**

(c) The point P on the Argand diagram represents the complex number z, where z satisfies **3**

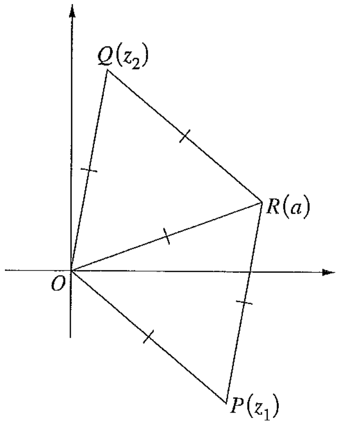
$$\frac{1}{z} + \frac{1}{\bar{z}} = 1.$$

Give a geometrical description of the locus of P as z varies.

Question 2 continues on page 5

Question 2 (continued) **Marks**

(d)



The points P, Q and R on the Argand diagram represent the complex numbers z_1, z_2 and a respectively.

The triangles OPR and OQR are equilateral with unit sides, so $|z_1| = |z_2| = |a| = 1$.

Let $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Explain why $z_2 = \omega a$. **1**

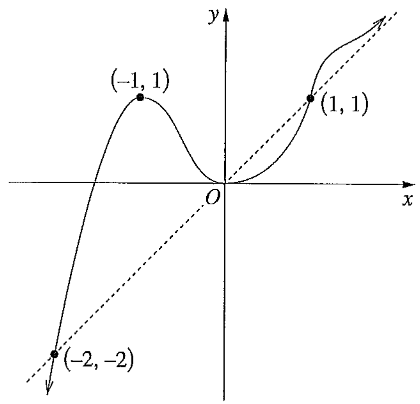
(ii) Show that $z_1 z_2 = a^2$. **1**

(iii) Show that z_1 and z_2 are the roots of $z^2 - az + a^2 = 0$. **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the graph of $y = f(x)$. The line $y = x$ is an asymptote.

Draw separate one-third page sketches of the graphs of the following:

(i) $f(-x)$ **1**

(ii) $f(|x|)$ **2**

(iii) $f(x) - x$. **2**

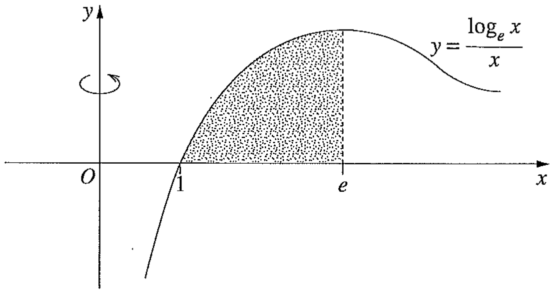
(b) The zeros of $x^3 - 5x + 3$ are α, β and γ. **2**

Find a cubic polynomial with integer coefficients whose zeros are 2α, 2β and 2γ.

Question 3 continues on page 7

Question 3 (continued) **Marks**

(c) 4

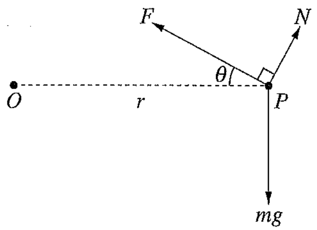


Use the method of cylindrical shells to find the volume of the solid formed when the shaded region bounded by

$$y = 0, \quad y = \frac{\log_e x}{x}, \quad x = 1 \text{ and } x = e$$

is rotated about the y-axis.

(d)



A particle P of mass m undergoes uniform circular motion with angular velocity ω in a horizontal circle of radius r about O. It is acted on by the force due to gravity, mg, a force F directed at an angle θ above the horizontal and a force N which is perpendicular to F, as shown in the diagram.

(i) By resolving forces horizontally and vertically, show that 3

$$N = mg\cos\theta - mr\omega^2\sin\theta.$$

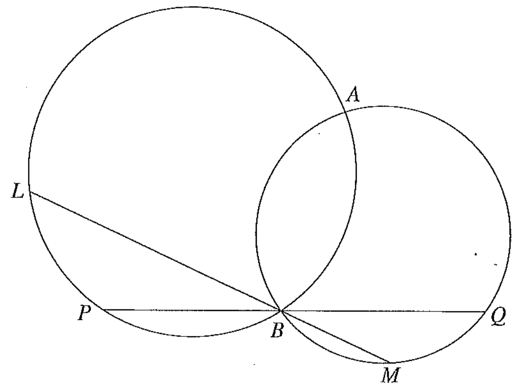
(ii) For what values of ω is $N > 0$? 1

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) 2



Two circles intersect at A and B.

The lines LM and PQ pass through B, with L and P on one circle and M and Q on the other circle, as shown in the diagram.

Copy or trace this diagram into your writing booklet.

Show that $\angle LAM = \angle PAQ$.

(b) (i) Show that $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$. 2

(ii) Show that $4\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right) = \sin 3\theta$. 2

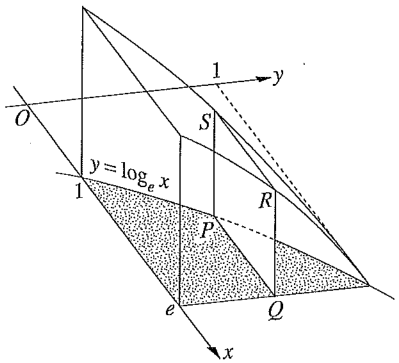
(iii) Write down the maximum value of $\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right)$. 1

Question 4 continues on page 9

Question 4 (continued)

Marks

(c) **3**



The base of a solid is the region bounded by the curve $y = \log_e x$, the x-axis and the lines $x = 1$ and $x = e$, as shown in the diagram.

Vertical cross-sections taken through this solid in a direction parallel to the x-axis are squares. A typical cross-section, $PQRS$, is shown.

Find the volume of the solid.

(d) The polynomial $P(x) = x^3 + qx^2 + rx + s$ has real coefficients. It has three distinct zeros, α, $-\alpha$ and β.

(i) Prove that $qr = s$. **3**

(ii) The polynomial does not have three real zeros. Show that two of the zeros are purely imaginary. (A number is purely imaginary if it is of the form iy, with y real and $y \neq 0$.) **2**

End of Question 4

Marks

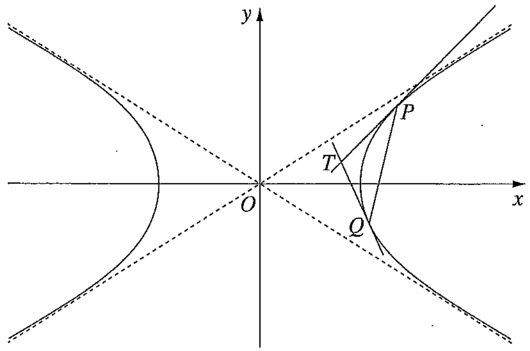
Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A bag contains 12 red marbles and 12 yellow marbles. Six marbles are selected at random without replacement.

(i) Calculate the probability that exactly three of the selected marbles are red. Give your answer correct to two decimal places. **1**

(ii) Hence, or otherwise, calculate the probability that more than three of the selected marbles are red. Give your answer correct to two decimal places. **2**

(b)



The points at $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The tangents at P and Q meet at $T(x_0, y_0)$.

(i) Show that the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. **2**

(ii) Hence show that the chord of contact, PQ, has equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. **2**

(iii) The chord PQ passes through the focus $S(ae, 0)$, where e is the eccentricity of the hyperbola. Prove that T lies on the directrix of the hyperbola. **1**

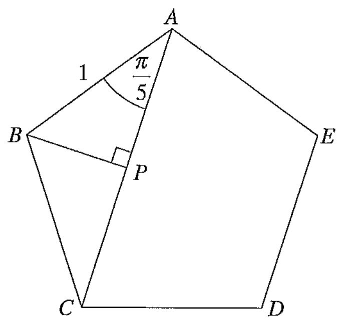
Question 5 continues on page 11

Question 5 (continued) **Marks**

(c) (i) Write $(x-1)(5-x)$ in the form $b^2-(x-a)^2$, where a and b are real numbers. **1**

(ii) Using the values of a and b found in part (i) and making the substitution $x-a=b\sin\theta$, or otherwise, evaluate $\int_1^5 \sqrt{(x-1)(5-x)}\,dx$. **2**

(d)



In the diagram, $ABCDE$ is a regular pentagon with sides of length 1. The perpendicular to AC through B meets AC at P.

Copy or trace this diagram into your writing booklet.

(i) Let $u=\cos\frac{\pi}{5}$. **2**

Use the cosine rule in ΔACD to show that $8u^3-8u^2+1=0$.

(ii) One root of $8x^3-8x^2+1=0$ is $\frac{1}{2}$. **2**

Find the other roots of $8x^3-8x^2+1=0$ and hence find the exact value of $\cos\frac{\pi}{5}$.

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use the binomial theorem **1**

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + b^n$$

to show that, for $n \geq 2$,

$$2^n > \binom{n}{2}.$$

(ii) Hence show that, for $n \geq 2$, **2**

$$\frac{n+2}{2^{n-1}} < \frac{4n+8}{n(n-1)}.$$

(iii) Prove by induction that, for integers $n \geq 1$, **3**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(iv) Hence determine the limiting sum of the series **1**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots.$$

Question 6 continues on page 13

Question 6 (continued)

Marks

(b) A raindrop falls vertically from a high cloud. The distance it has fallen is given by

$$x = 5\log_e\left(\frac{e^{1.4t} + e^{-1.4t}}{2}\right)$$

where x is in metres and t is the time elapsed in seconds.

(i) Show that the velocity of the raindrop, v metres per second, is given by **2**

$$v = 7\left(\frac{e^{1.4t} - e^{-1.4t}}{e^{1.4t} + e^{-1.4t}}\right).$$

(ii) Hence show that **2**

$$v^2 = 49\left(1 - e^{-\frac{2x}{5}}\right).$$

(iii) Hence, or otherwise, show that **2**

$$\ddot{x} = 9.8 - 0.2v^2.$$

(iv) The physical significance of the 9.8 in part (iii) is that it represents the acceleration due to gravity. **1**

What is the physical significance of the term $-0.2v^2$?

(v) Estimate the velocity at which the raindrop hits the ground. **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\sin x < x$ for $x > 0$. **2**

(ii) Let $f(x) = \sin x - x + \frac{x^3}{6}$. Show that the graph of $y = f(x)$ is concave up for $x > 0$. **2**

(iii) By considering the first two derivatives of $f(x)$, show that $\sin x > x - \frac{x^3}{6}$ for $x > 0$. **2**

Question 7 continues on page 16

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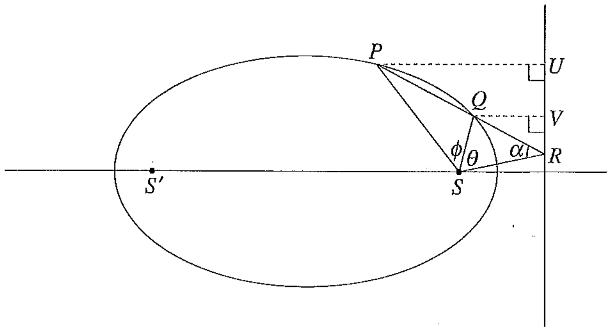
Marks

(b)



In the diagram the secant PQ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the directrix at R. Perpendiculars from P and Q to the directrix meet the directrix at U and V respectively. The focus of the ellipse which is nearer to R is at S.

Copy or trace this diagram into your writing booklet.

(i) Prove that $\frac{PR}{QR} = \frac{PU}{QV}$. **1**

(ii) Prove that $\frac{PU}{QV} = \frac{PS}{QS}$. **1**

(iii) Let $\angle PSQ = \phi$, $\angle RSQ = \theta$ and $\angle PRS = \alpha$. **2**

By considering the sine rule in ΔPRS and ΔQRS, and applying the results of part (i) and part (ii), show that $\phi = \pi - 2\theta$.

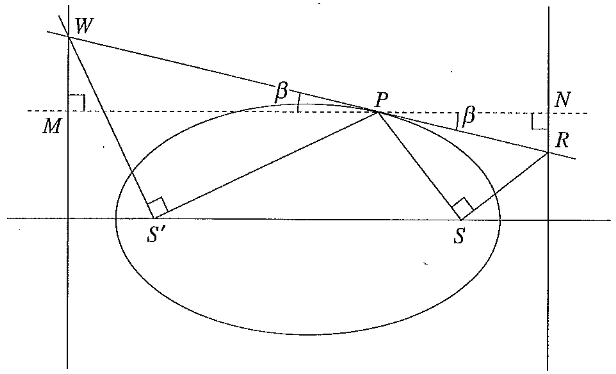
(iv) Let Q approach P along the circumference of the ellipse, so that $\phi \to 0$. **1**

What is the limiting value of θ?

Question 7 continues on page 17

Marks

(c)



The diagram shows an ellipse with eccentricity e and foci S and S'.

The tangent at P on the ellipse meets the directrices at R and W. The perpendicular to the directrices through P meets the directrices at N and M as shown. Both $\angle PSR$ and $\angle PS'W$ are right angles.

Let $\angle MPW = \angle NPR = \beta$.

(i) Show that **2**

$$\frac{PS}{PR} = e\cos\beta$$

where e is the eccentricity of the ellipse.

(ii) By also considering $\frac{PS'}{PW}$ show that $\angle RPS = \angle WPS'$. **2**

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Using a suitable substitution, show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

(ii) A function $f(x)$ has the property that $f(x)+f(a-x)=f(a)$. **2**

Using part (i), or otherwise, show that

$$\int_0^a f(x)\,dx = \frac{a}{2}f(a).$$

(b) (i) Let n be a positive integer. Show that if $z^2 \neq 1$ then **2**

$$1 + z^2 + z^4 + \cdots + z^{2n-2} = \left(\frac{z^n - z^{-n}}{z - z^{-1}}\right) z^{n-1}.$$

(ii) By substituting $z = \cos\theta + i\sin\theta$, where $\sin\theta \neq 0$, into part (i), show that **3**

$$1 + \cos 2\theta + \cdots + \cos(2n-2)\theta + i\left[\sin 2\theta + \cdots + \sin(2n-2)\theta\right]$$
$$= \frac{\sin n\theta}{\sin\theta}\left[\cos(n-1)\theta + i\sin(n-1)\theta\right].$$

(iii) Suppose $\theta = \frac{\pi}{2n}$. Using part (ii), show that **2**

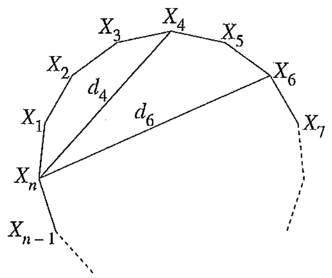
$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \cot\frac{\pi}{2n}.$$

Question 8 continues on page 19

Marks

Question 8 (continued)

(c)



The diagram shows a regular n-sided polygon with vertices $X_1, X_2, \cdots, X_n$. Each side has unit length. The length d_k of the 'diagonal' X_nX_k where $k = 1, 2, \cdots, n-1$ is given by

$$d_k = \frac{\sin\frac{k\pi}{n}}{\sin\frac{\pi}{n}}. \qquad \text{(Do NOT prove this.)}$$

(i) Show, using the result in part (b) (iii), that **2**

$$d_1 + \cdots + d_{n-1} = \frac{1}{2\sin^2\frac{\pi}{2n}}.$$

(ii) Let p be the perimeter of the polygon and $q = \frac{1}{n}(d_1 + \cdots + d_{n-1})$. **2**

Show that

$$\frac{p}{q} = 2\left(n\sin\frac{\pi}{2n}\right)^2.$$

(iii) Hence calculate the limiting value of $\frac{p}{q}$ as $n \to \infty$. **1**

End of paper

2007 Higher School Certificate Solutions
Mathematics Extension 2

Question 1

(a) *METHOD 1*

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c.$$

METHOD 2

Let $u = 2x$, $du = 2dx$

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{3^2-(2x)^2}}\,dx.$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{3^2-u^2}}\,du$$

$$= \frac{1}{2}\sin^{-1}\frac{u}{3} + c$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c.$$

(b) Let $u = \tan x$, $du = \sec^2 x\,dx$

$$\int \tan^2 x \sec^2 x\,dx = \int u^2\,du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{\tan^3 x}{3} + c.$$

(c) Let $u = x$, $du = dx$
$v = \sin x$, $dv = \cos x\,dx$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\therefore \int_0^{\pi} x\cos x\,dx = [x\sin x]_0^{\pi} - \int_0^{\pi} \sin x \, \ldots$$

$$= (\pi\sin\pi - 0) - [-\cos \ldots$$

$$= 0 - (-\cos\pi + \cos 0)$$

$$= -(1+1)$$

$$= -2.$$

(d) *METHOD 1*

Let $u = 1-x$, $du = -dx$
When $x = 0$, $u = 1$
When $x = \frac{3}{4}$, $u = \frac{1}{4}$

$$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx = \int_1^{\frac{1}{4}} \frac{1-u}{\sqrt{u}}\,(-du)$$

$$= \int_{\frac{1}{4}}^{1} \left(\frac{1}{\sqrt{u}} - \sqrt{u}\right) du$$

$$= \int_{\frac{1}{4}}^{1} \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{1}{4}}^{1}$$

$$= \left(2 - \frac{2}{3}\right) - \left(2\times\frac{1}{2} - \frac{2}{3}\times\left(\ldots\right.\right.$$

$$= 2 - \frac{2}{3} - 1 + \frac{1}{12}$$

$$= \frac{5}{12}.$$

METHOD 2

Let $u^2 = 1-x$, $2u\,du = -dx$
When $x = 0$, $u = 1$
When $x = \frac{3}{4}$, $u = \frac{1}{2}$

$$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx = \int_1^{\frac{1}{2}} \frac{(1-u^2)}{u}(-2u\,du)$$

$$= 2\int_{\frac{1}{2}}^{1} (1-u^2)\,du$$

$$= 2\left[u - \frac{u^3}{3}\right]_{\frac{1}{2}}^{1}$$

$$= 2\left[\left(1-\frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right)\right]$$

$$= \frac{5}{12}.$$

(e) $\displaystyle\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx$

$$= \int_{\frac{1}{2}}^{2} \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx,$$

using the given result

$$= \left[\ln(x+1) - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x\right]_{\frac{1}{2}}^{2}$$

$$= \left(\ln 3 - \frac{1}{2}\ln 5 + \tan^{-1} 2\right)$$

$$- \left(\ln\left(\frac{3}{2}\right) - \frac{1}{2}\ln\frac{5}{4} + \tan^{-1}\frac{1}{2}\right)$$

$$= \ln\left(3\times\frac{1}{\sqrt{5}}\times\frac{2}{3}\times\frac{\sqrt{5}}{2}\right) + \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}2 - \tan^{-1}\frac{1}{2}.$$

Note that $\tan^{-1}2 - \tan^{-1}\frac{1}{2}$ can be 'simplified' to $\tan^{-1}\frac{3}{4}$.

Question 2

(a) (i) $w = \overline{z}$
$= 4 - i.$

(ii) $w - z = 4 - i - (4 + i)$
$= 4 - i - 4 - i$
$= -2i.$

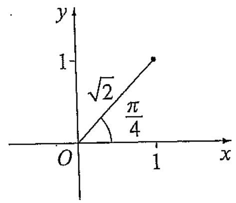
(iii)
$$\frac{z}{w} = \frac{4+i}{4-i}\times\frac{4-i}{4+i}$$

$$= \frac{(4+i)^2}{16+1}$$

$$= \frac{16+8i-1}{17}$$

$$= \frac{15}{17} + \frac{8i}{17}.$$

(b) (i)



$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

(ii)

$$(1+i)^{17} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{17}$$

$$= \left(\sqrt{2}\right)^{17}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$$

by de Moivre's theorem

$$= 256\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= 256(1+i)$$

$$= 256 + 256i.$$

(c) Let $z = x + iy$

$\therefore$ If $\dfrac{1}{z} + \dfrac{1}{\bar{z}} = 1$

$$\frac{1}{x+iy} + \frac{1}{x-iy} = 1$$

$$\frac{2x}{x^2+y^2} = 1$$

$$2x = x^2 + y^2$$

$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

$\therefore$ The locus of P as z varies is the circle, centre (1, 0) with radius 1.

(d) (i) $|z_2| = |a| = 1$

$\angle QOR = \dfrac{\pi}{3}$ (ΔOQR is equilateral)

Multiplication by $\omega = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$ is a rotation about O in the anticlockwise direction through an angle of $\dfrac{\pi}{3}$.

$$\therefore z_2 = a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= \omega a.$$

(ii) $|z_1| = |a| = 1$

$\angle ROP = \dfrac{\pi}{3}$ (ΔOPR is equilateral)

Similarly to (i),

$$a = z_1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= z_1\omega$$
$$\therefore \quad z_1 = \frac{a}{\omega}$$

Now $z_2 = \omega a$ from (i)

$$\therefore z_1z_2 = \frac{a}{\omega}\cdot\omega a$$
$$= a^2.$$

(iii) Observing that

$$z_1 = a\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z_1 + z_2 = a\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) + a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= a\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) + a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 2a\cos\frac{\pi}{3}$$
$$= a$$

Now $z_1z_2 = a^2$ from (ii).

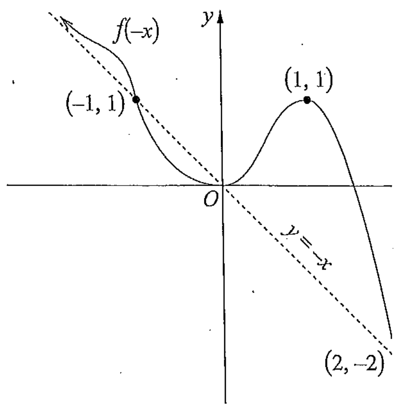
$\therefore$ The equation with roots z_1 and z_2

$$z^2 - (z_1 + z_2)z + z_1z_2 = 0$$

i.e. $z_1^2 - az + a^2 = 0$.

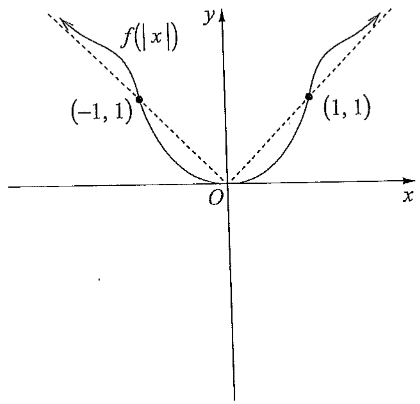
Question 3

(a) (i) The graph of $f(-x)$ is a reflectior the graph of $f(x)$ about the y-axi

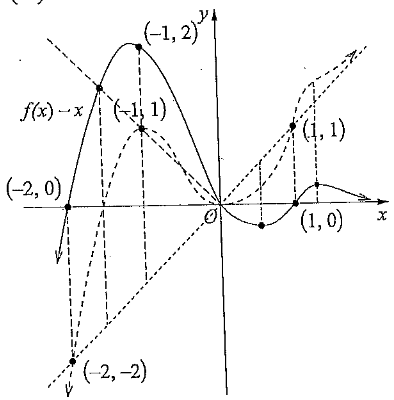


(ii) $f(|x|) = f(x)$ when $x \geq 0$.

$f(|x|) = f(-x)$ when $x < 0$.



(iii)



Note that the gradient is –1 at the points (–1, 2) and (0, 0).

(b) *METHOD 1*

Use the subsitution $m = 2x$ $\left(\text{i.e. } x = \dfrac{m}{2}\right)$ to create a polynomial with the required roots: $2\alpha, 2\beta, 2\gamma$.

$$x^3 - 5x + 3 = \left(\frac{m}{2}\right)^3 - 5\times\frac{m}{2} + 3$$
$$= \frac{m^3}{8} - \frac{5m}{2} + 3.$$

In terms of x this is $\dfrac{x^3}{8} - \dfrac{5x}{2} + 3$.

Any multiple of this polynomial would also be correct. The simplest example with integer coefficients is

$$8\left(\frac{x^3}{8} - \frac{5x}{2} + 3\right) = x^3 - 20x + 24.$$

METHOD 2

$$\alpha + \beta + \gamma = 0$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -5$$
$$\alpha\beta\gamma = -3$$

Let the new polynomial be

$$x^3 + bx^2 + cx + d$$

$$-b = 2\alpha + 2\beta + 2\gamma$$
$$= 2(0)$$
$$\therefore b = 0$$

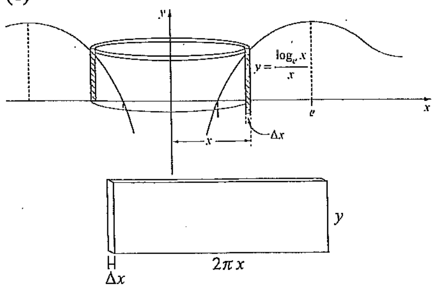
$$c = (2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma)$$
$$= 4(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= 4(-5)$$
$$= -20$$

$$-d = (2\alpha)(2\beta)(2\gamma)$$
$$= 8\alpha\beta\gamma$$
$$= 8(-3)$$
$$= -24$$
$$\therefore d = 24$$

Since these coefficients are all integers, a suitable polynomial is

$$x^3 + 0x^2 - 20x + 24$$
$$= x^3 - 20x + 24.$$

(c)



$$\Delta V = 2\pi x \times y \times \Delta x$$

$$V = \int_1^e 2\pi xy\, dx$$

$$= 2\pi \int_1^e x \frac{\log_e x}{x}\, dx$$

$$= 2\pi \int_1^e \log_e x\, dx$$

Let $u = \log_e dx, \quad du = \frac{dx}{x}$

$v = x, \qquad dv = dx$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

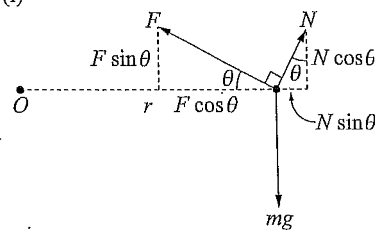
$$\therefore V = 2\pi \left[\left[x \log_e x \right]_1^e - \int_1^e x \times \frac{1}{x}\, dx \right]$$

$$= 2\pi \left[(e - 0) - [x]_1^e \right]$$

$$= 2\pi [e - e + 1]$$

2π units^3.

(d) (i)



Vertically: $F \sin\theta + N \cos\theta - mg = 0$

$F \sin\theta + N \cos\theta = mg$ —①

Horizontally: $F \cos\theta - N \sin\theta = m\omega^2 r$ —②

①× cos θ − ②× sin θ

$$F \cos\theta \sin\theta + N \cos^2\theta - F \cos\theta \sin\theta + N \sin^2 \theta$$

$$= mg \cos\theta - m\omega^2 r \sin\theta$$

$$N\left(\cos^2\theta + \sin^2\theta\right) = mg\cos\theta - m\omega^2 r \sin\theta$$

$$\therefore \quad N = mg\cos\theta - m\omega^2 r \sin\theta.$$

(ii) $N > 0$ implies

$$mg\cos\theta > m\omega^2 r \sin\theta$$

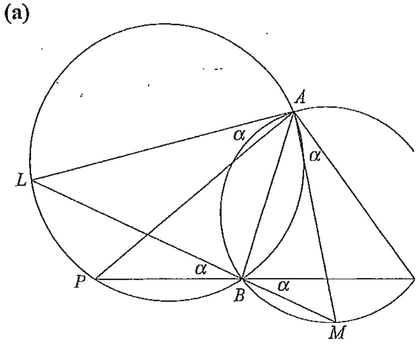
$$\frac{mg\cos\theta}{mr\sin\theta} > \omega^2$$

$$\omega^2 < \frac{g}{r\tan\theta}$$

$$|\omega| < \sqrt{\frac{g}{r\tan\theta}}.$$

Question 4

(a)



Let $\angle LAP = \alpha$

$\therefore \quad \angle LBP = \angle LAP$ (∠s in the same segment on

$= \alpha$

$\therefore \angle QBM = \angle LBP$ (Vertically opposite

$= \alpha$

$\therefore \angle QAM = \angle QBM$ (∠s in the same segment on

$= \alpha$

$\therefore \quad \angle LAP = \angle QAM$

Now $\angle LAM = \angle LAP + \angle PAM$ (from ab

$= \angle QAM + \angle PAM$

$\therefore \angle LAM = \angle PAQ.$

(b) (i) *METHOD 1*

$\cos 3\theta + i \sin 3\theta = (\cos\theta + i\sin\theta)^3$

by de Moivre's theorem

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

Equating imaginary parts,

$i\sin 3\theta = 3i\cos^2\theta i\sin\theta - i\sin^3\theta$

i.e. $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta.$

METHOD 2

$$\sin 3\theta = \sin(2\theta + \theta)$$

$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$

$$= 2\sin\theta\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)\sin\theta$$

$$= 3\sin\theta\cos^2\theta - \sin^3\theta.$$

(ii) $4\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right)$

$$= 4\sin\theta\left(\sin\theta\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\theta\right)\left(\sin\theta\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\cos\theta\right)$$

$$= 4\sin\theta\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)\left(-\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)$$

$$= 4\sin\theta\left(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\right)$$

$$= 3\sin\theta\cos^2\theta - \sin^3\theta$$

$= \sin 3\theta$ from (i) above.

(iii)

$$\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right)$$

$= \frac{1}{4}\sin 3\theta$ from (ii) above

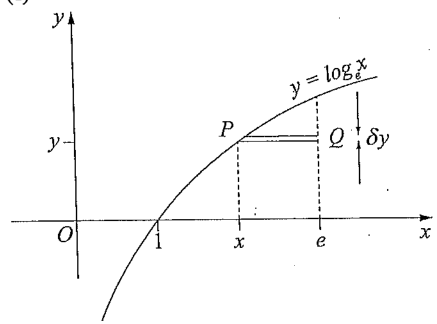
As the maximum value of $\sin 3\theta$ is 1,

the maximum value of $\frac{1}{4}\sin 3\theta$ is $\frac{1}{4}$.

∴ Maximum value of

$\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right)$ is $\frac{1}{4}$.

(c)



METHOD 1

$PQ = e - x$

∴ Area of square $PQRS = (e - x)^2$

The volume of the slice is

$\delta V = (e - x)^2\,\delta y$

∴ Volume of the solid is

$$V = \lim_{\delta y \to 0} \sum \delta V$$

$$= \int_{x=1}^{x=e} (e - x)^2\, dy$$

But $y = \log_e x$

$$dy = \frac{1}{x}\, dx$$

$$\therefore V=\int_1^e (e-x)^2 \cdot \frac{1}{x}\,dx$$

$$=\int_1^e \left(\frac{e^2}{x}-2e+x\right)dx$$

$$=\left[e^2\log_e x-2ex+\frac{x^2}{2}\right]_1^e$$

$$=\left(e^2\log_e e-2e^2+\frac{e^2}{2}\right)-\left(e^2\log_e 1-2e+\frac{1}{2}\right)$$

$$=-\frac{e^2}{2}+2e-\frac{1}{2}$$

$\therefore$ Volume of solid is $-\frac{e^2}{2}+2e-\frac{1}{2}$ units^3.

METHOD 2

$y=\log_e x$

$\therefore \quad x=e^y$

$\therefore PQ=e-e^y$

$\therefore$ Area of square $PQRS=\left(e-e^y\right)^2$

$\delta V=\left(e-e^y\right)^2\delta y$

$\therefore$ Volume of the solid is

$$V=\lim_{\delta y\to 0}\Sigma\delta V$$

$$=\int_0^1\left(e-e^y\right)^2 dy$$

$$=\int_0^1\left(e^2-2e^{y+1}+e^{2y}\right)dy$$

$$=\left[e^2y-2e^{y+1}+\frac{1}{2}e^{2y}\right]_0^1$$

$$=\left(e^2-2e^2+\frac{1}{2}e^2\right)-\left(0-2e+\frac{1}{2}e^0\right)$$

$$=-\frac{e^2}{2}+2e-\frac{1}{2}$$

$\therefore$ Volume of solid is $-\frac{e^2}{2}+2e-\frac{1}{2}$ units^3.

(d) (i) Sum of roots $=\alpha+-\alpha+\beta$

$=-q$

$\therefore \beta=-q$

Sum of roots two at a time

$=-\alpha^2+\alpha\beta-\alpha\beta$

$=r$

$\therefore a^2=r$

Sum of roots three at a time

$=-\alpha^2\beta$

$=-s$

METHOD 1

$r\beta=-s$

$r\times -q=-s$

$\therefore qr=s$

METHOD 2

Since $\beta=-q$ is a root, substitute $x=-q$ into the polynomial

$-q^3+q\cdot(-q)^2-qr+s=0$

$\therefore qr=s$

(ii) *METHOD 1*

Since the polynomial has real coefficients, complex zeros occur ir conjugate pairs.

We know that one zero, $\beta=-q$, is r since q is a coefficient

$\therefore \alpha$ and $-\alpha$ are complex, and must be conjugate pairs

i.e. $-\alpha=\overline{\alpha}$

Let $\alpha=a+ib$

$\therefore -(a+ib)=\overline{a+ib}$

$-a-ib=a-ib$

$2a=0$

$a=0$

$\therefore \alpha$ and $-\alpha$ are purely imaginary

METHOD 2

Since $\beta=-q$ is a zero then $x+q$ is a factor.

Using polynomial division and $qr=s$ from (i),

$x^3+qx^2+rx+s=(x+q)(x^2+r)$

$\therefore$ Zeros are $-q$ and $\pm i\sqrt{r}$

i.e. Two zeros are purely imaginar

Question 5

(a) (i) $P(3\text{ red, }3\text{ yellow})$

$$=\frac{{}^{12}C_3\times{}^{12}C_3}{{}^{24}C_6}$$

$=0.3595\ldots$

$\doteqdot 0.36$ (to 2 d.p.).

(ii) *METHOD 1*

$P(>3\text{ red})$

$=P(4\text{ red, }2\text{ yellow})$

$+P(5\text{ red, }1\text{ yellow})$

$+P(6\text{ red, }0\text{ yellow})$

$$=\frac{{}^{12}C_4\times{}^{12}C_2+{}^{12}C_5\times{}^{12}C_1+{}^{12}C_6\,{}^{12}C_0}{{}^{24}C_6}$$

$=0.3202\ldots$

$\doteqdot 0.32$ (to 2 d.p.).

METHOD 2

By symmetry,

$P(>3\text{ red})=P(<3\text{ red})$

$\therefore P(>3\text{ red})=\frac{1}{2}(1-P(3\text{ red}))$

$\doteqdot\frac{1}{2}(1-0.36)$ from (i)

$\doteqdot 0.32$ (to 2 d.p.).

(b) (i) Differentiating

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ implicitly

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$\therefore$ The gradient of the curve at $P(x_1,y_1)$ is $\frac{b^2x_1}{a^2y_1}$.

The equation of the tangent at P is

$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1y-a^2y_1^2=b^2x_1x-b^2x_1^2$$

$$b^2x_1x-a^2y_1y=b^2x_1^2-a^2y_1^2$$

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

$$\therefore \quad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$$

since (x_1,y_1) lies on the ellipse.

(ii) $T(x_0,y_0)$ belongs to TP and TQ.

$\therefore$ Its coordinates must satisfy the tangents at both P and Q,

i.e. $\frac{x_1x_0}{a^2}-\frac{y_1y_0}{b^2}=1$ for P

and

$\frac{x_2x_0}{a^2}-\frac{y_2y_0}{b^2}=1$ for Q

$\therefore$ The equation of PQ is

$$\frac{x\,x_0}{a^2}-\frac{y\,y_0}{b^2}=1$$

as (x,y) can be replaced by (x_1,y_1) or (x_2,y_2).

(iii) Substituting (x, y) with $(ae, 0)$ in the equation of PQ,

$\frac{ae\,x_0}{a^2}-0=1$ from (ii)

$\therefore \quad x_0=\frac{a}{e}$

$\therefore T$ lies on the directrix.

(c) (i) $(x-1)(5-x)=-5+6x-x^2$

$=-\left(x^2-6x+5\right)$

$=-\left[(x-3)^2-9+5\right]$

$=4-(x-3)^2$

$=2^2-(x-3)^2$

which matches the given form with $a=3$ and $b=2$ (or -2).

(ii)

$$\int_1^5 \sqrt{(x-1)(5-x)}\,dx = \int_1^5 \sqrt{2^2-(x-3)^2}\,dx$$

Let $x-3=2\sin\theta$

$dx = 2\cos\theta\,d\theta$

When $x=1$, $1-3=2\sin\theta$

$\sin\theta=-1$

$\theta=-\dfrac{\pi}{2}$

When $x=5$, $5-3=2\sin\theta$

$\sin\theta=1$

$\theta=\dfrac{\pi}{2}$

$$\therefore \int_1^5 \sqrt{2^2-(x-3)^2}\,dx$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2^2-2^2\sin^2\theta}\;2\cos\theta\,d\theta$$

$$=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$=8\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \quad \text{since } \cos^2\theta \text{ is even}$$

$$=8\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$=4\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$=4\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)-(0)\right]$$

$=2\pi.$

Note: The substitution and subsequent working can be avoic by realising that the curve is a semi-circle of radius 2 having arı $\dfrac{\pi\times 2^2}{2}=2\pi.$

(d) (i) $AD = AC = 2AP$

$=2\cos\dfrac{\pi}{5}$

$=2u$

The angle sum of a pentagon is

$(n-2)\pi=(5-2)\pi$

$=3\pi$

$\therefore\ \angle BAE = \dfrac{3\pi}{5}$

By symmetry,

$\angle DAE = \angle CAB$

$=\dfrac{\pi}{5}$

$\therefore\ \angle CAD = \dfrac{3\pi}{5} - 2\times\dfrac{\pi}{5}$

$=\dfrac{\pi}{5}$

Using the cosine rule in ΔACD,

$CD^2 = AC^2 + AD^2 - 2AC.AD$ co

$1^2 = (2u)^2 + (2u)^2 - 2.2u.2u.u$

$1 = 8u^2 - 8u^3$

$\therefore\ 8u^3 - 8u^2 + 1 = 0.$

(ii) $8x^3 - 8x^2 + 1 = 0$

Since $x=\dfrac{1}{2}$ is a root,

$2x-1$ is a factor.

$\therefore\ 8x^3-8x^2+1=(2x-1)(4x^2+bx\cdot$

where b can be determined from the coefficient of x (or x^2) on both sides

$\therefore\ 0=-1\times b+2\times -1$

$b=-2$

$\therefore\ 8x^3-8x^2+1=(2x-1)(4x^2-2x$

The other roots satisfy

$4x^2-2x-1=0$

Using the quadratic formula,

$x=\dfrac{2\pm\sqrt{20}}{8}$

$=\dfrac{1\pm\sqrt{5}}{4}$

i.e. $\cos\dfrac{\pi}{5}=\dfrac{1\pm\sqrt{5}}{4}$ from (i)

Since $\cos\dfrac{\pi}{5}>0$,

$\cos\dfrac{\pi}{5}=\dfrac{1+\sqrt{5}}{4}.$

Question 6

(a) (i) $(a+b)^n =$

$$a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n}b^n$$

Let $a=b=1$

$$\therefore (1+1)^n = 1+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}$$

Since $\dbinom{n}{r}>0$ for all $r=1, 2, 3\ldots\ n,$

$2^n>\dbinom{n}{2}$

(ii) $2^n>\dbinom{n}{2}$ for $n\ge 2$

$2^n>\dfrac{n!}{2!(n-2)!}$

$2^n>\dfrac{n(n-1)}{2}$

$\dfrac{1}{2^n}<\dfrac{2}{n(n-1)}$

$\dfrac{2(n+2)}{2^n}<\dfrac{4(n+2)}{n(n-1)}$

$\therefore\ \dfrac{n+2}{2^{n-1}}<\dfrac{4n+8}{n(n-1)}.$

(iii) When $n=1$,

LHS $=1$

RHS $=4-\dfrac{1+2}{2^0}$

$=1$

$=$ LHS

$\therefore$ The equation is true for $n=1$.

Let k be a value of n for which the result is true,

i.e. $1+2\left(\dfrac{1}{2}\right)+3\left(\dfrac{1}{2}\right)^2+\ldots+$ —①

$=4-\dfrac{k+2}{2^{k-1}}$

We need to show that the result is true for $n=k+1$,

i.e. $1+2\left(\dfrac{1}{2}\right)+3\left(\dfrac{1}{2}\right)^2+\ldots+k\left(\dfrac{1}{2}\right)^{k-1}$

$+(k+1)\left(\dfrac{1}{2}\right)^k=4-\dfrac{k+3}{2^k}$ —②

In ②:

LHS $=4-\dfrac{k+2}{2^{k-1}}+(k+1)\left(\dfrac{1}{2}\right)^k$ from ①

$=4-\dfrac{k+2}{2^{k-1}}+\dfrac{k+1}{2^k}$

$=4-\dfrac{2(k+2)-(k+1)}{2^k}$

$=4-\dfrac{2k+4-k-1}{2^k}$

$=4-\dfrac{k+3}{2^k}$

$=$ RHS

$\therefore$ When the result is true for $n=k$, it is also true for $n=k+1$.

$\therefore$ By the principle of mathematical induction, the result is true for all integers $n\ge 1$.

(iv)

From (ii): $\dfrac{n+2}{2^{n-1}} < \dfrac{4n+8}{n(n-1)}$

$$0 < \lim_{n\to\infty} \frac{n+2}{2^{n-1}} < \lim_{n\to\infty} \frac{4n+8}{n(n-1)} \quad \text{as } n>0$$

But $\displaystyle\lim_{n\to\infty} \frac{4n+8}{n(n-1)} = 0$

$$\therefore \quad \lim_{n\to\infty} \frac{n+2}{2^{n-1}} = 0$$

Now

$$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\ldots = \lim_{n\to\infty}\left(4-\frac{n+2}{2^{n-1}}\right)$$
$$= 4.$$

(b) (i)

$$x = 5\log_e\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)$$

$$\frac{dx}{dt} = 5\cdot\frac{2}{e^{1.4t}+e^{-1.4t}}\cdot\frac{1.4e^{1.4t}-1.4e^{-1.4t}}{2}$$
$$= \frac{5(1.4)\left(e^{1.4t}-e^{-1.4t}\right)}{e^{1.4t}+e^{-1.4t}}$$
$$\therefore \quad v = \frac{7\left(e^{1.4t}-e^{-1.4t}\right)}{e^{1.4t}+e^{-1.4t}}$$

(ii)

From (i): $v^2 = 7^2\left(\dfrac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)^2$

$$= 49\left(\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)^2 \quad \text{—①}$$

Now $1-e^{-\frac{2x}{5}} = 1-e^{-2\log_e\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)}$

$$= 1-e^{\log_e\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)^{-2}}$$
$$= 1-\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)^{-2}$$
$$= 1-\frac{4}{\left(e^{1.4t}+e^{-1.4t}\right)^2}$$
$$= \frac{\left(e^{1.4t}+e^{-1.4t}\right)^2-4}{\left(e^{1.4t}+e^{-1.4t}\right)^2}$$
$$= \frac{e^{2.8t}+2+e^{-2.8t}-4}{\left(e^{1.4t}+e^{-1.4t}\right)^2}$$
$$= \left(\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)^2$$

From ①: $v^2 = 49\left(\dfrac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)^2$

$$\therefore \; v^2 = 49\left(1-e^{-\frac{2x}{5}}\right).$$

(iii) $\dot{x}^2 = v^2$

$$2\ddot{x} = \frac{d}{dx}v^2$$
$$\therefore \ddot{x} = \frac{1}{2}\frac{d}{dx}v^2$$
$$= \frac{1}{2}\frac{d}{dx}\left(49\left(1-e^{-\frac{2x}{5}}\right)\right)$$
$$= \frac{49}{2}\left(\frac{2}{5}e^{-\frac{2x}{5}}\right)$$
$$= 9.8\,e^{-\frac{2x}{5}}$$
$$= -9.8\left[\left(1-e^{-\frac{2x}{5}}\right)-1\right]$$
$$= -9.8\left(\frac{v^2}{49}\right)+9.8$$
$$= 9.8-0.2v^2.$$

(iv) The term $-0\cdot 2v^2$ in $\ddot{x}$ represents the deceleration due to resistive forces su[ch] as air resistance.

(v) *METHOD 1*

Assuming that the particle hits the ground with terminal velocity,

$$\ddot{x} = 0$$

i.e. $9.8-0.2v^2 = 0$ from (ii)

$$v^2 = 49$$
$$|v| = 7$$

$\therefore$ the raindrop hits the ground at 7 ms^{-1}.

METHOD 2

$$\lim_{t\to\infty} v = \lim_{t\to\infty} 7\left(\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)$$
$$= 7$$

$\therefore$ the raindrop hits the ground at 7 ms^{-1}.

Question 7

(a) (i) Let $g(x) = \sin x - x$

$$g'(x) = \cos x - 1$$
$$< 0 \quad \text{for} \quad x > 0$$

$\therefore$ $g(x)$ is decreasing for all $x > 0$.

When $x = 0$, $g(0) = 0$

$\therefore$ When $x > 0$, $g(x) < 0$

$\therefore$ $\sin x - x < 0$

$\sin x < x$ for $x > 0$.

(ii)

$$f(x) = \sin x - x + \frac{x^3}{6}$$
$$f'(x) = \cos x - 1 + \frac{x^2}{2}$$
$$f''(x) = -\sin x + x$$

$\sin x < x$ for $x > 0$ from (i)

$\therefore f''(x) > 0$ for $x > 0$

$\therefore f(x)$ is concave up for $x > 0$.

(iii) As $f''(x) > 0$, $f'(x)$ is increasing.

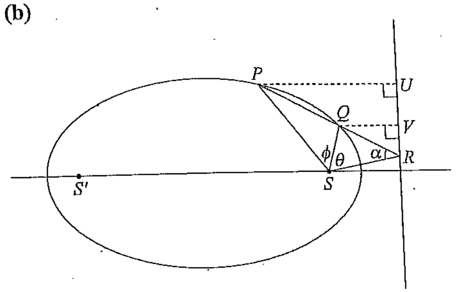
Also, when $x = 0$, $f'(0) = 0$

$\therefore$ $f'(x) > 0$

As $f(x)$ is concave up and its gradient is positive,

$$f(x) > 0$$
$$\therefore \sin x - x + \frac{x^3}{6} > 0$$
$$\therefore \; \sin x > x - \frac{x^3}{6} \quad \text{for } x > 0.$$

(b)



(i) In ΔPUR and ΔQVR,

$\angle R$ is common

$\angle PUR = \angle QVR = 90°$

$\therefore \Delta PUR \,|||\, \Delta QVR$ (equiangular)

$\therefore \dfrac{PR}{QR} = \dfrac{PU}{QV}$ (matching sides in similar triangles are proportional).

(ii) $PS = ePU$ and $QS = eQV$

$$\therefore \; \frac{PU}{QV} = \frac{ePU}{eQV}$$
$$= \frac{PS}{QS}.$$

(iii) In ΔPRS,

$$\frac{\sin(\phi+\theta)}{PR} = \frac{\sin\alpha}{PS}$$
$$\therefore \frac{\sin(\phi+\theta)}{\sin\alpha} = \frac{PR}{PS} \quad \text{—①}$$

In ΔQRS,

$$\frac{\sin\theta}{QR} = \frac{\sin\alpha}{QS}$$

$$\therefore \frac{\sin\theta}{\sin\alpha}=\frac{QR}{QS} \qquad \text{—②}$$

①÷②:

$$\frac{\sin(\phi+\theta)}{\sin\theta}=\frac{PR}{PS}\times\frac{QS}{QR}$$

$$=\frac{PR}{QR}\times\frac{QS}{PS}$$

$=1$ from (i) and (ii)

$$\therefore \sin(\phi+\theta)=\sin\theta$$

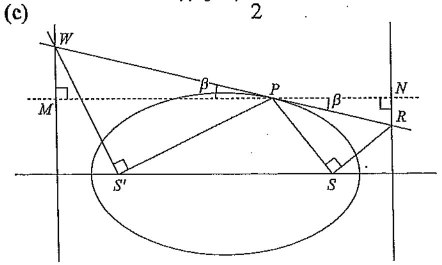
$$\therefore \phi+\theta=\pi-\theta \quad (\phi\neq 0)$$

$$\therefore \phi=\pi-2\theta.$$

(iv) As $\phi\to 0$ $\pi-2\theta\to 0$

$$\therefore \theta\to\frac{\pi}{2}$$

(c)



(i) Since $PS=ePN$

$$\frac{PS}{PR}=e\frac{PN}{PR}$$

$$=e\cos\beta.$$

(ii) Similarly, $\dfrac{PS'}{PR}=e\dfrac{PM}{PW}$

$$=e\cos\beta$$

$$\therefore \frac{PS}{PR}=\frac{PS'}{PW}$$

i.e. $\cos^{-1}\angle RPS=\cos^{-1}\angle WPS'$

Since these are both acute angles,

$$\angle RPS=\angle WPS'.$$

Question 8

(a) (i) Let $u=a-x,\ du=-dx$

When $x=a,\quad u=0$

When $x=0,\quad u=a$

$$\int_0^a f(x)\,dx=\int_a^0 f(a-u)(-du)$$

$$=\int_0^a f(a-u)\,du$$

$$=\int_0^a f(a-x)\,dx.$$

(ii) Given $f(x)+f(a-x)=f(a)$

$$\therefore \int_0^a f(x)\,dx+\int_0^a f(a-x)\,dx=\int_0^a f(a)\,dx$$

$$\int_0^a f(x)\,dx+\int_0^a f(x)\,dx=\int_0^a f(a)\,dx$$

from (i)

$$2\int_0^a f(x)\,dx=\left[x f(a)\right]_0^a$$

$$=a f(a)-0 f($$

$$=a f(a)$$

$$\therefore \int_0^a f(x)\,dx=\frac{a}{2}f(a).$$

(b) (i) $1+z^2+z^4+\ldots+z^{2n-2}$

is a geometric series with $r=z^2$, and n terms.

$$\therefore 1+z^2+z^4+\ldots+z^{2n-2}$$

$$=1+z^2+\left(z^2\right)^2+\ldots+\left(z^2\right)^{n-1}$$

$$=\frac{\left(z^2\right)^n-1}{z^2-1} \quad \text{for } z^2\neq 0$$

$$=\frac{z^{2n}-1}{z^2-1}\times\frac{z^{-1}}{z^{-1}}$$

$$=\frac{z^{2n-1}-z^{-1}}{z-z^{-1}}$$

$$=\frac{z^{n-1}\left(z^n-z^{-n}\right)}{z-z^{-1}}$$

$$=\left(\frac{z^n-z^{-n}}{z-z^{-1}}\right)z^{n-1}.$$

(ii) Let $z=\cos\theta+i\sin\theta$

Then $z^k=\cos k\theta+i\sin k\theta$

by de Moivre's theorem

and $z^n-z^{-n}=2i\sin n\theta$

Substituting both these results into (i),

$$1+(\cos 2\theta+i\sin 2\theta)+(\cos 4\theta+i\sin 4\theta)$$
$$+\ldots+(\cos(2n-2)\theta+i\sin(2n-2)\theta)$$

$$=\frac{2i\sin n\theta}{2i\sin\theta}\left[\cos(n-1)\theta+i\sin(n-1)\theta\right]$$

$$\therefore 1+\left[\cos 2\theta+\cos 4\theta+\ldots+\cos(2n-2)\theta\right]$$
$$+i\left[\sin 2\theta+\sin 4\theta+\ldots+\sin(2n-2)\theta\right]$$

$$=\frac{\sin n\theta}{\sin\theta}\left[\cos(n-1)\theta+i\sin(n-1)\theta\right].$$

(iii) Equating the imaginary parts of (ii),

$$\sin 2\theta+\sin 4\theta+\ldots+\sin(2n-2)\theta$$

$$=\frac{\sin n\theta}{\sin\theta}.\sin(n-1)\theta$$

Substituting $\theta=\dfrac{\pi}{2n}$,

$$\sin\frac{2\pi}{2n}+\sin\frac{4\pi}{2n}+\ldots+\sin\frac{(2n-2)\pi}{2n}$$

$$=\frac{\sin\frac{n\pi}{2n}}{\sin\frac{\pi}{2n}}\cdot\sin\frac{(n-1)\pi}{2n}$$

$$\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}$$

$$=\frac{1}{\sin\frac{\pi}{2n}}.\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)$$

$$=\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$$

$$=\cot\frac{\pi}{2n}.$$

(c) (i) $d_1+d_2+\ldots+d_{n-1}$

$$=\frac{1}{\sin\frac{\pi}{n}}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}\right)$$

$$=\frac{1}{\sin\frac{\pi}{n}}\left(\cot\frac{\pi}{2n}\right) \quad \text{using (b) (iii)}$$

$$=\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{n}\sin\frac{\pi}{2n}}$$

$$=\frac{\cos\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}}$$

$$=\frac{1}{2\sin^2\frac{\pi}{2n}}.$$

(ii) Since $X_1X_2=X_2X_3=\ldots=X_nX_1=1$

$$p=n$$

Also $q=\dfrac{1}{n}\left(d_1+d_2+\ldots+d_{n-1}\right)$

$$=\frac{1}{2n\sin^2\frac{\pi}{2n}} \quad \text{using (i)}$$

$$\therefore \frac{p}{q}=\frac{n}{\frac{1}{2n\sin^2\frac{\pi}{2n}}}$$

$$=2n^2\sin^2\frac{\pi}{2n}$$

$$=2\left(n\sin\frac{\pi}{2n}\right)^2.$$

(iii) $$\lim_{n\to\infty}\frac{p}{q}=\lim_{n\to\infty}2\left(n\sin\frac{\pi}{2n}\right)^2$$

$$=\lim_{n\to\infty}2\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\cdot\frac{\pi}{2}\right)^2$$

$$=2\left(\frac{\pi}{2}\right)^2$$

$$=\frac{\pi^2}{2}.$$

End Mathematics Extension 2 solutions