STUDENT NUMBER/NAME: ....

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board-approved calculators maybe used.
- A table of standard integrals is provided on a separate sheet.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Answers should be given in simplest exact form unless specified otherwise.
- Start each **NEW** question in a separate answer booklet.

Total Marks – 120

- Attempt Questions 1 – 10.
- All questions are of equal value.

Examiner: *A. Fuller*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (12 marks) Use a separate writing booklet

(a) Evaluate $2\sin\frac{\pi}{5}$ correct to three significant figures. **1**

(b) Solve the equation $\log_3 x = 2$. **1**

(c) Factorise $2x^2 - 3x - 2$. **2**

(d) Expand and simplify $(\sqrt{5}+1)(2\sqrt{5}-3)$. **2**

(e) Find the limiting sum of the geometric series $\frac{3}{2}+\frac{3}{8}+\frac{3}{32}+\cdots$ **2**

(f) Solve the equation $|1-3x| = 7$. **2**

(g) Find the exact value of $\sec\frac{5\pi}{6}$. **2**

Question 2 (12 marks) Use a separate writing booklet

(a) Find a primitive of $e^2 + x^2$ with respect to x. **1**

(b) Differentiate with respect to x:

(i) $\sin x + x^2$ **2**

(ii) $(e^x + x)^2$. **2**

(c) Evaluate $\int_1^3 \frac{6}{x^2}\,dx$. **3**

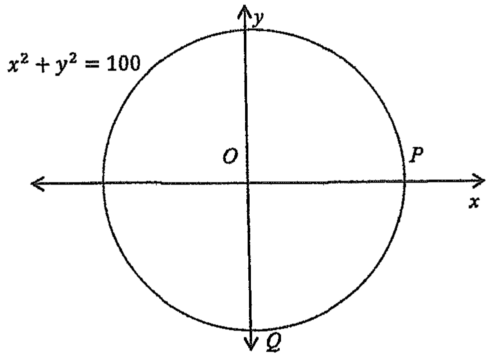
(d) Find $\int \frac{x^4+2}{2x}\,dx$. **2**

(e) Find the coordinates of the focus of the parabola $x^2 = 8(y+1)$. **2**

Question 3 (12 marks) Use a separate writing booklet

(a) Evaluate $\sum_{k=2}^{5} k(k+1)$. **1**

(b) The diagram shows the circle $x^2 + y^2 = 100$. P is a point where it meets the x-axis and Q is a point where it meets the y-axis, as shown.



(i) Copy the diagram to your answer booklet showing the coordinates of P and Q. **1**

(ii) Prove that $R(-6,8)$ also lies on the circle and plot it on the diagram in (i). **1**

(iii) Find the gradient of PR. **1**

(iv) M is the midpoint of the interval PR. O is the origin. Prove that OM and PR are perpendicular. **2**

(v) Show that the equation of the line joining PR is $x + 2y - 10 = 0$. **1**

(vi) Find the perpendicular distance of the point Q from the line joining PR. **2**

(vii) Calculate the distance PR. **1**

(viii) S lies on the circle. Find the maximum possible area of ΔPRS. **2**

Question 4 (12 marks) Use a separate writing booklet

(a) The n th term of a certain series is given by $T_n = 101 - 3n$.

(i) Show that $T_n - T_{n-1} = -3$ **1**

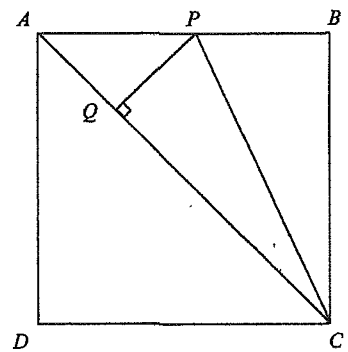
(ii) What type of series is it? **1**

(iii) What is the first negative term? **2**

(iv) Hence, or otherwise, what is the highest value for S_n, the sum of the series? **2**

(b) Differentiate $f(x) = 5 - x^2$ by first principles. **2**

(c) $ABCD$ is a square. PC bisects $\angle ACB$. Q is the foot of the perpendicular from P to AC.



(i) Prove that $\Delta PBC \equiv \Delta PQC$. **2**

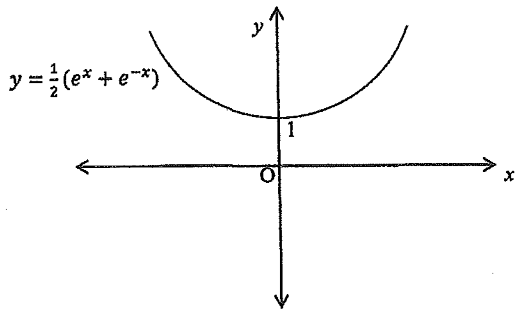
(ii) Hence, prove that $AQ = BP$. **2**

Question 5 (12 marks) Use a separate writing booklet

(a) Approximate $\int_{-1}^{3} f(x)dx$ using Simpson's Rule with five function values. **2**

x	-1	0	1	2	3
$f(x)$	5	2	-1	3	7

(b) The sketch below shows the curve $y = \frac{1}{2}(e^x + e^{-x})$, called a *catenary*.



(i) Determine the size of the area bound by the curve $y = \frac{1}{2}(e^x + e^{-x})$, **2**

the x-axis, the y-axis and the line $x = 1$.

(ii) Determine the volume generated when the area in (i) is rotated **3**

about the x-axis.

(c) If α and β are the roots of the equation $x^2 - 5x + 2 = 0$.

Find, without solving, the values of:

(i) $\alpha + \beta$ **1**

(ii) $\alpha^2 - 5\alpha$ **1**

(iii) $\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$ **1**

(iv) $\alpha^3 + \beta^3$. **2**

Question 6 (12 marks) Use a separate writing booklet

(a) Solve $4m^2 - 12m > 0$ **2**

(b) Consider the graph of $y = x^3 - mx^2 + mx$ where m is a constant.

(i) If the graph has only one stationary point what is this points' nature? **1**

(ii) Find $\frac{dy}{dx}$. **1**

(iii) For what values of m will the graph have:

(α) two distinct stationary points **1**

(β) only one stationary point? **1**

(iv) Sketch the graph when $m = 3$. **2**

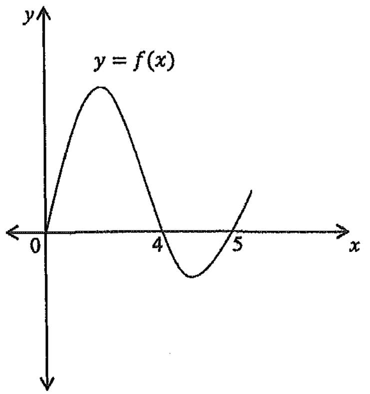
(c) (i) If $y = \log\left(\frac{1-\cos x}{1+\cos x}\right)$. Prove that $\frac{dy}{dx} = 2\operatorname{cosec} x$. **2**

(ii) Hence, Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec} x \, dx$. **2**

Question 7 (12 marks) Use a separate writing booklet

(a) Find the equation of the tangent to the curve $y = \frac{2}{2x+1}$ when $x = 0$. **3**

(b) The graph of $y = f(x)$ is shown for $x \geq 0$.



Given that $\int_0^4 f(x)\,dx = 13$ and that $\int_0^5 f(x)\,dx = 11$.

(i) What is the value of $\int_4^5 f(x)\,dx$? **1**

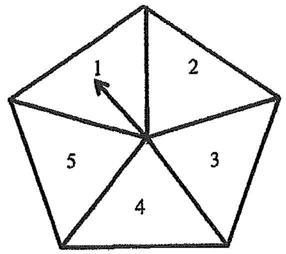
(ii) If $f(x)$ is an odd function, evaluate $\int_{-4}^{0} f(x)\,dx$. **1**

(iii) If $f(x) = \frac{d}{dx}(g(x))$ and $g(0) = 0$. **2**

Sketch $y = g(x)$ for $0 \leq x \leq 5$.

Question 7 continues on the next page

(c)



(i) The spinner above is spun twice. What is the probability of getting:

(α) two 5's? **1**

(β) the same number on both spins? **1**

(γ) a sum of 6 from the numbers that appear on each spin? **1**

(ii) How many times would the spinner need to be spun to have a 99% chance of getting at least one 5? **2**

End of Question 7

Question 8 (12 marks) Use a separate writing booklet

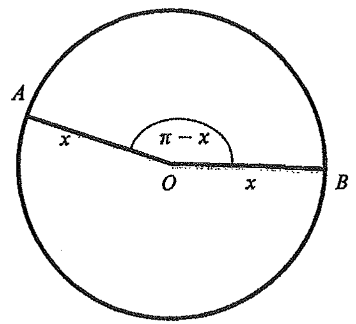
(a) The population P of a country town is decreasing at an increasing rate.

What does this imply about:

(i) $\frac{dP}{dt}$ **1**

(ii) $\frac{d^2P}{dt^2}$? **1**

(b)



A sector, OAB, of a circle is such that, when its radii are x cm, then

$\angle AOB = (\pi - x)$ radians, and x varies from 0 to π.

(i) Find the area of sector OAB in terms of x. **1**

(ii) Find the maximum perimeter of sector OAB. **3**

(iii) Prove that the area of $\triangle\, AOB$, T, is given by **1**

$T = \frac{x^2 \sin x}{2}$.

(iv) Show that, when T is a maximum, $x + 2\tan x = 0$. **3**

(v) The graph of $y = \tan x$ is provided on a separate sheet. **1**

On this sheet, show how the value of x which makes T a maximum can be approximated correct to one decimal place graphically and provide your answer in the space provided.

Make sure you place this separate sheet in your answer booklet for Question 8

Question 9 (12 marks) Use a separate writing booklet

(a) $I_1 = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x + \cos x}\,dx,\quad I_2 = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x}\,dx$

(i) Prove that $I_1 + I_2 = \frac{\pi}{4}$. **2**

(ii) Prove that $I_1 - I_2 = \frac{1}{2}\ln 2$. **2**

(iii) Find the value of I_1. **1**

(b) On the first of January next year, Murray will invest \$1000 in a superannuation scheme. On the subsequent nine January 1[st's] he will make further investments, increasing them by 4% each year to account for inflation. The scheme pays 10% per year interest and Murray will withdraw his funds when his final investment has been invested for 1 year.

(i) What is the value of his first investment when withdrawn? **1**

(ii) What is the amount of his last investment? **1**

(iii) What is the value of his last investment when withdrawn? **1**

(iv) How much has he invested in the superannuation scheme altogether? **2**

(v) How much has Murray's superannuation scheme earned from his investments? **2**

Question 10 (12 marks) Use a separate writing booklet

(a) The velocity of a particle moving in a straight line is given by $v = 1 - e^{-\frac{t}{2}}$. **2**

In which direction does the particle first move? Justify your answer.

(b) The velocity of a train increases from 0 to U at a constant rate a. The train remains at this velocity U until it decreases from U to 0 at a constant rate b. The distance of this journey is D and the time taken is T.

(i) Draw a velocity-time graph of the train's journey indicating **2**

how long it takes the train to reach a speed of U.

(ii) Show that the time taken for the journey is given by **2**

$T = \frac{D}{U} + \left(\frac{1}{2a} + \frac{1}{2b}\right)U.$

(c) If a, b, $a + b$ and ab are positive numbers that form four consecutive terms in a geometric series.

(i) Show that $a^2 + ab - b^2 = 0$. **2**

(ii) Hence, show that $\frac{a}{b} = \frac{-1+\sqrt{5}}{2}$. **2**

(iii) Find the value of a. **2**

End of paper

Sydney High Trial HSC 2011 2 unit

1 (a) $2\sin\frac{\pi}{5}$ $\qquad \frac{\pi}{5} = \frac{180}{5} = 36^\circ$

$= 2\sin 36^\circ \doteqdot 1.18$ (3 SF) ①

(b) $\log_3 x = 2$

$3^2 = x = 9$ ①

(c) $2x^2 - 3x - 2 = (2x+1)(x-2)$ ②

(d) $10 - 3\sqrt{5} + 2\sqrt{5} - 3 = 7 - 1\sqrt{5}$ ②

(e) $r = \frac{1}{4}$ $\quad -1 < r < 1$ ✓ $\quad a = \frac{3}{2}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{4}} = \frac{\frac{3}{2}}{\frac{3}{4}} = \frac{3}{2} \times \frac{4}{3} = 2.$ ②

(f) $|1-3x| = 7$

$1 - 3x = 7$ $\qquad 1 - 3x = -7$

$-6 = 3x$ $\qquad -3x = -8$

$x = -2$ ✓ $\qquad x = \frac{8}{3}$ $(2\frac{2}{3})$ ✓ ②

(g) $\sec\frac{5\pi}{6} = \frac{1}{\cos 150^\circ} = \frac{1}{-\cos 30^\circ} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$

$\frac{5 \times 180}{6} = 150^\circ$

also accept $-\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ ②

2 (a) $\int e^2 + x^2 \, dx = e^2 x + \frac{x^3}{3} + c$

(b) (i) $y = \sin x + x^2$

$\frac{dy}{dx} = \cos x + 2x$

(ii) $y = (e^x + x)^2$

Let $u = e^x + x$

$\frac{du}{dx} = e^x + 1$

now let $y = u^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 2u \cdot (e^x + 1)$

$= 2(e^x + x)(e^x + 1)$

(c) $\int_1^3 \frac{6}{x^2} \, dx$

$= 6 \int_1^3 x^{-2} \, dx$

$= 6 \left[\frac{x^{-1}}{-1} \right]_1^3$

$= -6 \left[3^{-1} - 1^{-1} \right]$

$= -6 \left[-\frac{2}{3} \right]$

$= 4$

(d) $\int \frac{x^4 + 2}{2x} \, dx$

$= \int \frac{x^3}{2} + \frac{1}{x} \, dx$

$= \frac{x^4}{8} + \ln x + c$

(e) $x^2 = 8(y+1)$

$x^2 = 4 \cdot 2 (y+1)$

Vertex is $(0,-1)$ Focal length $= 2$

$\therefore$ Focus is $(0, 1)$

2011 Mathematics Trial HSC

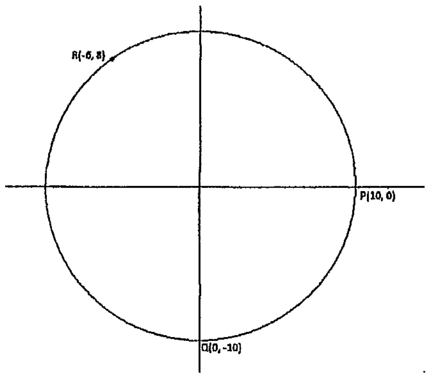
Question 3:

a)

$$\sum_{k=2}^{5} k(k+1)$$
$$= 2(3) + 3(4) + 4(5) + 5(6)$$
$$= 68$$

b)

(i)



(ii) Substitute $R(-6, 8)$ in $x^2 + y^2 = 100$

$(-6)^2 + 8^2 = 36 + 64 = 100$

(iii)

$$m_{PR} = \frac{8-0}{-6-10}$$
$$= \frac{8}{-16}$$
$$= -\frac{1}{2}$$

(iv)

$$M_{PR} = \left(\frac{10-6}{2}, \frac{0+8}{2}\right)$$
$$M = (2, 4)$$
$$m_{OM} = \frac{4-0}{2-0}$$
$$= 2$$
$$m_{PR} \times m_{OM} = -\frac{1}{2} \times 2 = -1$$

$\therefore$ PR and OM are perpendicular.

(v)

$$y - y_1 = m(x - x_1)$$
$$y - 0 = -\frac{1}{2}(x - 10)$$
$$2y = -x + 10$$
$$x + 2y - 10 = 0$$

(vi)

$$d = \frac{|0 + 2(-10) - 10|}{\sqrt{1^2 + 2^2}}$$
$$d = \frac{30}{\sqrt{5}}$$
$$d = 6\sqrt{5}$$

(vii)

$$d_{PR} = \sqrt{(10+6)^2 + (0-8)^2}$$
$$= \sqrt{256 + 64}$$
$$= \sqrt{320}$$
$$= 8\sqrt{5}$$

(viii)

$$d_{OM} = \sqrt{(2-0)^2 + (4-0)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

Height $= 2\sqrt{5} + 10$

Maximum Area $= \frac{1}{2}bh$

$$= \frac{1}{2}(8\sqrt{5})(2\sqrt{5} + 10)$$
$$= 4\sqrt{5}(2\sqrt{5} + 10)$$
$$= 40 + 40\sqrt{5}$$

2UNIT SOLUTIONS (Q4) 12 marks

Q4. (a) $T_n = 101 - 3n$

(i) $T_{n-1} = 101 - 3(n-1)$
$= 101 - 3n + 3$
$= 104 - 3n$

Then $T_n - T_{n-1} = 101 - 3n - 104 + 3n$ (1)
$= -3$ ✓

(ii) A.P. ✓ (1)

(iii) $101 - 3n < 0$
$-3n < -101$
$n > \frac{101}{3}$ (2)
$n > 33\frac{2}{3}$ ✓

∴ 1st negative term is 34^{th}
$T_{34} = 101 - 3 \times 34 = -1$ ✓

(iv) S_n has max value for first 33 terms
$a = 98$; $d = -3$ $n = 33$ ✓

$S_{33} = \frac{n}{2}\left[2a + \overline{n-1}\,d\right]$

$= \frac{33}{2}\left[2 \times 98 + 32 \times -3\right]$ (2)

$S_{33} = \frac{33}{2} \times 100$ ✓ $= 1650$

(b) $f(x) = 5 - x^2$

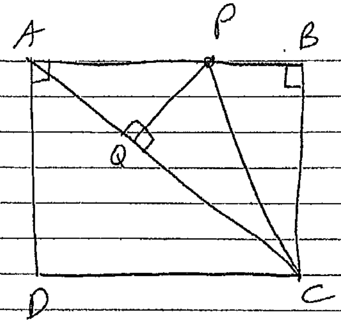
$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{5 - (x^2 + 2xh + h^2) - 5 + x^2}{h}$ ✓ (2)

$= \lim_{h\to 0} \frac{-2xh - h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(-2x - h)}{\cancel{h}}$

$f'(x) = -2x$ ✓

4(c)



(i) Prove $\Delta PBC \equiv \Delta PQC$

Proof PC in common
$\angle PQC = \angle PBC = 90°$ (given) ✓
$\angle PCQ = \angle PCB$ (given)

∴ $\Delta PBC \equiv \Delta PQC$ (AAS) ✓ (2)

(ii) Since AC is a diagonal of square, it bisects $\angle A$ and $\angle C$

$\Rightarrow \angle QAP = 45°$

Then $\angle APQ = 45°$ (Angle sum of $\Delta = 180°$)

∴ ΔAQP is isosceles ✓

$AQ = QP$ (sides opposite equal angles in isosceles Δ are equal)

Also $QP = PB$ (Matching sides in cong. Δ's)

∴ $AQ = PB$ # ✓ (2)

Sydney High Trial 2011 2 unit

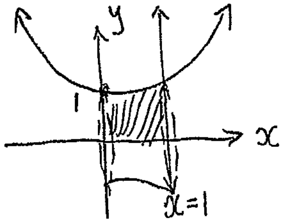
5 (a) $\int_{-1}^{3} f(x)\,dx$

x	-1	0	1	2	3
$f(x)$	5	2	-1	3	7
	y_0	y_1	y_2	y_3	y_n

$h = 1$

$$\doteqdot \frac{1}{3}\left[(5+7)+2(-1)+4(2+3)\right] = \frac{30}{3} = 10\,u^2$$ (2)

(b)



(i) $A = \frac{1}{2}\int_0^1 (e^x + e^{-x})\,dx$

$= \frac{1}{2}\left(e^x - e^{-x}\right)\Big]_0^1$

$= \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^0 - e^{-0})$

$= \frac{1}{2}\left(e^1 - \frac{1}{e}\right)$

$= \frac{e^2 - 1}{2e}\,u^2$ or $1.175\ldots\,u^2$ (2)

(ii) $V = \pi\int_0^1 \frac{1}{4}\left(e^x + e^{-x}\right)^2 dx$

$V = \frac{\pi}{4}\int_0^1 \left(e^{2x} + 2 + e^{-2x}\right) dx$

$= \frac{\pi}{4}\left(\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^1$

$= \frac{\pi}{8}\left(e^{2x} + 4x - e^{-2x}\right]_0^1$

$= \frac{\pi}{8}\left(e^2 + 4 - e^{-2}\right) - \frac{\pi}{8}\left(e^0 + 0 - e^{-0}\right)$

$= \frac{\pi}{8}\left(e^2 + 4 - \frac{1}{e^2}\right) u^3$

OR $\frac{\pi}{8e^2}\left(e^4 + 4e^2 - 1\right) u^3$

OR $\doteqdot 4.419\,u^3$. (3)

(c) (i) $\alpha + \beta = \frac{-b}{a} = \frac{-(-5)}{1} = 5$ (1)

(ii) $\alpha^2 - 5\alpha = -2$. (1)

$\beta = 5 - \alpha$

so $\alpha\beta = \alpha(5-\alpha) = -\alpha(\alpha - 5)$
$= -(\alpha^2 - 5\alpha)$
$= -2$.

since α is a root
$\alpha^2 - 5\alpha + 2 = 0$
$\alpha^2 - 5\alpha = -2$

(iii) $\alpha\beta + 2 + \frac{1}{\alpha\beta} = 2 + 2 + \frac{1}{2}$
$= 4\frac{1}{2}$ (1)

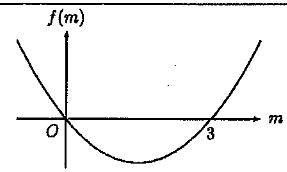
$\alpha\beta = \frac{c}{a}$
$= \frac{2}{1}$

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$
$= 125 - 3\times 2\times 5$
$= 95$ (2)

2011 Mathematics Trial HSC:
Solutions— Question 6

6. (a) Solve $4m^2 - 12m > 0$. [2]

Solution: $4m(m-3) > 0$.
Now, from the sketch,
$m < 0$ and $m > 3$.



(b) Consider the graph of $y = x^3 - mx^2 + mx$ where m is a constant.

(i) If the graph has only one stationary point, what is this point's nature? [1]

Solution: It is an horizontal point of inflexion.

(ii) Find $\frac{dy}{dx}$. [1]

Solution: $\frac{dy}{dx} = 3x^2 - 2mx + m$.

(iii) For what vaues of m will the graph have:

(α) two distinct stationary points, [1]

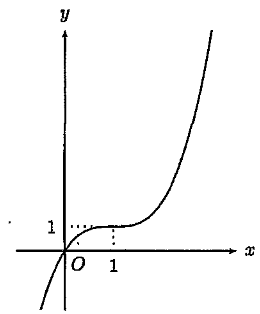
Solution: $\Delta = 4m^2 - 12m > 0$,
i.e., $m < 0$ and $m > 3$.

(β) only one stationary point? [1]

Solution: $m = 0$ or $m = 3$.

(iv) Sketch the graph when $m = 3$. [2]

Solution:
$$\begin{aligned} y &= x^3 - 3x^2 + 3x,\\ y' &= 3x^2 - 6x + 3,\\ &= 3(x-1)^2,\\ &= 0 \text{ when } x = 1. \end{aligned}$$



(c) (i) If $y = \log\left(\frac{1-\cos x}{1+\cos x}\right)$, prove that $\frac{dy}{dx} = 2\operatorname{cosec} x$. [2]

Solution:
$$\begin{aligned}
y &= \log(1-\cos x) - \log(1+\cos x),\\
\frac{dy}{dx} &= \frac{\sin x}{1-\cos x} - \frac{-\sin x}{1+\cos x},\\
&= \frac{\sin x + \sin x\cos x + \sin x - \sin x \cos x}{1-\cos^2 x},\\
&= \frac{2\sin x}{\sin^2 x},\\
&= \frac{2}{\sin x},\\
&= 2\operatorname{cosec} x.
\end{aligned}$$

Alternative Method:
$$y = \log u, \qquad u = \frac{1-\cos x}{1+\cos x},$$
$$\frac{dy}{du} = \frac{1}{u}, \qquad \frac{du}{dx} = \frac{(1+\cos x).\sin x - (-\sin x)(1-\cos x)}{(1+\cos x)^2},$$
$$= \frac{\sin x + \sin x\cos x + \sin x - \sin x\cos x}{(1+\cos x)^2},$$
$$= \frac{2\sin x}{(1+\cos x)^2}.$$
$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cancel{(1+\cos x)}}{(1-\cos x)} \times \frac{2\sin x}{(1+\cos x)^{\cancel{2}}},\\
&= \frac{2\sin x}{1-\cos^2 x},\\
&= \frac{2\sin x}{\sin^2 x},\\
&= \frac{2}{\sin x},\\
&= 2\operatorname{cosec} x.
\end{aligned}$$

(ii) Hence evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec} x \, dx$. [2]

Solution:
$$\begin{aligned}
\tfrac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\operatorname{cosec} x\,dx &= \tfrac{1}{2}\left[\log\left(\tfrac{1-\cos x}{1+\cos x}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}},\\
&= \tfrac{1}{2}\left\{\log\left(\tfrac{1}{1}\right) - \log\left(\tfrac{1/2}{3/2}\right)\right\},\\
&= \log\sqrt{3} \ \left(\text{or } \tfrac{1}{2}\log 3, \text{ or even } -\tfrac{1}{2}\log\left(\tfrac{1}{3}\right)\right).
\end{aligned}$$

Q.7. (a) $y = \frac{2}{2x+1}$ when $x=0$.

$= 2(2x+1)^{-1}$

$\frac{dy}{dx} = -2(2x+1)^{-2}.2$

$= \frac{-4}{(2x+1)^2}$ At $x=0$, $y' = -4$ = gradient

When $x=0$, $y=2 \Rightarrow P(0,2)$ and $m=-4$

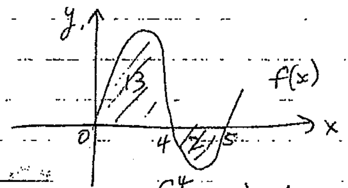
Eqn of tangent:

$y - y_1 = m(x - x_1)$

$y - 2 = -4(x - 0)$

$\boxed{y = -4x + 2}$ 3

(b)



Given $\int_0^4 f(x)\,dx = 13$ and $\int_0^5 f(x)\,dx = 11$

(i) $\int_4^5 f(x)\,dx = -2.$ 1

(ii) $f(x)$ is odd $\Rightarrow f(-x) = -f(x)$

Then $\int_{-4}^0 f(x)\,dx = -13$ 1

7(b)

(iii) $f(x) = \frac{d}{dx}(g(x))$ and $g(0) = 0$.

Sketch integral fn. $g(x)$ for $0 \le x \le 5$.

$f(x)$ = derivative fn

When $f(x) = 0 \Rightarrow$ stat. pts on $g(x)$

$\Rightarrow$ stat pts at $x = 0, 4, 5$

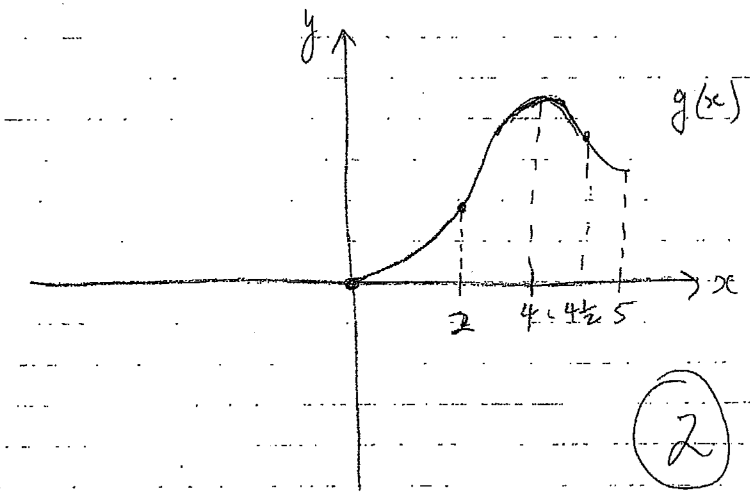
From $x = 0 \to 2$, $f(x)$ +ve $\Rightarrow g(x)$ increasing.

$x = 2 \to 4\frac{1}{2}$, $f(x)$ −ve $\Rightarrow g(x)$ decreasing

$x = 4\frac{1}{2} \to 5$, $f(x)$ +ve $\Rightarrow g(x)$ increasing

Also at $x = 2$, $f(x)$ has a max $\Rightarrow$ change in concavity in $g(x)$

$x = 4\frac{1}{2}$, $f(x)$ has a min $\Rightarrow$ " " in $g(x)$



✓ turning pts.

✓ general shape

2

7(c) (i) (α) $P(5,5) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$ ✓ 1

(β) $P(2 \text{ same}) = 5 \times \frac{1}{25} = \frac{1}{5}$ ✓ 1

(γ) $P(1,6 \text{ or } 2,\cancel{4} \text{ or } 3,3)$

$= 2 \times \frac{1}{5} \times \frac{1}{5} + 2 \times \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5}$

$= \frac{5}{25} = \frac{1}{5}$. ✓ 1

(ii) $P(\text{at least one } 5) \ge .99$

$\Rightarrow P(\text{no 5's in } n \text{ trials}) \le .01$.

$\left(\frac{4}{5}\right)^n \le .01$ ✓

$(.8)^n \le .01$

2

$n \log(.8) \le \log(.01)$

$n \le \frac{\log(.01)}{\log(.8)}$

$n \le 20.63$ ✓

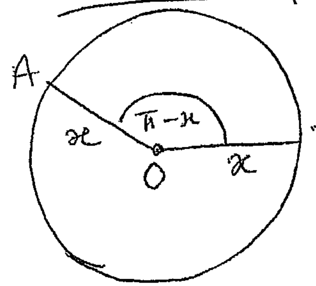
$\therefore$ must spin at least 21 times

Question 8

(a) (i) $\frac{dP}{dt} < 0$ [1]

(ii) $\frac{d^2P}{dt^2} < 0$ $\left(\text{or } \frac{d^2P}{dt^2} > 0\right)$ [1]

(b)



$0 \leq x \leq \pi$

(i) $A = \frac{1}{2}x^2(\pi - x)$ [1]

(ii) $P = 2x + x(\pi - x)$

$= 2x + \pi x - x^2$

$\frac{dP}{dx} = 2 + \pi - 2x$

T.P. when $2x = 2 + \pi$

$x = 1 + \frac{\pi}{2}$

$\frac{d^2P}{dt^2} = -2$

$\therefore$ Max when $x = 1 + \frac{\pi}{2}$

$\therefore P_{max} = 2(1+\frac{\pi}{2}) + \pi(1+\frac{\pi}{2}) - (1+\frac{\pi}{2})^2$

$= 2 + \pi + \pi + \frac{\pi^2}{2} - (1 + \pi + \frac{\pi^2}{4})$

$\doteqdot 6.609$ $\quad = 1 + \pi + \frac{\pi^2}{4}$ [3]

(iii) $T = \frac{1}{2}x^2 \sin(\pi - x)$

$= \frac{1}{2}x^2 \sin x$ as reqd [1]

(iv) $\frac{dT}{dx} = x\sin x + \frac{1}{2}x^2\cos x$

SP when $0 = x\sin x + \frac{1}{2}x^2\cos x$

$x(\sin x + \frac{1}{2}x\cos x) = 0$

Noting $0 \leq x \leq \pi$

$x = 0$ or $\sin x + \frac{1}{2}x\cos x = 0$

i.e. $\tan x + \frac{1}{2}x = 0$

or $x + 2\tan x = 0$

Nature of Stationary Points:

At $x = 0$ $\quad T'(-1) = 1.111$ $\quad$ 0 +/

$T'(1) = 1.111$ +/

$\therefore$ Point of Inflexion

At $x = -2\tan x$ ($x \simeq 2.3$ from calculator)

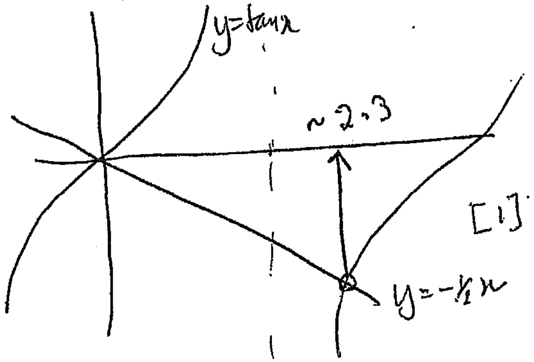
$T'(2) = 0.98$ $\quad$ +/ 0 \ −

$T'(3) = -4.03$

$\therefore$ Rel Max Turning Point

When $x + 2\tan x = 0$ [3]

(v) Graph $y = -\frac{1}{2}x$ on a graph of $y = \tan x$



[1]

QUESTION 9.

(a) $I_1 = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x + \cos x}\,dx$ $\qquad I_2 = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x}\,dx$

(i) $I_1 + I_2 = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x}\,dx$.

$= \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$

$= \int_0^{\frac{\pi}{4}} 1\,dx$

$= x \Big|_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}$

(ii) $I_1 - I_2 = \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$

$= \ln(\sin x + \cos x)\Big|_0^{\frac{\pi}{4}}$

$= \ln(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - \ln(1)$.

$= \ln\sqrt{2}$

$= \frac{1}{2}\ln 2$.

(iii) $I_1 + I_2 = \frac{\pi}{4}$

$I_1 - I_2 = \frac{1}{2}\ln 2$.

$2I_1 = \frac{\pi}{4} + \frac{1}{2}\ln 2$.

$I_1 = \frac{\pi}{8} + \frac{1}{4}\ln 2$.

(b) (i) $1000\times\left(1+\frac{10}{100}\right)^{10} = \$2593.74$

(ii) $1000\times\left(1+\frac{4}{100}\right)^{9} = \$1423.31$

(iii) $1000\times\left(1+\frac{4}{100}\right)^{9}\times\left(1+\frac{10}{100}\right) = \$1565.64$

(iv) $1000 + 1000(1.04) + 1000(1.04)^2 + \dots + 1000(1.04)^9$

$= 1000\left(1 + 1.04 + \dots + 1.04^9\right)$

$= 1000\left(\frac{1-1.04^{10}}{1-1.04}\right)$

$= \$12006.11$

(v) Let $P = 1000$, $I = 1.04$, $R = 1.1$

$A_{10} = PR^{10} + PIR^9 + PI^2R^8 + \dots + PI^8R^2 + PI^9R.$

$= PR^{10}\left(1 + \frac{I}{R} + \frac{I^2}{R^2} + \frac{I^3}{R^3} + \dots + \frac{I^8}{R^8} + \frac{I^9}{R^9}\right)$

$= PR^{10}\left(\frac{1-\left(\frac{I}{R}\right)^{10}}{1-\frac{I}{R}}\right)$

$= 1000\times 1.1^{10}\left(\frac{1-\left(\frac{1.04}{1.1}\right)^{10}}{1-\frac{1.04}{1.1}}\right)$

$= \$20414.13$

QUESTION 10.

(a) $v = 1 - e^{-\frac{t}{2}}$.

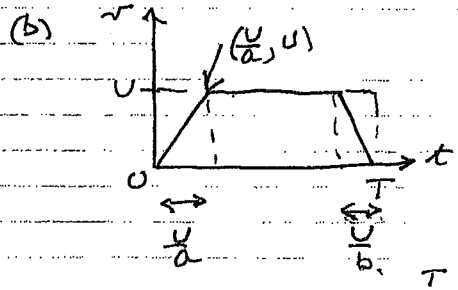
when $t=0$, $v = 1-1 = 0$ $\therefore$ stationary

when $t=0$. $\left(\text{accel.} = \frac{dv}{dt} = \frac{1}{2}e^{-\frac{t}{2}}\right)$

$\dot{v} = \frac{1}{2}e^0$

$= \frac{1}{2}$ which is positive $\therefore$ initially the velocity is +ve ie to the right.

(b)



now $D = \int_0^T v\,dt$. (this equivalent to the area of the trapezium

$= UT - \frac{1}{2}\frac{U}{a}\times U - \frac{1}{2}\frac{U}{b}\times U$

$\therefore D = UT - \frac{1}{2}U^2\left(\frac{1}{a}+\frac{1}{b}\right)$

$\therefore UT = D + U^2\left(\frac{1}{2a}+\frac{1}{2b}\right)$

$T = \frac{D}{U} + U\left(\frac{1}{2a}+\frac{1}{2b}\right)$

(c) (i). $\frac{b}{a} = \frac{a+b}{b}$

$\therefore b^2 = a^2 + ab.$

$\therefore a^2 + ab - b^2 = 0.$

(II) $a = \dfrac{-b \pm \sqrt{b^2 + 4b^2}}{2}$

$= \dfrac{-b \pm b\sqrt{1+4}}{2}$

$a = \dfrac{-b \pm b\sqrt{5}}{2}$

$\dfrac{a}{b} = \dfrac{-1 \pm \sqrt{5}}{2}$ (now $a, b > 0$)

$\therefore \dfrac{a}{b} = \dfrac{-1+\sqrt{5}}{2}$

(III) now $\dfrac{a+b}{ab} = \dfrac{a}{b} = \dfrac{\sqrt{5}-1}{2}$ — ①

i.e. $\dfrac{1}{b} + \dfrac{1}{a} = \dfrac{\sqrt{5}-1}{2}$ — Ⓐ

$\therefore$

From ① $\dfrac{1}{b} = \dfrac{\sqrt{5}-1}{2a}$

$\therefore$ in Ⓐ

$\dfrac{\sqrt{5}-1}{2a} + \dfrac{1}{a} = \dfrac{\sqrt{5}-1}{2}$

$\dfrac{1}{2a}\left(\sqrt{5}-1+2\right) = \dfrac{\sqrt{5}-1}{2}$

$\dfrac{1}{a}\left(\sqrt{5}+1\right) = \sqrt{5}-1$

$\dfrac{1}{a} = \dfrac{\sqrt{5}-1}{\sqrt{5}+1}$

$a = \dfrac{\sqrt{5}+1}{\sqrt{5}-1} = \dfrac{(\sqrt{5}+1)^2}{4}$

$= \dfrac{3+\sqrt{5}}{2}$