Sydney Grammar School

2012 Trial Examination

FORM VI
MATHEMATICS

Monday 6th August 2012

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 80 boys

Examiner
TCW

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ \ x > 0$

Tear-off pages follow ...

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

When written in radians, 200° is equal to:

(A) $\pi + 20$

(B) $\frac{6\pi}{5}$

(C) $\frac{9\pi}{10}$

(D) $\frac{10\pi}{9}$

QUESTION TWO

At what angle is the line $y = -\sqrt{3}x$ inclined to the positive side of the x-axis?

(A) 30°

(B) 60°

(C) 120°

(D) 150°

QUESTION THREE

Which of the following is the point of intersection of the two lines $3x - 4y + 6 = 0$ and $x - y - 1 = 0$?

(A) $(0, 0)$

(B) $(-2, -3)$

(C) $(10, 9)$

(D) $(11, 10)$

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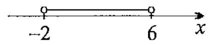
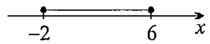
QUESTION FOUR

Which of the following graphs represents the solution to $|x - 2| \le 4$?

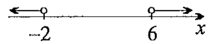
(A)



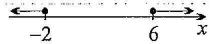
(B)



(C)



(D)



QUESTION FIVE

The equation of the normal to the curve $y = x^3 - 4x$ at the point $(1, -3)$ is:

(A) $y = x + 4$

(B) $y = x - 4$

(C) $y = -x + 2$

(D) $y = -x - 2$

QUESTION SIX

Suppose that the point $P\big(a, f(a)\big)$ lies on the curve $y = f(x)$. If $f'(a) = 0$ and $f''(a) < 0$, which of the following statements describes the point P on the graph of $y = f(x)$?

(A) P is a minimum turning point.

(B) P is a maximum turning point.

(C) P is a stationary point of inflexion.

(D) P is a non-stationary point of inflexion.

Exam continues overleaf ...

QUESTION SEVEN

The equation $3x^2 + 2x - 1 = 0$ has roots α and β. The value of $2\alpha + 2\beta$ is:

(A) 10

(B) $-\frac{1}{3}$

(C) $-\frac{2}{3}$

(D) $-\frac{4}{3}$

QUESTION EIGHT

Which of the following statements is true for the geometric sequence $24, 12, 6, \ldots$?

(A) The fourth term is 0.

(B) The sum of the first four terms is 44.

(C) The sum of the series will never exceed 48.

(D) There are infinitely many negative terms.

QUESTION NINE

A parabola has its focus at $(2, -2)$ and the equation of its directrix is $y = 2$. Which of the following is the equation of the parabola?

(A) $(x-2)^2 = 8y$

(B) $(x-2)^2 = -8y$

(C) $(x-2)^2 = 8(y+2)$

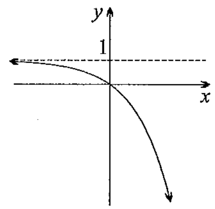
(D) $(x-2)^2 = -8(y+2)$

Exam continues next page ...

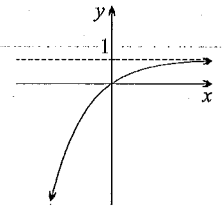
QUESTION TEN

Which of the following graphs could have equation $y = 1 - 2^x$?

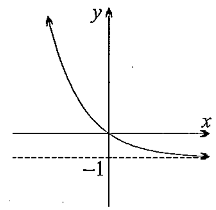
(A)



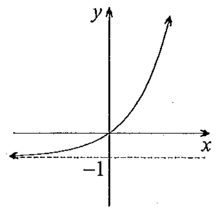
(B)



(C)



(D)



End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

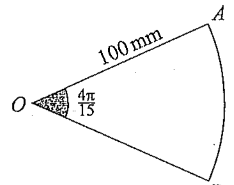
Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Write $\dfrac{6}{\sqrt{5}-\sqrt{3}}$ with a rational denominator and simplify. [2]

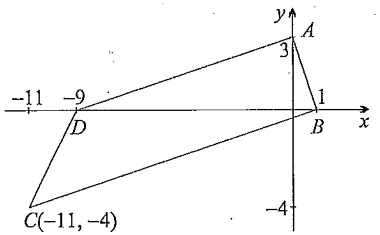
(b)



The diagram above shows a sector AOB with radius 100 mm and $\angle AOB = \dfrac{4\pi}{15}$. [2]

Find the length of arc AB correct to the nearest millimetre.

(c)



The diagram above shows a quadrilateral with vertices $A(0,3)$, $B(1,0)$, $C(-11,-4)$ and $D(-9,0)$.

(i) Show that $AB = \sqrt{10}$ units and $BC = 4\sqrt{10}$ units. [2]

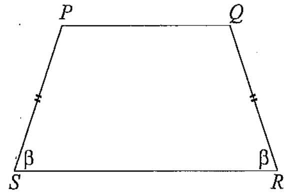
(ii) Show that $AD \parallel BC$. [1]

(iii) Show that $AB \perp BC$. [1]

(iv) Find AD and hence find the area of the trapezium $ABCD$. [2]

Exam continues next page ...

(d)



In the diagram above, $PS = QR$ and $\angle PSR = \angle QRS = \beta$.

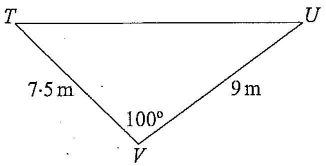
(i) Prove that $\triangle PRS \equiv \triangle QSR$. [3]

(ii) Hence prove that $\angle PSQ = \angle QRP$. Let $\angle PRS = \alpha$. [2]

Exam continues overleaf ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, $TV = 7{\cdot}5\,\text{m}$, $UV = 9\,\text{m}$ and $\angle V = 100°$.

(i) Find the length of TU correct to 1 decimal place. **2**

(ii) Find the area of $\triangle TUV$ correct to 1 decimal place. **2**

(b) Differentiate:

(i) $y = \dfrac{3}{x^2}$ **1**

(ii) $y = (x^3 - 2)^{10}$ **1**

(iii) $y = \dfrac{x}{\cos x}$ **2**

(c) Evaluate:

(i) $\displaystyle\int_1^e \frac{6}{x}\,dx$ **2**

(ii) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$ **2**

(d) Solve $\cos x(2\sin x - 1) = 0$, for $0 \le x \le 2\pi$. **3**

Exam continues next page ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The line ℓ has equation $3x + 4y - 2 = 0$. The point $(2, -1)$ lies on ℓ. Find the perpendicular distance from the line ℓ to the line with equation $3x + 4y + 5 = 0$. **2**

(b) For what values of x is the curve $y = 2x^3 - 9x^2 + 5$ increasing? **2**

(c) A particle is moving along a straight line. Its displacement, x metres, from a fixed point O after t seconds is given by $x = 2 + 2\sin 2t$.

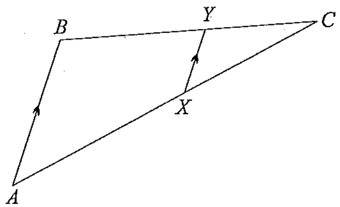
(i) What is the particle's initial position? **1**

(ii) Sketch the particle's displacement-time graph for the first 2π seconds of motion. **2**

(iii) Find when and where the particle first comes to rest. **2**

(iv) Find the maximum speed of the particle and write down a time when this maximum speed occurs. **2**

(d)



In the diagram above $AB \parallel XY$.

(i) Prove that $\triangle ABC\ |||\ \triangle XYC$. **2**

(ii) Given that $AB = XC = 18\,\text{cm}$ and $XY = 8\,\text{cm}$, find AX giving a reason. **2**

Exam continues overleaf ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Copy and complete the following table for $f(x) = (\log_e x)^2$. Write the function values correct to 3 decimal places. [1]

x	1	1·5	2
$f(x)$			

(ii) Use Simpson's rule with three function values to find an approximation of [2]

$$\int_1^2 (\log_e x)^2\,dx.$$

Give your answer correct to 2 decimal places.

(b) (i) Evaluate $1 + 2 + 3 + \cdots + 300$. [1]

(ii) Find the sum of all integers from 1 to 300 which are not divisible by 3. [2]

(c) The function $f(x)$ has derivative $f'(x) = 12x - kx^2$. The curve $y = f(x)$ has a point of inflexion at $(1, -4)$.

(i) Show that $k = 6$. [1]

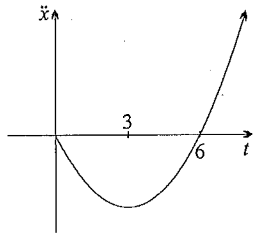
(ii) Find the equation of the curve $y = f(x)$. [2]

(d) Consider the function $y = x \log_e x$.

(i) Find $\dfrac{dy}{dx}$. [1]

(ii) Hence find the minimum value of $x \log_e x$ and justify your answer. [3]

(e)



The diagram above shows a particle's acceleration-time graph. Draw a possible sketch of the particle's velocity-time graph, given that initially the particle is stationary. [2]

Exam continues next page ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) A certain grasshopper plague is following the law of natural growth. The grasshopper population G satisfies the equation

$$G = G_o e^{kt}.$$

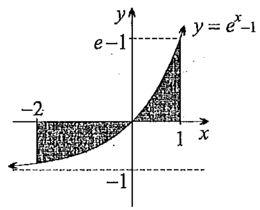
Time t is measured in months and G_o and k are constants.
Initially there were 10 000 grasshoppers in the plague and after 8 months there were 40 000.

(i) Show that $k = \frac{1}{4}\ln 2$. [2]

(ii) Find the number of grasshoppers in the plague after 2 years. [2]

(iii) After how many whole months would the population exceed 10 million? [2]

(b)



The diagram above shows the region bounded by the curve $y = e^x - 1$ and the x-axis from $x = -2$ to $x = 1$. Find the exact area of the shaded region. [3]

(c) Atticus makes a deposit of \$5000 at the start of each <u>year</u> into a savings account. He earns <u>monthly</u> compound interest on his savings account at 4·8% per annum.
Let A_n be the value of the account at the end of n years.

(i) Show that $A_1 = \$5245{\cdot}35$. [2]

(ii) Show that $A_2 = \$5000(1{\cdot}004^{12} + 1{\cdot}004^{24})$. [1]

(iii) Show that $A_n = \dfrac{\$5000 \times 1{\cdot}004^{12} \times (1{\cdot}004^{12n} - 1)}{1{\cdot}004^{12} - 1}$. [1]

(iv) Find the amount of interest Atticus earns on his savings account over 10 years. [2]

Exam continues overleaf ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the quadratic equation $2x^2 + (m+1)x + (m-1) = 0$.

(i) Find the discriminant in terms of m. [1]

(ii) For what values of m will the quadratic have real roots? [2]

(b) The rate at which fuel is being pumped from a full tank is given by

$$\frac{dF}{dt} = 1 + \frac{5}{1+3t} \text{ kL/min},$$

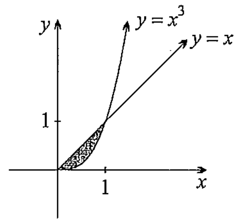
where F kilolitres is the amount of fuel pumped out in the first t minutes.

(i) Find the rate at which the fuel is being pumped out after 8 minutes. [1]

(ii) Draw a sketch of $\frac{dF}{dt}$ as a function of time. [2]

(iii) Find the amount of fuel pumped out after 8 minutes, correct to the nearest litre. [2]

(c)



The diagram above shows the region bounded by $y = x$ and $y = x^3$ from $x = 0$ to $x = 1$.

(i) Find the volume generated when the shaded region is rotated about the x-axis. [2]

(ii) Show that $y = x^{2n-1}$ and $y = x^{2n+1}$ intersect at the origin and the point $(1, 1)$ for $x \geq 0$. [1]

(iii) Suppose that n is a positive integer. Consider the volume V_n of the solid generated when the closed region bounded by the curves $y = x^{2n-1}$ and $y = x^{2n+1}$ is rotated about the x-axis. Show that [2]

$$V_n = \pi\left(\frac{1}{4n-1} - \frac{1}{4n+3}\right).$$

(iv) Give a geometric description and the dimensions of a single solid with volume [1]

$$V_1 + V_2 + V_3 + \cdots.$$

(v) Hence find the sum of the infinite series [1]

$$\frac{1}{3\times 7} + \frac{1}{7\times 11} + \frac{1}{11\times 15} + \cdots.$$

——— End of Section II ———

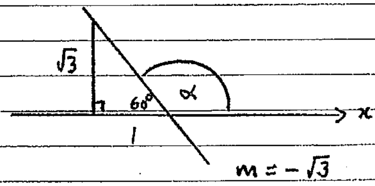
END OF EXAMINATION

VI Mathematics Trial 2012 – Solutions

SECTION I – MULTIPLE CHOICE

(D) 1. $200^\circ = 200^\circ \times \frac{\pi}{180^\circ} = \frac{10\pi}{9}$

(C) 2.



$\tan\alpha = -\sqrt{3}$

$\alpha = 120^\circ$

(C) 3.

$3x - 4y + 6 = 0$ —— ①

$x - y - 1 = 0$ —— ②

$y = x - 1$ —— (2A)

sub (2A) into ①:

$3x - 4(x-1) + 6 = 0$

$3x - 4x + 4 + 6 = 0$

$x = 10$

$y = 9$

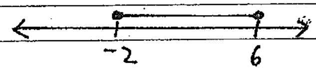
The point of intersection is $(10, 9)$

(B) 4.

$|x-2| \leq 4$

$-4 \leq x - 2 \leq 4$

$-2 \leq x \leq 6$



(B) 5.

$y = x^3 - 4x$

$y' = 3x^2 - 4$

At $(1,-3)$ $y' = 3 - 4 = -1$

Equation of normal: $y + 3 = 1(x-1)$

$y = x - 4$

(B) 6. $f'(a) = 0$, so P is a stationary point.

$f''(a) < 0$, so the curve is concave down at P.

$\therefore$ P is a maximum turning point.

(D) 7.

$3x^2 + 2x - 1 = 0$

$2\alpha + 2\beta = 2(\alpha+\beta)$

$= 2 \times -\frac{2}{3}$

$= -\frac{4}{3}$

or $(3x-1)(x+1) = 0$

$x = \frac{1}{3}$ or -1

$2(\alpha+\beta) = 2\left(\frac{1}{3} - 1\right)$

$= -\frac{4}{3}$

(C) 8.

GP: 24, 12, 6, ... $a = 24$, $r = \frac{1}{2}$

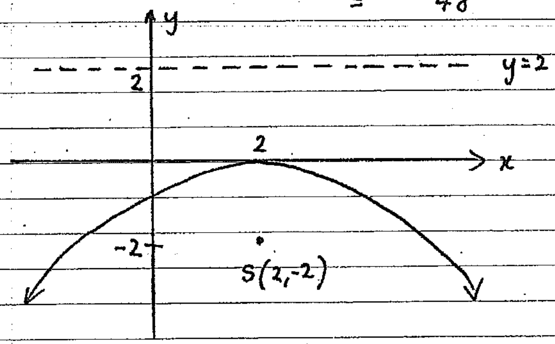
$T_4 = 3$

$S_4 = 45$

$S_\infty = \frac{24}{1 - \frac{1}{2}}$

$= 48$

(B) 9.



$V(2,0)$

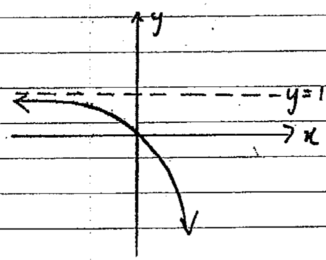
$a = 2$

$(x-h)^2 = -4a(y-k)$

$(x-2)^2 = -8y$

(A) 10.

$y = 1 - 2^x$



This is the curve $y = 2^x$ reflected in the x-axis and then shifted up 1 unit.

NOTE:

as $x \to -\infty$, $2^x \to 0$

$1 - 2^x \to 1$

Question 11

(a) $\dfrac{6}{\sqrt5-\sqrt3} = \dfrac{6}{\sqrt5-\sqrt3} \times \dfrac{\sqrt5+\sqrt3}{\sqrt5+\sqrt3}$ ✓

$= \dfrac{6(\sqrt5+\sqrt3)}{5-3}$

$= 3(\sqrt5+\sqrt3)$ ✓

(b) $l = r\theta$

arc $AB = 100\text{ mm} \times \dfrac{4\pi}{15}$ ✓

$= \dfrac{80\pi}{3}$ mm

$\doteqdot 84$ mm ✓

(c) (i) $AB^2 = 3^2 + 1^2$ ✓

$AB^2 = 10$

$AB = \sqrt{10}$ units

$BC^2 = (1+11)^2 + (0+4)^2$ ✓

$BC^2 = 160$

$BC = \sqrt{160}$

$BC = 4\sqrt{10}$ units

(ii) $m_{AD} = \dfrac{3-0}{0+9}$

$= \dfrac{1}{3}$

$m_{BC} = \dfrac{0+4}{1+11}$

$= \dfrac{1}{3}$ ✓

$\therefore AD \parallel BC$

(iii) $m_{AB} = -\dfrac{3}{1} = -3$

$m_{AB} \times m_{BC} = -3 \times \dfrac{1}{3} = -1$ ✓

$\therefore AB \perp BC$

(iv) $AD^2 = 9^2 + 3^2$

$AD^2 = 90$

$AD = 3\sqrt{10}$ units ✓

$A = \frac{1}{2}h(a+b)$

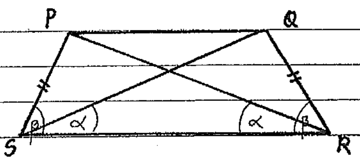
$= \dfrac{AB}{2}(AD+BC)$

$= \dfrac{\sqrt{10}}{2}(3\sqrt{10}+4\sqrt{10})$

$= \dfrac{\sqrt{10}}{2} \times 7\sqrt{10}$

$= 35$ square units ✓

(d)



(i) In Δs PRS and QSR

PS = QR (given)

SR is common ✓✓

∠PSR = ∠QRS = β (given)

∴ ΔPRS ≡ ΔQSR (SAS) ✓

(ii) ~~Let ∠PRS = α.~~

∠QSR = α (matching ∠s of congruent Δs) ✓

∠PSQ = ∠PSR − ∠QSR (adj. ∠s)

= β − α

= ∠QRS − ∠PRS = ∠QRP ✓

15

Question 12

(a) (i) $TU^2 = 7.5^2 + 9^2 - 2\times7.5\times9\times\cos100^\circ$ ✓ (cos rule)

$TU = 12.7$ m (1dp) ✓

(no penalty for incorrect rounding (i) and (ii))

(ii) Area of $\Delta TUV = \frac{1}{2}\times7.5\times9\times\sin100^\circ$ ✓

$= 33.2\text{ m}^2$ (1dp) ✓

(b) (i) $y = 3x^{-2}$

$\dfrac{dy}{dx} = -6x^{-3}$ ✓

$= -\dfrac{6}{x^3}$

(ii) $y = (x^3-2)^{10}$

$\dfrac{dy}{dx} = 30x^2(x^3-2)^9$ ✓

(iii) $y = \dfrac{x}{\cos x}$

$\dfrac{dy}{dx} = \dfrac{\cos x + x\sin x}{\cos^2 x}$ ✓ (numerator) ✓ (denominator)

(c) (i) $\displaystyle\int_1^e \frac{6}{x}\,dx = \Big[6\log_e x\Big]_1^e$ ✓

$= 6\log_e e - 6\log_e 1$

$= 6 - 0$

$= 6$ ✓

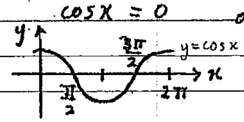
(ii) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx = \Big[\tfrac{1}{2}\tan 2x\Big]_0^{\frac{\pi}{8}}$ ✓

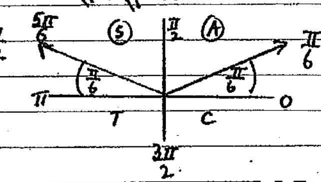
$= \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{2}\tan 0$

$= \dfrac{1}{2}$ ✓

(d) $\cos x(2\sin x - 1) = 0$, $0 \le x \le 2\pi$

$\cos x = 0$ or $\sin x = \frac{1}{2}$





✓ (equations)

$x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ or $\dfrac{3\pi}{2}$ ✓✓

15

Question 13

(a) $(2,-1)$ lies on $\ell : 3x+4y-2=0$

Distance from $(2,-1)$ to $3x+4y+5=0$:

$$\text{Distance} = \frac{|3(2)+4(-1)+5|}{\sqrt{9+16}}$$ ✓

$$= \frac{|7|}{5}$$

$$= 1\tfrac{2}{5} \text{ units}$$ ✓

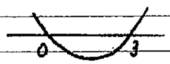
(b) $y = 2x^3 - 9x^2 + 5$

$y' = 6x^2 - 18x$ ✓

The curve is increasing when $y' > 0$

$6x(x-3) > 0$

$x < 0$ or $x > 3$



✓

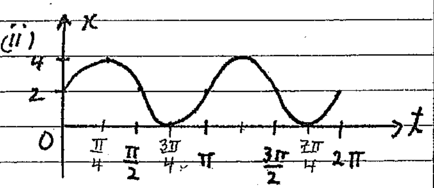
(c) (i) $x = 2 + 2\sin 2t$

when $t=0$, $x = 2 + 2\sin 0$

$x = 2$ m ✓

ie 2 m to the right of O.

(ii)



Period $= \frac{2\pi}{2} = \pi$ ✓ period

✓ intercepts shape

(iii) From the graph,

$\frac{dx}{dt} = 0$ when $t = \frac{\pi}{4}$ s ✓

for the first time.

when $t = \frac{\pi}{4}$, $x = 2 + 2\sin\frac{\pi}{2}$

$x = 2 + 2$

$x = 4$ m ✓

(OR solve $v = 0$
$4\cos 2t = 0$.)

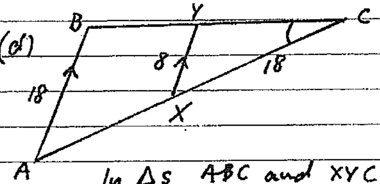
(iv) $v = 4\cos 2t$

$\max|v| = 4$ m/s ✓

when $t = 0, \frac{\pi}{2}, \pi,$ OR ... ✓

(one time required)

(d)



(i) In Δs ABC and XYC

$\angle C$ is common

$\angle ABC = \angle XYC$ (corresp. $\angle$s, $AB \parallel XY$) ✓

$\therefore \Delta ABC \parallel\!\mid \Delta XYC$ (AA) ✓

(ii) $\frac{AX+18}{18} = \frac{18}{8}$

(matching sides of similar Δs in the same ratio) ✓

$AX + 18 = \frac{9}{4} \times 18$

$AX = \frac{81}{2} - 18$

$AX = 22\frac{1}{2}$ cm ✓

15

Question 14

(a) (i)

x	1	1.5	2
$f(x)$	0	0.164	0.480

✓ (3dp)

$f(x) = (\log_e x)^2$

(ii) $\int_1^2 (\log_e x)^2\,dx \doteq \frac{2-1}{6}\left(0 + 4(\log_e 1.5)^2 + (\log_e 2)^2\right)$ ✓

$\doteq 0.19$ (2 dp) ✓

(b) (i) $1+2+3+\ldots+300 = \frac{300}{2}(1+300)$

$= 45\,150$ ✓

(ii) Integers divisible by 3 :

$3+6+9+\ldots+300 = \frac{100}{2}(3+300)$

$= 50 \times 303$

$= 15\,150$ ✓

Sum of integers not divisible by 3 (from 1 to 300) $= 45150 - 15150$

$= 30\,000$ ✓

(c) (i) $f'(x) = 12x - kx^2$ and $f''(x) = 12 - 2kx$

inflexion at $(1,-4)$

$f''(1) = 0$

$12 - 2k = 0$

$2k = 12$

$k = 6$ ✓

note: there is a change in concavity at $(1,-4)$.

$f''(x) = 12 - 12x$

x	0	1	2
$f''(x)$	12	0	-12

Question 14 (continued)

(c) (ii) $f'(x) = 12x - 6x^2$

$f(x) = 6x^2 - 2x^3 + c$ ✓

$(1,-4)$ lies on $y = f(x)$, so $f(1) = -4$:

$6 - 2 + c = -4$

$c = -8$

$\therefore \quad f(x) = 6x^2 - 2x^3 - 8$ ✓

(d) (i) $y = x \log_e x$

$y' = 1 \times \log_e x + x \times \frac{1}{x}$

$= \log_e x + 1$ ✓

(ii) when $y' = 0$

$\log_e x + 1 = 0$

$\log_e x = -1$

$x = e^{-1}$ ✓

$x = \frac{1}{e}$

when $x = \frac{1}{e}$, $y = \frac{1}{e} \log_e \frac{1}{e}$

$= -\frac{1}{e}$

$y'' = \frac{1}{x}$

when $x = \frac{1}{e}$, $y'' = \frac{1}{\frac{1}{e}}$

$= e$

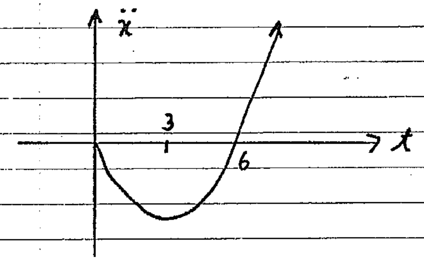
> 0 ✓

so minimum y occurs when $x = \frac{1}{e}$.

(This is the absolute minimum for the natural domain $x > 0$.)

The minimum value of $x \log_e x$ is $-\frac{1}{e}$. ✓

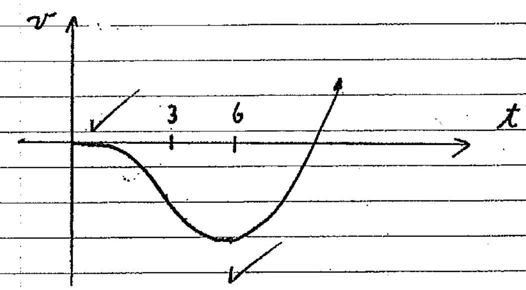
Question 14 (continued)



$\frac{dv}{dt} = 0$ when $t = 0$ and $t = 6$. (stationary points)

minimum $\frac{dv}{dt}$ occurs when $t = 3$. (point of inflexion)

Given $v = 0$ when $t = 0$:



Question 15

(a) (i) $G = G_0 e^{kt}$

when $t=0$, $G_0 = 10\,000$

when $t=8$, $40\,000 = 10000\, e^{8k}$ ✓

$e^{8k} = 4$

$8k = \ln 4$

$k = \frac{1}{8} \times 2\ln 2$ ✓

$k = \frac{1}{4}\ln 2$

(ii) 2 years = 24 months

when $t = 24$, $G = 10\,000\, e^{24 \times \frac{1}{4}\ln 2}$ ✓

$= 10\,000\, e^{6\ln 2}$

$= 10\,000 \times 2^6$

$= 640\,000$ ✓

(iii) when $10\,000\,000 = 10\,000\, e^{kt}$

$e^{kt} = 1000$

$kt = \ln 1000$

$t = \dfrac{\ln 1000}{\frac{1}{4}\ln 2}$ ✓

$t \doteqdot 39.86$

So the population exceeds 10 million after 40 whole months. ✓

(b) $\text{Area} = \int_0^1 (e^x - 1)\,dx - \int_{-2}^0 (e^x - 1)\,dx$ ✓

$= \left[e^x - x\right]_0^1 - \left[e^x - x\right]_{-2}^0$ ✓

$= e^1 - 1 - 1 - \left(1 - \left(\frac{1}{e^2} + 2\right)\right)$

$= e - 2 + 1 + \frac{1}{e^2}$

$= e + \frac{1}{e^2} - 1$ square units ✓

Question 15 (continued)

(c) 4.8% p.a. $= \frac{4.8}{12}\%$ per month

$= 0.004$ per month

(i) $A_1 = \$5000\,(1+R)^{12}$

$= \$5000\,(1.004)^{12}$ ✓

$= \$5245.35$ ✓

(ii) $A_2 = (A_1 + \$5000) \times 1.004^{12}$

$= (\$5000 \times 1.004^{12} + \$5000) \times 1.004^{12}$ ✓

$= \$5000 \times 1.004^{24} + \$5000 \times 1.004^{12}$

$= \$5000\,(1.004^{12} + 1.004^{24})$

(iii) $A_3 = \$5000\,(1.004^{12} + 1.004^{24} + 1.004^{36})$

$A_n = \$5000\,(1.004^{12} + 1.004^{24} + 1.004^{36} + \dots + 1.004^{12n})$

GP: $a = 1.004^{12}$, $r = 1.004^{12}$, n terms; $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$A_n = \$5000 \times \dfrac{1.004^{12}\left((1.004^{12})^n - 1\right)}{1.004^{12} - 1}$ ✓

$\therefore A_n = \dfrac{\$5000 \times 1.004^{12} \times (1.004^{12n} - 1)}{1.004^{12} - 1}$

(iv) Interest $= A_{10} - \$5000 \times 10$ ✓

$= \$65\,689.84 - \$50\,000$

$= \$15\,689.84$ ✓

/15

Question 16

(a) $2x^2 + (m+1)x + (m-1) = 0$

(i) $\Delta = (m+1)^2 - 4 \times 2 \times (m-1)$

$= m^2 + 2m + 1 - 8m + 8$

$= m^2 - 6m + 9$

(ii) Real roots occur when $\Delta \geq 0$

$(m-3)^2 \geq 0$

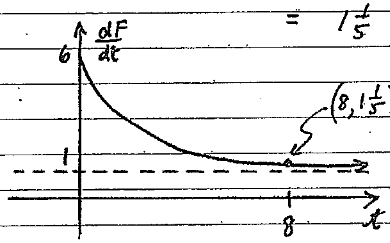
So the quadratic will have real roots for <u>all real m</u>.

(b) (i) $\dfrac{dF}{dt} = 1 + \dfrac{5}{1+3t}$ kL/min

when $t = 8$, $\dfrac{dF}{dt} = 1 + \dfrac{5}{1+24}$

$= 1\frac{1}{5}$ kL/min

(ii)



shape, (0,6)

$t \geq 0$, asymptote

(iii) From $t = 0$ to $t = 8$:

$F = \int_0^8 1 + \dfrac{5}{1+3t}\, dt$

$= \left[t + \frac{5}{3}\log(1+3t) \right]_0^8$

$= 8 + \frac{5}{3}\log 25 - \left(0 + \frac{5}{3}\log 1\right)$

$= 8 + \frac{5}{3}\log 25$ kL

$\doteqdot 13\ 365$ L (nearest L)

Question 16 (continued)

(c) (i) $V = \pi \int_0^1 x^2 - x^6\, dx$

$= \pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1$

$= \pi \left(\frac{1}{3} - \frac{1}{7} \right)$

$= \dfrac{4\pi}{21}$ cubic units

(ii) when $x^{2n-1} = x^{2n+1}$ (OR BY SUBSTITUTION)

$x^{2n-1}(1 - x^2) = 0$

$x = 0$ or 1, for $x \geq 0$

when $x = 0$, $y = 0^{2n+1} = 0$

when $x = 1$, $y = 1^{2n+1} = 1$

} so the curves intersect at (0,0) and (1,1).

(iii) $V_n = \pi \int_0^1 (x^{2n-1})^2\, dx - \pi \int_0^1 (x^{2n+1})^2\, dx$

$= \pi \int_0^1 \left(x^{4n-2} - x^{4n+2} \right) dx$

$= \pi \left[\dfrac{x^{4n-1}}{4n-1} - \dfrac{x^{4n+3}}{4n+3} \right]_0^1$

$= \pi \left(\dfrac{1}{4n-1} - \dfrac{1}{4n+3} \right)$

(iv) $V_1 + V_2 + V_3 + \ldots$ gives the volume of a <u>cone</u> with height 1 unit and radius 1 unit.

(v) From part (iv), $V_1 + V_2 + V_3 + \ldots = \frac{1}{3}\pi(1)^2(1)$

$V_1 + V_2 + V_3 + \ldots = \dfrac{\pi}{3}$

From part (iii), $\pi\left(\frac{1}{3} - \frac{1}{7} + \frac{1}{7} - \frac{1}{11} + \frac{1}{11} - \frac{1}{15} + \ldots \right) = \dfrac{\pi}{3}$

$\dfrac{4}{3 \times 7} + \dfrac{4}{7 \times 11} + \dfrac{4}{11 \times 15} + \ldots = \dfrac{1}{3}$

$\therefore \dfrac{1}{3 \times 7} + \dfrac{1}{7 \times 11} + \dfrac{1}{11 \times 15} + \ldots = \dfrac{1}{12}$.