2009 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Trialmaths Enterprises

Mathematics Extension I

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of past HSC papers. Moreover, some questions have been adapted from previous HSC examinations as well as from trial examinations from a variety of schools, in an attempt to provide students with exposure to a broad range of possible questions.

However, there is no guarantee whatsoever that the 2009 HSC examination will have similar content, style or format. This paper is intended only as a trial for the HSC examination or as revision leading to the examination.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax dx, = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(x+\sqrt{x^2-a^2}),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x+\sqrt{x^2+a^2})$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (12 marks) Use a separate page/booklet **Marks**

(a) One of the roots of $2x^2 - ax + 4 = 0$ is $x = -1$. What is the value of a? **1**

(b) Differentiate with respect to x: $y = \tan^{-1}\left(\frac{1}{x}\right)$ **3**

(c) State the domain and range of the function

$$y = \cos^{-1} 2x$$ **2**

(d) A ball is thrown from the origin with an initial velocity of 30m/s at an angle $\theta = \tan^{-1}\frac{3}{4}$ to the horizontal. If air resistance is neglected and the acceleration due to gravity is taken as 10 m/s^2

(i) find the greatest height attained. **2**

(ii) the velocity of the ball after $\frac{1}{2}$ second. **1**

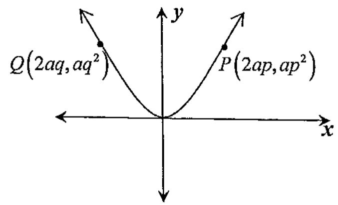
(e) The sides of a cube are decreasing at the rate of 3cm/s. Find at what rate the volume is decreasing, when the sides are each 20 cm. **3**

Question 2 (12 marks) Use a separate page/booklet **Marks**

(a) (i) Show that $\frac{1+\cos 2A}{\sin 2A} = \cot A$ **2**

(ii) Hence find the exact value of $\cot 15^0$ **2**

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

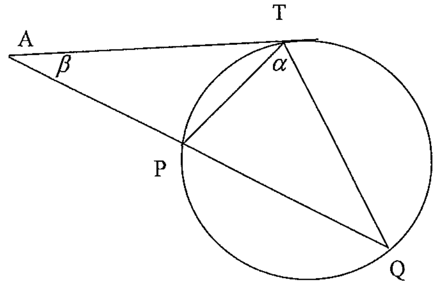


(i) Show that the equation of the tangent at Q is given by $y - qx + aq^2 = 0$ **2**

(ii) If the tangent at P and the tangent at Q intersect at 45^0 show that

$$|1 + pq| = |p - q|$$ **2**

(c) AT is a tangent to the circle, touching it at T. If $\angle TAP = \beta$ and $\angle PTQ = \alpha$ express the angles TQA and TPA in terms of α and β. **4**



NOT TO SCALE

Question 3 (12 marks) Use a separate page/booklet **Marks**

(a) Consider the function $f(x) = 3x - x^3$

(i) Sketch $y = f(x)$, showing the x and y intercepts and the coordinates of the stationary points. **3**

(ii) Find the largest domain containing the origin for which $f(x)$ has an inverse function $f^{-1}(x)$. **1**

(iii) State the domain of $f^{-1}(x)$ **1**

(iv) Find the gradient of the inverse function at $x = 0$ **2**

(b) A polynomial $P(x)$ of degree three, has zeros at $x = 1, x = -1,$ and $x = 2$ and a remainder of 16 when divided by $(x-3)$. Find $P(x)$, expressing it in the form $P_0x^3 + P_1x^2 + P_2x + P_3$ **2**

(c) Using the substitution $u = x^4$, or otherwise, evaluate $\int_0^1 \frac{x^3 dx}{1+x^8}$ **3**

Question 4 (12 marks) Use a separate page/booklet. **Marks**

(a) Find the term independent of x in the expansion $\left(\sqrt{x} - \frac{1}{x}\right)^{12}$ **3**

(b) A particle is moving in simple harmonic motion in a straight line. It's speed v cm/s when at a distance x cm from the centre of oscillation, 0, is given by

$$v^2 = \pi^2\left(9 - x^2\right)$$

(i) Between which two points on the line does it oscillate? **1**

(ii) What is the acceleration as a function of x? **1**

(iii) State the period. **1**

(iv) If the particle is initially at $x = 1.5$, find how many seconds has elapsed before it reaches $x = \frac{3\sqrt{2}}{2}$ for the first time. **3**

(c) Evaluate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}}$ **3**

Question 5 (12 marks) Use a separate page/booklet. **Marks**

(a) Use mathematical induction to prove that, for every positive integer n,

$$3^n \geq 1+2n$$ **3**

(b) (i) Show that the function $f(x)=x^3-\frac{3}{2}x^2-x+\frac{3}{2}$ has a zero for x between $1\frac{1}{4}$ and 2. **1**

(ii) Taking $x=2$ as a first approximation to this zero, use Newton's method to calculate a second approximation. **2**

(c) Prove $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\left(\frac{\theta}{2}\right)$ where $0<\theta<\frac{1}{2}\pi$ **3**

(d) When x cm from the origin, the acceleration of a particle moving in a straight line is given by:

$$\frac{d^2x}{dt^2}=\frac{-5}{(x+2)^3}$$

It has an initial velocity of 2 cm/s at $x=0$. If the velocity is V cm/s, find V in terms of x. **3**

Question 6 (12 marks) Use a separate page/booklet **Marks**

(a) If α,β,θ are the roots of the equation

$$3x^3-3x^2-4x+6=0$$

Find the values of

(i) $\alpha^{-1}+\beta^{-1}+\theta^{-1}$ **2**

(ii) $\alpha^2+\beta^2+\theta^2$ **2**

(b) If $\sin(\ln x)=a$ and $\cos(\ln y)=\frac{1}{b}, b\neq 0$, and $a\neq b$

(i) Using, $\sin^2 A+\cos^2 A=1$, show that $\cos(\ln x)=\pm\sqrt{1-a^2}$ **1**

(ii) Show that $\tan(\ln x)=\pm\frac{a}{\sqrt{1-a^2}}$ and $\tan(\ln y)=\pm\sqrt{b^2-1}$ **2**

(iii) Find an expression for $\tan[\ln(xy)]\times\tan\left[\ln\left(\frac{x}{y}\right)\right]$ in terms of a and b. **3**

(iv) Hence find the possible values of a when $\tan[\ln(xy)]\times\tan\left[\ln\left(\frac{x}{y}\right)\right]=1$ **2**

Question 7 (12 marks) Use a separate page/booklet **Marks**

(a) Three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio 6:3:1.

(i) Find the value of n. **4**

(ii) State which terms have their coefficients in the ratio 6:3:1. **1**

(b) There are three boxes A, B, C into which different cards are placed without regard to the order of cards within any box.

(i) Find the total number of ways in which 10 cards may be distributed with 5 in A, 3 in B and 2 in C. **2**

(ii) Four cards are distributed at random between A, B and C. Find the probability that each box will contain at least one card. **3**

(iii) Three cards are distributed into the boxes with each card having probability of $\frac{3}{10}$ of being placed in A and $\frac{2}{5}$ probability of being placed in B. Find the probability that each of the three boxes will contain a card. **2**

2009 TRIAL HSC EXTENSION 1 MATHEMATICS

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- **Solutions including marking scale**
- **Mapping grid**

These suggested solutions and marking schemes are issued as a guide only. Individual teachers may find many acceptable responses and employ different marking schemes.

ANSWERS QUESTION 1

Question1 (a) –(e)

Criteria
• (a) One mark for the answer. (b) One mark for $\frac{dy}{dx}=\frac{1}{1+u^2}\times\frac{-1}{x^2}$, one for $\frac{dy}{dx}$ in terms of x and one for simplification. (c) One for the domain and one for the range. • (d) (i) One for time of flight and one for the height. (ii) One for the answer. (e) One for the use of the chain rule, one for finding x and one for finding the volume.

Answers:

(a) $2x^2-ax+4=0$

let $x=-1$

$\therefore 2(-1)^2-a(-1)+4=0 \therefore a=-6$

(b)

$y=\tan^{-1}\left(\frac{1}{x}\right)$

let $y=\tan^{-1}u$ where $u=\frac{1}{x}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=\frac{1}{1+u^2}\times\frac{-1}{x^2}=\frac{1}{1+\frac{1}{x^2}}\times\frac{-1}{x^2}$

$=\frac{x^2}{x^2+1}\times\frac{-1}{x^2}=\frac{-1}{x^2+1}$

(c)

$y=\cos^{-1}2x$

Domain : $\frac{-1}{2}\le x\le\frac{1}{2}$

Range : $0\le y\le\pi$

(d) (i)

$\ddot{y}=-10 \Rightarrow \dot{y}=-10t+v\sin\theta=-10t+30\times\frac{3}{5}=-10t+18$

$y=-5t^2+18t+c_1$

$t=0, y=0 \therefore c_1=0 \Rightarrow y=-5t^2+18t$

greatest height when $\dot{y}=0$

$0=-10t+18 \therefore t=1.8$

$y=-5(1.8)^2+18\times1.8=16.2m$

(ii)

$v^2=13^2+24^2$

$v=\sqrt{745}$

$=27.29$m/s

(e)

Let x be the length of each edge of the cube

$v=x^3 \therefore \frac{dv}{dx}=3x^2$

$\frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}$

$=3x^2\times-3$

when $x=20 \therefore \frac{dv}{dt}=3\times20^2\times-3=-3600\text{cm}^3/\text{s}$

$\therefore$ The volume of the block is then decreasing at the rate of $3600\text{cm}^3/\text{s}$

$\ddot{x}=0$

$\dot{x}=v\cos\theta$

$=30\times\frac{4}{5}\left(\tan\theta=\frac{3}{4}\therefore\sin\theta=\frac{3}{5}\text{ and }\cos\theta=\frac{4}{5}\right)$

$=24$

$x=24t+c$

$t=0,x=0\therefore c=0\therefore x=24t$

ANSWERS QUESTION 2

Question 2 (a)- (c)

Criteria
(a) (i) one for $\frac{1+\cos 2A}{\sin 2A}=\frac{1+2\cos^2 A-1}{2\sin A\cos A}$ and one for simplification (ii) one for correct substitution of ratios and one for simplification. (b) (i) One for gradient and one for equation. (ii) one for $\tan 45^0=\left\lvert\frac{p-q}{1+pq}\right\rvert$ and one for simplification.(c) One each for $\angle QTA=180^0-(\theta+\beta)$, $\angle TQA=90^0-\frac{(\alpha+\beta)}{2}$, $\angle TPA=180^0-\left[\beta+90^0-\frac{(\alpha+\beta)}{2}\right]$ and $\angle TPA=90^0+\frac{(\alpha-\beta)}{2}$

(a)(i) $\text{LHS}=\frac{1+2\cos^2 A-1}{2\sin A\cos A}$

$=\frac{2\cos^2 A}{2\sin A\cos A}$

$=\cot A=\text{RHS}$

(ii) sub $A=15^0$

$\cot 15^0=\frac{1+\cos 30^0}{\sin 30^0}$

$=\frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$=\frac{2+\sqrt{3}}{2}\times 2$

$=2+\sqrt{3}$

(b)(i) $x^2=4ay\therefore y=\frac{x^2}{4a}$

$\frac{dy}{dx}=\frac{x}{2a}$ at $x=2aq\therefore\frac{dy}{dx}=\frac{2aq}{2a}=q$

Eqn of tangent $y-aq^2=q(x-2aq)$

$y-qx+aq^2=0$

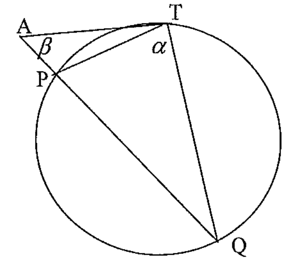
(b) (ii)

$\tan\theta=\left\lvert\frac{m_1-m_2}{1+m_1m_2}\right\rvert$

$\tan 45^0=\left\lvert\frac{p-q}{1+pq}\right\rvert$

$1=\left\lvert\frac{p-q}{1+pq}\right\rvert\Rightarrow\lvert 1+pq\rvert=\lvert p-q\rvert$

(c)



Let $\angle TQA=\theta$

$\therefore\angle PTA=\theta$ (alt.angle thm)

$\angle QTA=180^0-(\theta+\beta)$ [angle sum of ΔTAQ]

$\angle QTA=\alpha+\theta$

$\therefore 180^0-(\theta+\beta)=\alpha+\theta$

$180^0-\theta-\beta=\alpha+\theta$

$180^0-\alpha-\beta=2\theta$

$\theta=\angle TQA=90^0-\frac{(\alpha+\beta)}{2}$

$\angle TPA=180^0-(\beta+\theta)$

$=180^0-\left[\beta+90^0-\frac{(\alpha+\beta)}{2}\right]$

$=180^0-\beta-90^0+\frac{\alpha}{2}+\frac{\beta}{2}$

$=90^0+\frac{(\alpha-\beta)}{2}$.

ANSWERS QUESTION 3

Question 3 (a) (b) (c)

Criteria

- (a) (i) one each for stationary points and nature and one for sketch showing intercepts, turning points. (ii) and (iii) one for the correct answer (iv) One for $\frac{dy}{dx}=\frac{1}{3-3y^2}$ and one for the answer.
- (b) One for finding k and one for simplification. (c) One for limits of integrand, one for integration and one for simplification.

(a) (i) $f(x)=3x-x^3$

$f'(x)=3-3x^2$

st pts $0=3-3x^2$

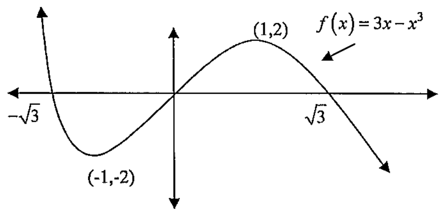
$0=3(1-x^2)$

$\therefore (1,2)$ and $(-1,-2)$

$f''(x)=-6x$

$f''(1)<0 \therefore$ rel max at $(1,2)$

$f''(-1)>0 \therefore$ rel min at $(-1,-2)$



(ii) $-1\le x\le 1$

(iii) range of $f(x)$ is $-2\le y\le 2$

$\therefore$ domain of $f^{-1}(x)$ is $-2\le x\le 2$

(iv) Inverse function

$x=3y-y^3$

$\frac{dx}{dy}=3-3y^2$

$\therefore \frac{dy}{dx}=\frac{1}{3-3y^2}$ if $x=0 \therefore y=0$

$\frac{dy}{dx}=\frac{1}{3-3(0)^2}=\frac{1}{3}$

(b)

$P(x)=k(x-1)(x+1)(x-2)=0$

Since $P(3)=16$

$\therefore 16=k(3-1)(3+1)(3-2)$

$\therefore k=2$

$P(x)=2(x-1)(x+1)(x-2)$

$=2(x^2-1)(x-2)$

$=2(x^3-x-2x^2+2)$

$=2(x^3-2x^2-x+2)$

(c)

If $u=x^4$

$\frac{du}{dx}=4x^3$

$du=4x^3dx$

when $x=0 \therefore u=0$

when $x=1 \therefore u=1$

$\int_0^1\frac{x^3dx}{1+x^8}=\frac{1}{4}\int_0^1\frac{4x^3dx}{1+(x^4)^2}$

$=\frac{1}{4}\int_0^1\frac{du}{1+u^2}$

$=\frac{1}{4}\left[\tan^{-1}u\right]_0^1$

$=\frac{1}{4}\left(\frac{\pi}{4}-0\right)$

$=\frac{\pi}{16}$

ANSWERS QUESTION 4

Question 4

Criteria

- One for $U_{r+1}={}^{12}C_r\left(x^{6-\frac{3r}{2}}\right)(-1)^r$, one for finding r and one for the term (b) (i)-(iii) one for correct answer (iv) One for finding α, one for $\cos\left(\pi t+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$ and one for the time.(c) One for $\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{9-4x^2}}$ $=\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\frac{3}{2}}$ $\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{9-4x^2}}=\frac{1}{2}\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{\left(\frac{3^2}{2^2}-x^2\right)}}$ and one for simplification

(a)

$U_{r+1}={}^{12}C_r\left(x^{\frac{1}{2}}\right)^{12-r}\left(-\frac{1}{x}\right)^r$

$={}^{12}C_r\left(x^{6-\frac{r}{2}}\right)\frac{(-1)^r}{x^r}$

$={}^{12}C_r\left(x^{6-\frac{3r}{2}}\right)(-1)^r$

term independent of x has power of $x=0$

$\therefore 6-\frac{3r}{2}=0\Rightarrow 3r=12\Rightarrow r=4$

$\therefore$ term independent of x is ${}^{12}C_4(-1)^4=495$

(b)(i)

$v^2=n^2(a^2-x^2)$

$=\pi^2(9-x^2)$

$\therefore n=\pi, a=3$

$\therefore$ oscillates between -3 and 3

(b)(ii)

$x=a\cos(nt+\alpha)$

$x=3\cos(\pi t+\alpha)$

$\dot{x}=-3\pi\sin(\pi t+\alpha)$

$\ddot{x}=-3\pi^2\cos(\pi t+\alpha)$

$=-\pi^2x$

(b)(iii)

period $=\frac{2\pi}{n}=\frac{2\pi}{\pi}=2$

(b)(iv)

$t=0, x=1.5$

$x=3\cos(\pi t+\alpha) \therefore 1.5=3\cos\alpha$

$\cos\alpha=\frac{1}{2} \therefore \alpha=\frac{\pi}{3}$

$x=3\cos\left(\pi t+\frac{\pi}{3}\right)$

$\frac{3\sqrt{2}}{2}=3\cos\left(\pi t+\frac{\pi}{3}\right)$

$\cos\left(\pi t+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$

$\pi t+\frac{\pi}{3}=\frac{\pi}{4}$ or $\pi t+\frac{\pi}{3}=\frac{7\pi}{4}$

$t=\frac{-1}{12}$ or $1\frac{5}{12} \therefore t>0 \therefore t=1\frac{5}{12}$

$\therefore$ reaches $\frac{3\sqrt{2}}{2}$ cm for the first time after $1\frac{5}{12}$ secs

(c)

$$\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{9-4x^2}}=\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}$$

$$=\frac{1}{2}\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{\left(\frac{3^2}{2^2}-x^2\right)}}$$

$$=\left[\frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right)\right]_0^{\frac{3}{2}}$$

$$=\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\frac{3}{2}}$$

$$=\frac{1}{2}\sin^{-1}1-\frac{1}{2}\sin^{-1}0$$

$$=\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{4}$$

Question 5

Criteria
• (a) one for the $3^{k+1}\geq 1+2(k+1)$,one for $3^{k+1}=3\times 3^k$ and one for $3^{k+1}\geq 3+2k+4k$ (b) (i) One for the correct answer. (ii) One for $f'(x_1)=3x^2-3x-1$ and one for simplification (c) one for $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$,one for $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\frac{2t^2+2t}{2+2t}$ and one for conclusion.
• (d) One for $\frac{1}{2}v^2=\frac{-5}{-2(x+2)^2}+c$, one for finding the constant and one for simplification.

(a) $3^n\geq 1+2n$

Step 1 prove true for n=1

$3^1\geq 1+2\times 1\ \therefore$ true

Step 2 assume true for n=k

assume $3^k\geq 1+2k$

Step 3 Prove true for n=k+1

$3^{k+1}\geq 1+2(k+1)$

$\geq 2k+3$

$3^{k+1}=3\times 3^k$

$\geq 3(1+2k)$

$\geq 3+6k$

$\geq 3+2k+4k$

$\therefore$ true

Step 4 conclusion

By assuming the result is true for n=k we have proved it true for n=k+1. Since it is true for n=1 $\therefore$ it is true for n=1+1=2, n=2+1=3 and so on $\therefore$ true for all positive integers.

(b) (i)

$$f(x)=x^3-\frac{3}{2}x^2-x+\frac{3}{2}$$

$$f\left(1\frac{1}{4}\right)=\left(1\frac{1}{4}\right)^3-\frac{3}{2}\left(1\frac{1}{4}\right)^2-1\frac{1}{4}+\frac{3}{2}$$

$$=\frac{125}{64}-\frac{3}{2}\times\frac{25}{16}-\frac{5}{4}+\frac{3}{2}<0$$

$$f(2)=2^3-\frac{3}{2}(2)^2-2+\frac{3}{2}=\frac{3}{2}>0$$

$\therefore$ a zero between $x=1\frac{1}{4}$ and x=2

(c) $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\left(\frac{\theta}{2}\right)$

$\sin\theta=\frac{2t}{1+t^2}$, $\cos\theta=\frac{1-t^2}{1+t^2}$ where $t=\tan\frac{\theta}{2}$

$$\text{LHS}=\frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$=\frac{\frac{1+t^2+2t-1+t^2}{1+t^2}}{\frac{1+t^2+2t+1-t^2}{1+t^2}}$$

$$=\frac{2t^2+2t}{2+2t}$$

$$=\frac{2t(t+1)}{2(1+t)}=t=\tan\frac{\theta}{2}$$

(b)(ii)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ where } f'(x_1) = 3x^2 - 3x - 1$$

$$= 2 - \frac{\frac{3}{2}}{5}$$

$$= 1\frac{7}{10}$$

(d) $\frac{d^2x}{dt^2} = \frac{-5}{(x+2)^3}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-5}{(x+2)^3} = -5(x+2)^{-3}$$

$$\frac{1}{2}v^2 = \frac{-5}{-2(x+2)^2} + c$$

$v = 2$ at $x = 0$

$$\therefore 2 = \frac{5}{8} + c$$

$$\frac{1}{2}v^2 = \frac{5}{2(x+2)^2} + \frac{11}{8}$$

$$v^2 = \frac{5}{(x+2)^2} + \frac{11}{4}$$

$$= \frac{20 + 11(x+2)^2}{4(x+2)^2}$$

$$v = \frac{\sqrt{20 + 11(x+2)^2}}{2(x+2)} \text{ since } v > 0 \text{ when } x = 2$$

Question 6

Criteria
• (a) (i) One for $\beta\theta + \alpha\theta + \alpha\beta = \frac{-4}{3}$ and one for simplification (ii) one for sum of roots is one and one for simplification (b) (i) One for correct answer. (ii) one for $\tan(\ln y) = \pm\sqrt{b^2 - 1}$ and one for simplification (iii) one for $\tan[\ln(xy)] \times \tan\left[\ln\left(\frac{x}{y}\right)\right] = \frac{\tan^2(\ln x) - \tan^2(\ln y)}{1 - \tan^2(\ln x) \times \tan^2(\ln y)}$, one for $\tan[\ln(xy)] \times \tan\left[\ln\left(\frac{x}{y}\right)\right] = \frac{\frac{a^2}{1-a^2} - (b^2 - 1)}{1 - \frac{a^2}{1-a^2} \times (b^2 - 1)}$ and one for simplification. (iv) one for $2a^2b^2 - b^2 = 0$ and one for simplification.

(a) (i)

$3x^3 - 3x^2 - 4x + 6 = 0$

$$\alpha^{-1} + \beta^{-1} + \theta^{-1} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\theta}$$

$$= \frac{\beta\theta + \alpha\theta + \alpha\beta}{\alpha\beta\theta}$$

$$= \frac{\frac{-4}{3}}{\frac{-6}{3}} = \frac{2}{3}$$

(a)(ii) $\alpha^2 + \beta^2 + \theta^2 = (\alpha + \beta + \theta)^2 - 2(\alpha\beta + \alpha\theta + \beta\theta)$

$$= 1^2 - 2 \times \frac{-4}{3} = 3\frac{2}{3}$$

(b)(i) Using $\sin^2 A + \cos^2 A = 1$

$$\cos A = \pm\sqrt{1 - \sin^2 A}$$

$$\cos(\ln x) = \pm\sqrt{1 - \sin^2(\ln x)}$$

$$\cos(\ln x) = \pm\sqrt{1 - a^2}$$

(b)(ii)

$$\sin A = \pm\sqrt{1 - \cos^2 A}$$

$$\sin(\ln y) = \pm\sqrt{1 - \cos^2(\ln y)}$$

$$\cos(\ln y) = \frac{1}{b} \Rightarrow \sin(\ln y) = \pm\sqrt{1 - \frac{1}{b^2}}$$

$$\sin(\ln y) = \frac{\pm\sqrt{b^2 - 1}}{b}$$

$$\tan(\ln y) = \frac{\sin(\ln y)}{\cos(\ln y)}$$

$$= \frac{\pm\sqrt{b^2 - 1}}{b} \div \frac{1}{b}$$

$$= \pm\sqrt{b^2 - 1}$$

$$\tan(\ln x) = \frac{\sin(\ln x)}{\cos(\ln x)}$$

$$= \frac{\pm a}{\sqrt{1 - a^2}}$$

(b)(iii)

$$\tan[\ln(xy)] \times \tan\left[\ln\left(\frac{x}{y}\right)\right]$$

$$= \tan(\ln x + \ln y) \times \tan(\ln x - \ln y)$$

$$= \frac{\tan(\ln x) + \tan(\ln y)}{1 - \tan(\ln x) \times \tan(\ln y)} \times \frac{\tan(\ln x) - \tan(\ln y)}{1 + \tan(\ln x) \times \tan(\ln y)}$$

$$= \frac{\tan^2(\ln x) - \tan^2(\ln y)}{1 - \tan^2(\ln x) \times \tan^2(\ln y)}$$

$$= \frac{\frac{a^2}{1-a^2} - (b^2 - 1)}{1 - \frac{a^2}{1-a^2} \times (b^2 - 1)}$$

$$= \frac{a^2 - (1 - a^2)(b^2 - 1)}{1 - a^2 - a^2(b^2 - 1)}$$

$$= \frac{a^2b^2 - b^2 + 1}{1 - a^2b^2}$$

(iv)

$$\tan[\ln(xy)] \times \tan\left[\ln\left(\frac{x}{y}\right)\right] = 1$$

$$\frac{a^2b^2 - b^2 + 1}{1 - a^2b^2} = 1$$

$$a^2b^2 - b^2 + 1 = 1 - a^2b^2$$

$$2a^2b^2 - b^2 = 0$$

$$b^2(2a^2 - 1) = 0$$

$$\therefore b = 0 \text{ or } a^2 = \frac{1}{2}$$

$$b \neq 0, \therefore a = \pm\frac{1}{\sqrt{2}}$$

ANSWERS QUESTION 7

Question 1 (a) (i) (ii) (iii) (iv)

Criteria
• (a) one for $\frac{r!(n-r)!}{(r-1)!(n-r+1)!}=2$, one for $\frac{r}{(n-r+1)}=2$, one for $4r=3n-1$ and one for finding n (ii) one for the correct answer. (b) (i) One for indicating $^{10}C_3$ or $^{5}C_3$ ways and one for conclusion. (ii) one for indicating number of ways of distributing the 4 cards is 81, one for indicating number of ways of distributing 2 cards is 12 and one for the final answer. (iii) One for $P(\text{1st card in A, 2nd card in B, 3rd card in C})=\frac{3}{10}\times\frac{2}{5}\times\frac{3}{10}=\frac{9}{250}$ and one for the final answer.

(a)(i)

$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2......+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+{}^nC_{r+1}x^{r+1}+...+{}^nC_nx^n$

Three consecutive coefficients are ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$

The first two coefficients are in the ratio 6:3

$$\therefore \frac{{}^nC_{r-1}}{{}^nC_r}=\frac{6}{3}=2$$

$$\frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}}=2$$

$$\frac{r!(n-r)!}{(r-1)!(n-r+1)!}=2$$

$$\frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}=2$$

$$\frac{r}{(n-r+1)}=2$$

$$r=2(n-r+1)$$

$$=2n-2r+2$$

$$3r=2n+2$$

$$r=\frac{2n+2}{3}$$

The second two coefficients are in the ratio 3:1

$$\frac{{}^nC_r}{{}^nC_{r+1}}=\frac{3}{1}\therefore {}^nC_r=3\,{}^nC_{r+1}$$

$$\frac{n!}{r!(n-r)!}=\frac{3n!}{(r+1)!(n-r-1)!}$$

$$\frac{(r+1)!}{r!}=\frac{3(n-r)!}{(n-r-1)!}$$

$$\frac{(r+1)r!}{r!}=\frac{3(n-r)(n-r-1)!}{(n-r-1)!}$$

$$r+1=3n-3r$$

$$4r=3n-1$$

$$\therefore 4\left(\frac{2n+2}{3}\right)=3n-1$$

$$8n+8=9n-3$$

$$n=11$$

(ii)

The terms with these coefficients are the r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms

$r=\frac{2n+2}{3}$ where $n=11$

$=\frac{2\times11+2}{3}=8$

The terms are 8^{th}, 9^{th} and 10^{th} terms

(b) (i) The five cards in A can be chosen in ${}^{10}C_5$ ways, and the three cards in B can be chosen in ${}^5C_3$ ways. The two remaining cards then go into C.

$\therefore {}^{10}C_5\times{}^5C_3\times1=2520$

(ii)

Since each card can be placed in one of the three boxes, there are $3^4=81$ ways of distributing the 4 cards. If each box is to contain at least 1 card, then one of the boxes must contain 2 cards and the other two 1 card each.

Number of ways of distributing 2 cards in A, one in B and 1 in C is

${}^4C_2\times{}^2C_1\times1=12$

Since the two cards can be in B or C, the number of ways of achieving a 2:1:1 split is $3\times12=36$

$\therefore P(\text{at least 1 card in each box})=\frac{36}{81}=\frac{4}{9}$

(iii)

Consider

$P(\text{1st card in A, 2nd card in B, 3rd card in C})=\frac{3}{10}\times\frac{2}{5}\times\frac{3}{10}=\frac{9}{250}$

There are $3!=6$ ways of placing 1 card in each box

(i.e. ABC, ACB, BCA, CBA, CAB, BAC)

$\therefore P(\text{1 card in each box})=6\times\frac{9}{250}=\frac{27}{125}$