NSW INDEPENDENT SCHOOLS

2017
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A reference sheet is provided
- All necessary working should be shown in Question 11 – 14
- Write your student number and/or name at the top of every page

Total marks – 70

Section I - Pages 2 – 5
10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Section II - Pages 6 – 9
60 marks
Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

Student name / number ____________________

Mark

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1. If $\sin x = \frac{1}{3}$ and $0 < x < \frac{\pi}{2}$, what is the value of $\sin 2x$? 1

(A) $\frac{2}{3}$

(B) $\frac{2\sqrt{2}}{3}$

(C) $\frac{2\sqrt{2}}{9}$

(D) $\frac{4\sqrt{2}}{9}$

2. The equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0,\ b \neq 0,\ c \neq 0$ and $d \neq 0$, has roots α, β and γ. Which of the following is an expression for $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$? 1

(A) $-\frac{a}{b}$

(B) $-\frac{c}{b}$

(C) $-\frac{c}{d}$

(D) $-\frac{b}{c}$

3. In how many ways can the letters of the word SQUARE be arranged in a line so that the three vowels occur next to each other in alphabetical order? 1

(A) 20

(B) 24

(C) 120

(D) 144

4. Which of the following is an expression for the inverse of the function $f(x) = e^{x-2} + 1$? 1

(A) $f^{-1}(x) = \log_e(x-2)+1$

(B) $f^{-1}(x) = \log_e(x+2)-1$

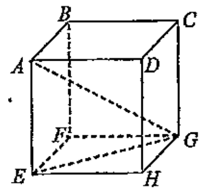
(C) $f^{-1}(x) = \log_e(x-1)+2$

(D) $f^{-1}(x) = \log_e(x+1)-2$

Student name / number ____________________

Mark

5.



In the diagram $ABCDEFGH$ is a cube of side 1 metre.
What is the size of $\angle AGE$ correct to the nearest degree? 1

(A) 35°

(B) 37°

(C) 39°

(D) 41°

6. Which of the following is an expression for $\frac{d}{dx}\tan^{-1}(2x+1)$? 1

(A) $\frac{1}{4x^2+4x+2}$

(B) $\frac{1}{2x^2+2x+1}$

(C) $\frac{1}{4x^2+2}$

(D) $\frac{1}{2x^2+1}$

7. Which of the following is an expression for $\int \cos^2 \frac{x}{2}\, dx$? 1

(A) $\frac{x}{2} + \frac{1}{2}\sin x + c$

(B) $\sin^2 \frac{x}{2} + c$

(C) $2\sin^2 \frac{x}{2} + c$

(D) $x + \sin x + c$

Student name / number ____________________

Mark

8. One of the two shorter sides of a right-angled triangle has fixed length 3 metres. The other of the two shorter sides has variable length x metres and is increasing at a rate of $0 \cdot 25$ metres per minute. What is the rate of increase of the length of the hypotenuse when $x = 4$? 1

(A) $0 \cdot 1$ metres per minute

(B) $0 \cdot 2$ metres per minute

(C) $0 \cdot 3$ metres per minute

(D) $0 \cdot 4$ metres per minute

9. The number $N(t)$ of individuals in a population is given by $N(t) = 100 + 200\, e^{-0 \cdot 1t}$ after t years. What is the time for the population size to fall to half its initial value? 1

(A) $10 \log_e 2$ years

(B) $10 \log_e 3$ years

(C) $10 \log_e 4$ years

(D) $10 \log_e 6$ years

10. A particle is performing Simple Harmonic Motion about a fixed point O on a line. At time t seconds it has displacement x metres from O given by $x = a \cos nt$ for some constants $a > 0$ and $n > 0$. The period of the motion is T seconds. What is the time taken by the particle to move from its starting position to a point half-way towards O? 1

(A) $\frac{T}{12}$

(B) $\frac{T}{9}$

(C) $\frac{T}{8}$

(D) $\frac{T}{6}$

Student name / number ____________________

Mark

Section II

60 Marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

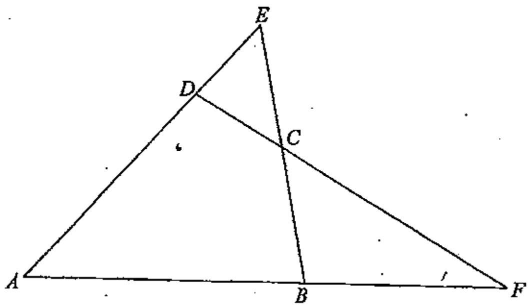
(a) $A(-5, 10)$ and $B(1, -2)$ are two points. Find the coordinates of the point P that divides the interval AB internally in the ratio $3 : 1$. 2

(b) Find in simplest exact form the value of $\int_{-\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}\, dx$. 3

(c) Use the substitution $t = \tan\frac{x}{2}$ to show that $\sec x + \tan x = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$. 3

(d) Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^9$. 3

(e)



In the diagram $ABCD$ is a cyclic quadrilateral. AD produced and BC produced meet at E. AB produced and DC produced meet at F. $\angle DEC = \angle BFC$.

(i) Show that $\angle ADC = \angle ABC$. 2

(ii) Show that AC is a diameter of the circle through A, B, C and D. 2

Student name / number ______________

Question 12 (15 marks) **Use a separate writing booklet.** Mark

(a) Find correct to the nearest degree the acute angle between the lines $3x-y=0$ and $x-3y=0$. 2

(b)(i) Find the domain and range of the function $f(x)=2\cos^{-1}(1-x)$. 2

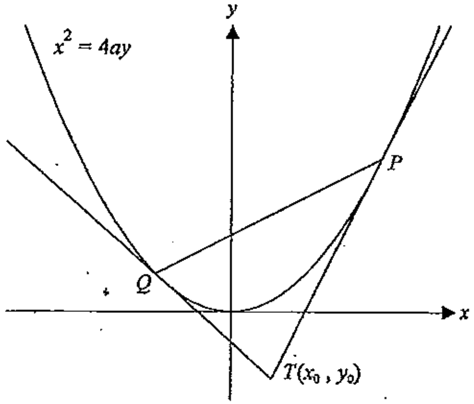
(ii) Sketch the graph of the function $f(x)=2\cos^{-1}(1-x)$ showing the coordinates of the end points. 1

(c)(i) Show that $(x+2)$ is a factor of $P(x)=2x^3+7x^2+4x-4$. 1

(ii) Express $P(x)$ as a product of three linear factors. 2

(d) Use the substitution $u=x+1$ to evaluate in simplest exact form $\int_2^5 \frac{x+2}{(x+1)^2}\,dx$. 3

(e)



In the diagram the tangents to the parabola from $T(x_0, y_0)$ touch the parabola at points P and Q. The chord PQ has equation $xx_0=2a(y+y_0)$. (DO NOT PROVE THIS RESULT)
If the point T lies on the directrix of the parabola, show that

(i) the chord PQ passes through the focus F of the parabola. 2

(ii) TF is perpendicular to PQ. 2

Student name / number ______________

Question 13 (15 marks) **Use a separate writing booklet.** Mark

(a) Use Mathematical Induction to show that $5^n > 4^n + 3^n$ for all integers $n \geq 3$. 3

(b)(i) Given that the function $f(x)=e^x-\tan x$ is continuous for $0<x<\frac{\pi}{2}$, show that the equation $f(x)=0$ has a root between $x=1$ and $x=1\cdot5$. 1

(ii) Use one application of Newton's method with an initial approximation $1\cdot3$ to find the next approximation to this root, giving your answer correct to 2 decimal places. 2

(c) A particle is moving in a straight line. At time t seconds it has displacement x metres to the right of a fixed point O on the line and velocity v ms^{-1} given by $v=\frac{16-x^2}{x}$.
Initially the particle is 1 metre to the right of O.

(i) Show that $x=\sqrt{16-15e^{-2t}}$. 3

(ii) Find the limiting position of the particle. 1

(d) A particle is moving in a straight line performing Simple Harmonic Motion.
At time t seconds it has displacement x metres from a fixed point O on the line given by $x=1+2\cos\left(2t-\frac{\pi}{3}\right)$.

(i) Show that $\ddot{x}=-4(x-1)$. 1

(ii) Find the centre of the motion and the time taken for the particle to first reach maximum speed. 2

(iii) Find the first time the particle is at rest and the amplitude of the motion. 2

Student name / number ____________________

Mark

Question 14 (15 marks) **Use a separate writing booklet.**

(a)(i) Write down the expansion of $(1+x)^n$ in ascending powers of x. **1**

(ii) Hence show that ${}^nC_0+\frac{1}{2}{}^nC_1+\frac{1}{3}{}^nC_2+\ldots+\frac{1}{n+1}{}^nC_n=\frac{2^{n+1}-1}{n+1}$. **3**

(b) Bill and Ben play a game in which a fair die is rolled. Bill wins the game if the die shows 1, 2, 3 or 4. Ben wins the game if the die shows 5 or 6.

(i) If they play the game 6 times, find in simplest fraction form the probability that Bill wins two games more than Ben. **2**

(ii) If they continue playing until one wins two games more than the other, find in simplest fraction form the probability that Bill is the eventual winner. **3**

(c) A particle is projected from a point O on horizontal ground with speed of projection V ms^{-1} at an angle α above the horizontal. It moves in a vertical plane under gravity where the acceleration due to gravity is $9\cdot8$ ms^{-2}. At time t seconds its horizontal and vertical displacements from O, x metres and y metres respectively, are given by $x=Vt\cos\alpha$ and $y=Vt\sin\alpha-4\cdot9t^2$. (DO NOT PROVE THESE RESULTS)

The particle hits the ground X metres from O after time T seconds.

(i) Show that $4\cdot9^2T^4-V^2T^2+X^2=0$. **2**

(ii) Find the two possible values of T if $V=49$ and $X=196$. **2**

(iii) Show that there are two possible values of T provided $V^2>9\cdot8X$. **2**

Section I Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	D	$\sin 2x = 2\sin x\cos x = 2\times\frac{1}{3}\times\sqrt{1-\left(\frac{1}{3}\right)^2} = 2\times\frac{1}{3}\times\sqrt{\frac{8}{9}} = \frac{4\sqrt{2}}{9}$	H5
2	C	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \frac{\left(\frac{c}{a}\right)}{\left(-\frac{d}{a}\right)} = -\frac{c}{d}$	PE3
3	B	S, Q, R, (AEU) can be arranged in $4! = 24$ ways	PE3
4	C	$y = e^{x-2}+1$ $\quad y-1 = e^{x-2}$ $\quad \therefore x-2 = \log_e(y-1)$ $\quad x = \log_e(y-1)+2$ $\quad \therefore f^{-1}(x) = \log_e(x-1)+2$	HE4
5	A	$\angle AEG = 90°,\ AE = 1,\ EG = \sqrt{2}.\ \ \therefore \tan\angle AGE = \frac{1}{\sqrt{2}}\ \ \therefore \angle AGE \approx 35°$	H5
6	B	$\frac{d}{dx}\tan^{-1}(2x+1) = \frac{2}{1+(2x+1)^2} = \frac{2}{4x^2+4x+2} = \frac{1}{2x^2+2x+1}$	HE4
7	A	$\int \cos^2\frac{x}{2}\,dx = \frac{1}{2}\int(1+\cos x)\,dx = \frac{1}{2}(x+\sin x)+c = \frac{x}{2}+\frac{1}{2}\sin x + c$	H8
8	B	$\frac{d}{dt}\sqrt{x^2+3^2} = \frac{2x}{2\sqrt{x^2+3^2}}\frac{dx}{dt} = \frac{4}{5}\times 0\cdot 25 = 0\cdot 2 \quad \therefore 0\cdot 2\text{ m/min}$	HE5
9	C	$N(t) = \frac{1}{2}N(0) = \frac{1}{2}(100+200) = 150$ $\quad \therefore 200e^{-0\cdot 1t} = 50$ $\quad \therefore e^{0\cdot 1t} = 4$ $\quad \therefore 0\cdot 1t = \log_e 4$ $\quad t = 10\log_e 4$	HE3
10	D	$T = \frac{2\pi}{n}$ and $\cos nt = \frac{1}{2}$ $\quad \therefore nt = \frac{\pi}{3}$ $\quad \therefore t = \frac{T}{2\pi}\times\frac{\pi}{3} = \frac{T}{6}$	HE3

Section II

Question 11

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P	1

Answer

$A(-5,10) \quad B(1,-2)$

3 : 1

$$\left(\frac{3-5}{3+1}, \frac{-6+10}{3+1}\right) \qquad \therefore P\left(-\tfrac{1}{2}, 1\right)$$

Q11 (cont)

b. Outcomes assessed: HE4

Marking Guidelines

Criteria	Marks
• finds the primitive function	1
• substitutes limits and simplifies into form appropriate for deducing exact value	1
• evaluates in simplest exact form	1

Answer

$$\int_{-\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\tfrac{x}{2}\right]_{-\sqrt{2}}^{\sqrt{3}} = \sin^{-1}\tfrac{\sqrt{3}}{2} - \sin^{-1}\left(\tfrac{-1}{\sqrt{2}}\right) = \tfrac{\pi}{3}+\tfrac{\pi}{4} = \tfrac{7\pi}{12}$$

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• writes $\sec x + \tan x$ in terms of t	1
• simplifies resulting algebraic fraction	1
• uses expression for tan of the sum of two angles to complete proof	1

Answer

$$t = \tan\tfrac{x}{2} \Rightarrow \sec x + \tan x = \frac{(1+t^2)+2t}{1-t^2} = \frac{(1+t)^2}{(1+t)(1-t)} = \frac{1+t}{1-t}$$

$$\text{and } \tan\left(\tfrac{\pi}{4}+\tfrac{x}{2}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}} = \frac{1+t}{1-t}$$

$$\therefore \sec x + \tan x = \tan\left(\tfrac{\pi}{4}+\tfrac{x}{2}\right)$$

d. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
• writes an expression for the general term in the expansion	1
• identifies the constant term	1
• evaluates the constant term	1

Answer

Expansion of $\left(x^2-\frac{2}{x}\right)^9$ has general term ${}^9C_r\left(-\frac{2}{x}\right)^r\left(x^2\right)^{9-r} = {}^9C_r(-2)^r x^{18-3r}$, $r = 0, 1, \ldots, 9$.

$18-3r = 0 \Rightarrow r = 6$. Hence term independent of x is ${}^9C_6(-2)^6 = 5376$.

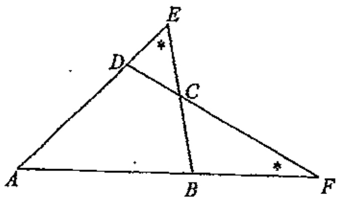
Q11 (cont)

e. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • uses exterior angle theorem for triangles to write expressions for $\angle ADC,\ \angle ABC$	1
• deduces these angles are equal using given angle equality and vertically opposite angles at C	1
ii • uses fact quadrilateral $ABCD$ is cyclic to deduce $\angle ADC,\ \angle ABC$ supplementary	1
• deduces $\angle ABC$ is a right angle and hence AC is a diameter of circle $ABCD$	1

Answer



i. In ΔDEC,
$\angle ADC = \angle DEC + \angle ECD$ (*Ext. ∠ is sum of int. opp. ∠'s*)
Similarly, in ΔBFC
$\angle ABC = \angle BFC + \angle FCB$
But $\angle ECD = \angle FCB$ (*vert. opp. ∠'s are equal*)
$\therefore \angle ADC = \angle ABC$ (given $\angle DEC = \angle BFC$)

ii. $\angle ADC + \angle ABC = 180°$ (*opp. ∠'s of cyclic quad are supplementary*)
$\therefore \angle ABC = 90°$
$\therefore AC$ is diameter of circle $ABCD$.

Question 12

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• writes a numerical expression for $\tan\theta$ in terms of the gradients of the lines	1
• evaluates the acute angle	1

Answer

$3x - y = 0$ has gradient 3
$x - 3y = 0$ has gradient $\frac{1}{3}$

$\tan\theta = \left|\dfrac{3-\frac{1}{3}}{1+3\times\frac{1}{3}}\right| = \frac{4}{3}$ $\quad \therefore \theta \approx 53°$ (to the nearest degree)

b. Outcomes assessed: HE4

Marking Guidelines

Criteria	Marks
i • states the domain	1
• states the range	1
ii • sketches the graph showing the coordinates of the endpoints	1

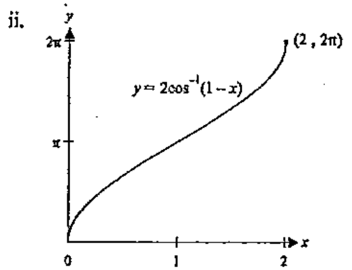
Answer

i.

$-1 \le 1 - x \le 1$ $\qquad 0 \le \cos^{-1}(1-x) \le \pi$
$1 \ge x - 1 \ge -1$ $\qquad 0 \le 2\cos^{-1}(1-x) \le 2\pi$
$2 \ge x \ge 0$ $\qquad 0 \le y \le 2\pi$

Domain $\{x : 0 \le x \le 2\}$
Range $\{y : 0 \le y \le 2\pi\}$

ii.



Q12(cont)

c. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • applies the factor theorem	1
ii • finds the quadratic quotient on division by $(x+2)$ or finds a second linear factor	1
• completes the factorisation	1

Answer

i. $P(x) = 2x^3 + 7x^2 + 4x - 4$ $\quad \therefore P(-2) = -16 + 28 - 8 - 4 = 0$ $\quad \therefore (x+2)$ is a factor of $P(x)$

ii. $P(x) = (x+2)(2x^2 + 3x - 2) = (x+2)(2x-1)(x+2)$

d. Outcomes assessed: HE6

Marking Guidelines

Criteria	Marks
• converts the integral by substitution to a definite integral with variable u	1
• finds the primitive function	1
• substitutes and evaluates in simplest form	1

Answer

$u = x + 1$
$du = dx$

$x = 2 \Rightarrow u = 3$
$x = 5 \Rightarrow u = 6$

$$\int_2^5 \frac{x+2}{(x+1)^2}\,dx = \int_3^6 \frac{u+1}{u^2}\,du$$
$$= \int_3^6 \left(\frac{1}{u} + \frac{1}{u^2}\right) du$$
$$= \left[\ln u - \tfrac{1}{u}\right]_3^6$$
$$= (\ln 6 - \ln 3) - \left(\tfrac{1}{6} - \tfrac{1}{3}\right)$$
$$= \ln 2 + \tfrac{1}{6}$$

e. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • states the y coordinate of T	1
• uses the equation of PQ to show the chord passes through F	1
ii • writes the gradients of TF and PQ in terms of a and x_0	1
• shows the product of the gradients is -1 and deduces the result	1

Answer

i. T on directrix $\Rightarrow y_0 = -a$

Then PQ has equation $xx_0 = 2a(y - a)$
Then PQ cuts y-axis at $F(0, a)$

ii. $m_{PQ} \cdot m_{TF} = \dfrac{x_0}{2a} \cdot \dfrac{(y_0 - a)}{x_0} = \dfrac{x_0}{2a} \cdot \dfrac{(-2a)}{x_0} = -1$

$\therefore TF \perp PQ$

Question 13

a. Outcomes assessed: HE2

Marking Guidelines

Criteria	Marks
• shows that the first member of the sequence of statements is true	1
• writes an appropriate inequality for 5^{k+1} conditional on the truth of the k^{th} statement	1
• completes the algebraic manipulation to show the conditional truth of the $(k+1)^{th}$ statement	1

Answer

Let $S(n)$, $n = 3, 4, 5, \ldots$, be the sequence of statements defined by $S(n):\ 5^n > 4^n + 3^n$.

Consider $S(3)$: $\quad 4^3 + 3^3 = 91 < 125 = 5^3 \quad \therefore S(3)$ is true

If $S(k)$ is true: $\quad 5^k > 4^k + 3^k \quad *$

Consider $S(k+1)$ for $k \geq 3$:

$$\begin{aligned} 5^{k+1} &= 5.5^k \\ &> 5\left(4^k + 3^k\right) \quad \text{if } S(k) \text{ is true, using } * . \\ &= 5.4^k + 5.3^k \\ &> 4.4^k + 3.3^k \\ &= 4^{k+1} + 3^{k+1} \end{aligned}$$

Hence if $S(k)$ is true for some $k \geq 3$, then $S(k+1)$ is true. But $S(3)$ is true and then $S(4)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all integers $n \geq 3$.

b. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • verifies that $f(1)$ and $f(1 \cdot 5)$ have opposite signs	1
ii • finds the derivative of f with respect to x.	1
• applies Newton's method to get the next approximation	1

Answer

i. $f(x) = e^x - \tan x$, $\quad f(1) = e - \tan 1 \approx 1 \cdot 16 > 0 \quad \therefore f(x) = 0$ for some x

f continuous for $0 < x < \frac{\pi}{2}$, $\quad f(1 \cdot 5) = e^{1 \cdot 5} - \tan 1 \cdot 5 \approx -9 \cdot 62 < 0 \quad$ between 1 and $1 \cdot 5$

ii. $f'(x) = e^x - \sec^2 x$. Next approximation is $x_1 = 1 \cdot 3 - \dfrac{e^{1 \cdot 3} - \tan 1 \cdot 3}{e^{1 \cdot 3} - \sec^2 1 \cdot 3} \approx 1 \cdot 31$ to 2 decimal places.

c. Outcomes assessed: HE5

Marking Guidelines

Criteria	Marks
i • writes the primitive function giving t as a function of x including a constant of integration	1
• uses the initial conditions to write t as a function of x	1
• finds x as a function of t	1
ii • finds the limiting value of x as $t \to +\infty$	1

Answer

i. $\dfrac{dx}{dt} = \dfrac{16 - x^2}{x}$

$\dfrac{dt}{dx} = -\frac{1}{2} \cdot \dfrac{-2x}{16 - x^2}$

$t = -\frac{1}{2}\ln\left(16 - x^2\right) + c$

$\left.\begin{array}{l} t = 0 \\ x = 1 \end{array}\right\} \Rightarrow \quad 0 = -\frac{1}{2}\ln 15 + c$

$t = -\frac{1}{2}\ln\left(\dfrac{16 - x^2}{15}\right)$

$e^{-2t} = \dfrac{16 - x^2}{15}$

$x^2 = 16 - 15e^{-2t}$

$x > 0$ for $t = 0 \Rightarrow x = \sqrt{16 - 15e^{-2t}}$

ii. As $t \to +\infty$, $e^{-2t} \to 0 \quad \therefore x \to \sqrt{16} = 4$. Particle moves right towards a limiting position 4m right of O.

Q13(cont)

d. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • finds velocity and acceleration in terms of t by differentiation to deduce result.	1
ii • states centre of motion	1
• finds time when first at centre	1
iii • finds time when velocity first zero	1
• states amplitude	1

Answer

i. $x = 1 + 2\cos\left(2t - \frac{\pi}{3}\right)$

$\dot{x} = -4\sin\left(2t - \frac{\pi}{3}\right)$

$\ddot{x} = -8\cos\left(2t - \frac{\pi}{3}\right)$

$= -4(x - 1)$

ii. Centre of motion is 1m from O in the positive direction.

$x = 1 \Rightarrow \cos\left(2t - \frac{\pi}{3}\right) = 0$

$2t - \frac{\pi}{3} = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2}, \ldots$

First such $t > 0$ is given by $2t = \frac{\pi}{2} + \frac{\pi}{3}$ $\quad \therefore \frac{5\pi}{12}$ s to first reach max. speed.

iii. $\dot{x} = 0 \Rightarrow \sin\left(2t - \frac{\pi}{3}\right) = 0$

$2t - \frac{\pi}{3} = 0,\ \pm\pi,\ \pm 2\pi \ldots$

First such $t > 0$ given by $2t - \frac{\pi}{3} = 0$

First at rest at time $\frac{\pi}{6}$ s.

$t = \frac{\pi}{6} \Rightarrow x = 3$. Amplitude is 2m.

Question 14

a. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • writes the expansion in ascending powers of x	1
ii • writes a definite integral between appropriate limits, finding one primitive function	1
• finds the remaining primitive function	1
• evaluates to complete the proof	1

Answer

i. $\quad (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

ii. $$\int_0^1 \left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n\right) dx = \int_0^1 (1+x)^n\, dx$$

$${}^nC_0\left[x\right]_0^1 + {}^nC_1\left[\tfrac{1}{2}x^2\right]_0^1 + {}^nC_2\left[\tfrac{1}{3}x^3\right]_0^1 + \ldots + {}^nC_n\left[\tfrac{1}{n+1}x^{n+1}\right]_0^1 = \tfrac{1}{n+1}\left[(1+x)^{n+1}\right]_0^1$$

$${}^nC_0(1-0) + \tfrac{1}{2}\,{}^nC_1\left(1^2 - 0\right) + \tfrac{1}{3}\,{}^nC_2\left(1^3 - 0\right) + \ldots + \tfrac{1}{n+1}\,{}^nC_n\left(1^{n+1} - 0\right) = \tfrac{1}{n+1}\left(2^{n+1} - 1^{n+1}\right)$$

$${}^nC_0 \quad + \quad \tfrac{1}{2}\,{}^nC_1 \quad + \quad \tfrac{1}{3}\,{}^nC_2 \quad + \ldots + \quad \tfrac{1}{n+1}\,{}^nC_n \quad = \frac{2^{n+1} - 1}{n+1}$$

Q14(cont)

b. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • identifies Binomial distribution and writes numerical expression for the required probability	1
• calculates the probability in simplest fraction form	1
ii • writes numerical expressions for Bill finally winning on 2nd, 4th, 6th games	1
• generalises and forms sum of terms in a GP for required probability	1
• calculates limiting sum in simplest fraction form.	1

Answer

i. Number of wins for Bill follows a Binomial probability distribution with $n=6,\ p=\frac{2}{3}, q=\frac{1}{3}$.

$P(\textit{Bill wins exactly 4 games}) = {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 = 15\times\frac{2^4}{3^6} = \frac{80}{243}$

ii. If last game is the n^{th} game, then n must be even. (If out of n games, Ben wins k and Bill wins $k+2$, then $n=2k+2=2(k+1)$ which is an even number). Considering the pairs (game 1 and game 2), (game 3 and game 4), ... , (game $n-3$ and game $n-2$), none of these paired games can have the same winner or they would have stopped playing before game n. Also if Bill is the overall winner, then Bill must win both games $n-1$ and n.

Let p_n be probability Bill is overall winner with n^{th} game the last game.

Then $p_2=\left(\frac{2}{3}\right)^2,\ p_4=2\times\frac{2}{3}\times\frac{1}{3}\times\left(\frac{2}{3}\right)^2,\ p_6=\left(2\times\frac{2}{3}\times\frac{1}{3}\right)\times\left(2\times\frac{2}{3}\times\frac{1}{3}\right)\times\left(\frac{2}{3}\right)^2\ \ldots$

$$P(\textit{Bill is the eventual winner}) = p_2+p_4+p_6+p_8\ldots$$

$$=\left(\frac{2}{3}\right)^2\left\{1+\frac{4}{9}+\left(\frac{4}{9}\right)^2+\left(\frac{4}{9}\right)^3+\ldots\right\} \qquad \leftarrow \textit{Limiting sum of GP}$$

$$\therefore P(\textit{Bill is the eventual winner}) = \left(\frac{2}{3}\right)^2\times\frac{1}{1-\frac{4}{9}}=\frac{4}{5}$$

c. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • writes relations linking T, V, α and X, T, V, α.	1
• eliminates α to obtain required equation	1
ii • writes and solves quadratic equation in T^2	1
• finds two values for T.	1
iii • finds discriminant of quadratic equation in T^2	1
• uses quadratic theory to deduce required result	1

Answer

i. $y=t(V\sin\alpha-4\cdot9t)$

$y=0$ for $t=T>0,\ x=X$

$\therefore V\sin\alpha=4\cdot9T$ and $X=VT\cos\alpha$

$(VT\sin\alpha)^2+(VT\cos\alpha)^2=(4\cdot9T^2)^2+X^2$

$V^2T^2(\sin^2\alpha+\cos^2\alpha)=4\cdot9^2T^4+X^2$

$4\cdot9^2T^4-V^2T^2+X^2=0 \qquad **$

ii.

$49^2T^4-100V^2T^2+100X^2=0$

$49^2T^4-100\times49^2T^2+100\times196^2=0$

$T^4-100T^2+1600=0$

$(T^2-80)(T^2-20)=0$

$T^2=80 \quad or \quad T^2=20$

$\therefore T=4\sqrt{5} \quad or \quad T=2\sqrt{5}$

iii. Two distinct real values for T^2 provided **, considered as a quadratic in T^2, has discriminant $\Delta>0$. Then the values of T^2 must both be positive, since the sum and product of the roots of ** are positive, giving two distinct positive values for T. $\Delta=V^4-4\times4\cdot9^2X^2=V^4-9\cdot8^2X^2 \quad \therefore \Delta>0 \Rightarrow V^2>9\cdot8X$