Student Name: ____________

South Sydney High School

2017

YEAR 12

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- Show relevant mathematical reasoning and/or calculations in Questions 11-16

Total marks - 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1 What is the value of $\frac{6}{iz}$ if $z=-1+i$?

(A) $-3-3i$
(B) $-3+3i$
(C) $3-3i$
(D) $3+3i$

2 Which of the following parametric equations represent the hyperbola $x^2-y^2=4$?

(A) $x=2\tan\theta$ and $y=2\sec\theta$
(B) $x=4\tan\theta$ and $y=4\sec\theta$
(C) $x=2\sec\theta$ and $y=2\tan\theta$
(D) $x=4\sec\theta$ and $y=4\tan\theta$

3 What is the value of the indefinite integral $\int\frac{x^2}{(1-x^2)^{\frac{3}{2}}}dx$?

(Use the substitution $x=\sin\theta$ with $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$).

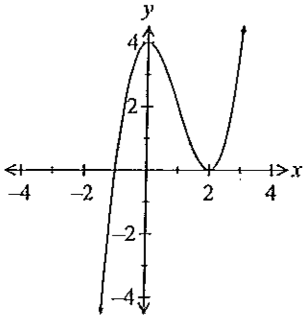
(A) $\frac{1}{(1-x^2)^{\frac{1}{2}}}-\cos^{-1}x+c$

(B) $\frac{x}{(1-x^2)^{\frac{1}{2}}}-\cos^{-1}x+c$

(C) $\frac{1}{(1-x^2)^{\frac{1}{2}}}-\sin^{-1}x+c$

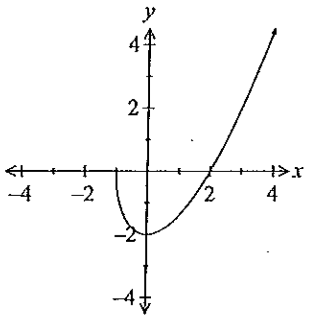
(D) $\frac{x}{(1-x^2)^{\frac{1}{2}}}-\sin^{-1}x+c$

4 The diagram below shows the graph of the function $y=f(x)$.

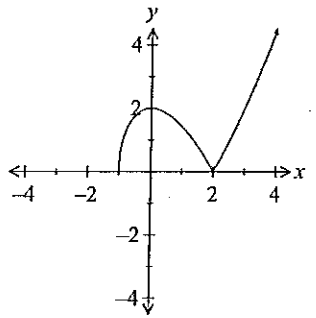


Which of the following is the graph of $y=|f(x)|$?

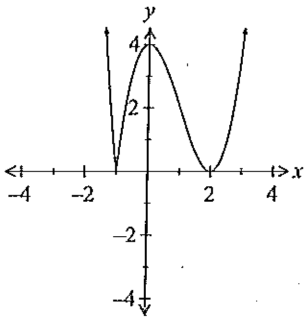
(A)



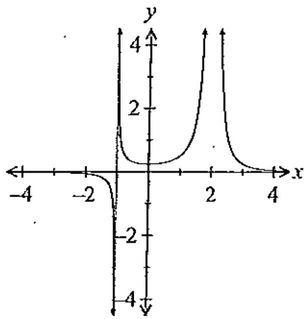
(B)



(C)



(D)



5 What is the square root of $8+6i$?

(A) $-3-i$

(B) $-3+i$

(C) $3-i$

(D) $5-3i$

6 A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $2m(v+v^2)$ Newtons when its speed is v ms^{-1}. At time t seconds the particle has a displacement x metres from a fixed point O on the line and velocity v ms^{-1}. Which of the following is an expression for x in terms of v?

(A) $-\frac{1}{2}\int\frac{1}{1+v}\,dv$

(B) $-\frac{1}{2}\int\frac{1}{v(1+v)}\,dv$

(C) $\frac{1}{2}\int\frac{1}{1+v}\,dv$

(D) $\frac{1}{2}\int\frac{1}{v(1+v)}\,dv$

7 The polynomial $P(z)$ has the equation $P(z)=z^4-4z^3+Az+20$, where A is real. Given that $3+i$ is a zero of $P(z)$, which of the following expressions is $P(z)$ as a product of two real quadratic factors?

(A) $(z^2-2z+2)(z^2-6z+10)$

(B) $(z^2+2z+2)(z^2-6z+10)$

(C) $(z^2-2z+2)(z^2+6z+10)$

(D) $(z^2+2z+2)(z^2+6z+10)$

8 The region enclosed by $y=\sin x$, $y=0$ and $x=\frac{\pi}{2}$ is rotated around the y-axis to produce a solid. What is the volume of this solid using the method of cylindrical shells?

(A) π units^3

(B) $\frac{\pi}{2}$ units^3

(C) $\frac{3\pi}{2}$ units^3

(D) 2π units^3

9 What is the indefinite integral of $\int \frac{\sec^2(\log_e x)}{x} dx$?

(A) $\tan(\log_e x)+c$

(B) $\tan(\cos x)+c$

(C) $\sec(\log_e x)+c$

(D) $\sec(\cos x)+c$

10 The equation $x^3-y^3+3xy+1=0$ defines y implicitly as a function of x.

Which of the following is the expression for $\frac{dy}{dx}$?

(A) $\frac{y^2-x}{x^2+y}$

(B) $\frac{y^2+x}{x^2-y}$

(C) $\frac{x^2+y}{y^2-x}$

(D) $\frac{x^2-y}{y^2+x}$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) For the hyperbola $\frac{x^2}{9}-\frac{y^2}{72}=1$ find the:

(i) eccentricity. 1

(ii) coordinates of the foci. 1

(iii) equations of the directrices. 1

(b) (i) What is the expansion of $(1+ia)^4$ in ascending powers of a? 1

(ii) Hence, find the values of a such that $(1+ia)^4$ is real. 2

(c) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3-\cos x-2\sin x} dx$. 4

(d) Sketch the locus of z on the Argand diagram where the inequalities $|z-1|\leq 3$ and $\text{Im}(z)\geq 3$ hold simultaneously. 3

(e) Prove that $z=2i$, $w=\sqrt{3}-i$ and $v=-\sqrt{3}-i$ are the vertices of an equilateral triangle. 2

Question 12 (15 marks) **Marks**

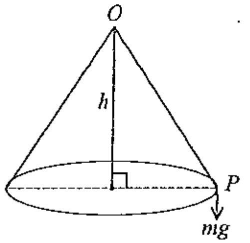
(a) (i) Find the real numbers a, b and c such that **3**

$$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$$

(ii) Hence, find $\int \frac{7x+4}{\left(x^2+1\right)\left(x+2\right)}dx$ **2**

(b) The polynomial $P(x)=x^3-6x^2+9x+c$ has a double zero. **2**
What are the values of c?

(c) A mass of m kg at P is suspended by a light inextensible string from point O. It describes a circle with a constant speed in a horizontal plane whose vertical distance below O is h metres.



(i) Show that $\omega=\sqrt{\frac{g}{h}}$ by resolving the forces. **3**

(ii) What is the period of motion? **1**

(d) Let $f(x)=(x+1)(x-2)(3-x)$. Draw separate one-third page sketches of these functions. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y=\left[f(x)\right]^2$ **2**

(ii) $y=e^{f(x)}$ **2**

Question 13 (15 marks) **Marks**

(a) The polynomial equation $x^3-3x^2-x+2=0$ has roots α, β and γ.

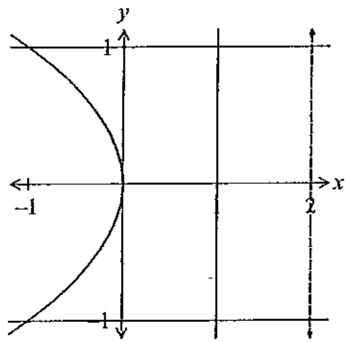
(i) Find the polynomial equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

(ii) Find the polynomial equation with roots $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$ and $\alpha+\beta+2\gamma$. **2**

(b) (i) Write $z=1+i$ in modulus-argument form. **1**

(ii) Find $\left|z^{10}\right|$ and $\arg(z^{10})$. **2**

(c) The region shown below is bounded by the lines $x=1$, $y=1$, $y=-1$ and the curve $x=-y^2$. The region is rotated through 360° about the line $x=2$ to form a solid. Calculate the volume of the solid using the method of slicing? **3**



(d) The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the same branch of the hyperbola $xy=c^2$ $(p\neq q)$. The tangents at P and Q meet at the point T.

(i) What is the gradient of the tangent at P? **2**

(ii) Find the equation of the tangent at P. **1**

(iii) Find the coordinates of T. **2**

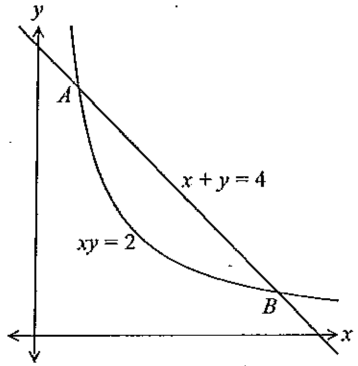
Question 14 (15 marks) **Marks**

(a) (i) How many different six digit numbers can be formed from the digits 1, 2, 3, 4, 5 and 6 without repetition? **1**

(ii) How many of these numbers are greater than 564,321? **1**

(iii) How many of these numbers are less than 564,321? **1**

(b) A solid is formed by rotating about the y-axis the region bounded by the line $x+y=4$ and the hyperbola $xy=2$ between $2-\sqrt{2} \le x \le 2+\sqrt{2}$. **4**



Find the volume of this solid using the method of of cylindrical shells.

(c) Point $P(x_0, y_0)$ is on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

(i) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**

(ii) Show the equation of the tangent at P is $\frac{x_0 x}{25}+\frac{y_0 y}{9}=1$. **2**

(iii) Let the tangent at P meet a directrix at a point Q. Show that ΔPSQ is right angled, where S is the corresponding focus. **2**

(d) Let $z=1-i$ be a root of the polynomial $z^3+pz+q=0$ where p and q are real numbers. Find the value of p and q. **2**

Question 15 (15 marks) **Marks**

(a) (i) Show that $\sin x+\sin 3x=2\sin 2x\cos x$ **1**

(ii) Hence or otherwise solve $\sin x+\sin 2x+\sin 3x=0$ for $0 \le x \le 2\pi$. **2**

(b) A body of mass m falls from rest in a medium with resistive force R = kv, where k is the coefficient of air resistance and v is the speed of the object, (k is a constant.) Prove that the distance x fallen when the velocity is v, is

(i) $x=\frac{mv}{k}-\frac{m^2 g}{k^2}\log_e\left(1-\frac{kv}{mg}\right)$ **4**

(ii) Find the terminal velocity for a falling 70 kg sky-driver, if k = 14 and g = 10 m.s^{-2}. Express your answer in km/h. **3**

(c) Consider the functions $f(x)=|x|+2$ and $g(x)=\frac{8}{|x|}$.

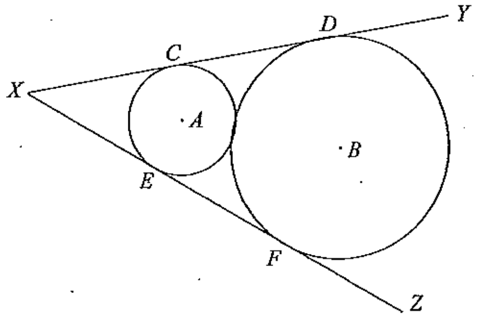
(i) Solve the equation $f(x)=g(x)$. **2**

(ii) Sketch the graph of $y=f(x)$ and $y=g(x)$ on the same set of axes. **2**

(iii) Hence solve the inequality $f(x)<g(x)$. **1**

Question 16 (15 marks) **Marks**

(a) Two circles with centres A and B are in contact with each other. Two straight lines XY and XZ are tangents to the circle with $\angle YXZ = 2\theta$.



(i) Prove that line AX bisects $\angle YXZ$. **2**

(ii) Prove that points X, A and B are collinear. **2**

(b) (i) Let $I_n = \int_0^x \cos^n t\,dt$, where $0 \le x \le \frac{\pi}{2}$. **2**

Show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ with $n \ge 2$.

(ii) Hence, otherwise, find the exact value I_4. **2**

(c) (i) Use the principle of mathematical induction to prove that: **3**

$$\sum_{i=1}^{n} i^2 = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$

(ii) Hence evaluate $\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$ **1**

(d) When a polynomial $P(x)$ is divided by (x^2+1) the remainder is $Ax+B$

(i) Show that $A = \frac{P(i) - P(-i)}{2i}$ and $B = \frac{P(i) + P(-i)}{2}$ **2**

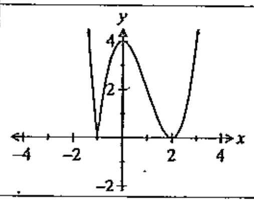
(ii) If $P(x)$ is odd, find the remainder when $P(x)$ is divided by (x^2+1) **1**

End of paper

ACE Examination 2017

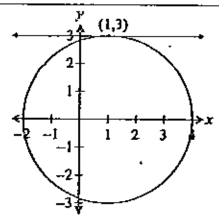
HSC Mathematics Extension 2 Yearly Examination

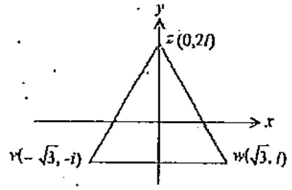
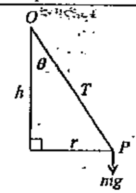
Worked solutions and marking guidelines

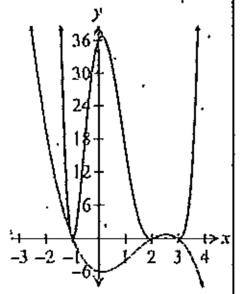
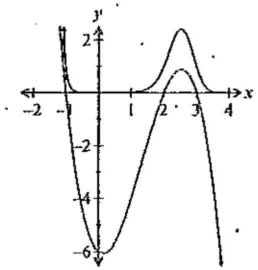
Section I		
	Solution	**Criteria**
1	$\frac{6}{iz}=\frac{6}{-i+i^2}$ $=\frac{6}{-1-i}\times\frac{-1+i}{-1+i}$ $=\frac{-6+6i}{1+1}$ $=-3+3i$	1 Mark: B
2	$x^2-y^2=4$ or $\frac{x^2}{4}-\frac{y^2}{4}=1$. Therefore $a=2$ and $b=2$ $x=2\sec\theta$ and $y=2\tan\theta$ with $\theta\neq\pm\frac{\pi}{2}$	1 Mark: C
3	Let $x=\sin\theta$, $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$ $dx=\cos\theta d\theta$ $(1-x^2)^{\frac{3}{2}}=(1-\sin^2\theta)^{\frac{3}{2}}$ $=(\cos^2\theta)^{\frac{3}{2}}=\cos^3\theta$ $\int\frac{x^2}{(1-x^2)^{\frac{3}{2}}}dx=\int\frac{\sin^2\theta}{\cos^3\theta}\cos\theta d\theta$ $=\int\tan^2\theta d\theta=\int(\sec^2\theta-1)d\theta$ $=\tan\theta-\theta+c=\frac{x}{(1-x^2)^{\frac{1}{2}}}-\sin^{-1}x+c$	1 Mark: D
4		1 Mark: C

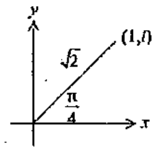
5	Let $(a+ib)^2=8+6i$ $a^2+2abi-b^2=8+6i$ $a^2-b^2=8$ and $2ab=6$ Solving these equations simultaneous $a=+3$ and $b=+1$. Therefore the root is $3+i$. $a=-3$ and $b=-1$. Therefore the root is $-3-i$.	1 Mark: A
6	$F=ma=mv\frac{dv}{dx}=-2m(v+v^2)$ $\therefore\frac{dx}{dv}=-\frac{1}{2(1+v)}$ or $x=-\frac{1}{2}\int\frac{1}{1+v}dv$	1 Mark: A
7	Roots are $3+i$, $3-i$, α and β $(3+i)(3-i)\alpha\beta=\frac{20}{1}$ $(3+i)+(3-i)+\alpha+\beta=\frac{4}{1}$ $(9-i^2)\alpha\beta=20$ $\alpha+\beta=-2$ $10\alpha\beta=20$ $\alpha=-2-\beta$ $\alpha\beta=2$ Hence $(-2-\beta)\beta=2$ $\beta^2+2\beta+2=0$ or $\beta=-1\pm i$ $P(z)=[z-(-1+i)][z-(-1-i)][z-(3+i)][z-(3-i)]$ $=(z^2+2z+2)(z^2-6z+10)$	1 Mark: B
8	Cylindrical shells radius is x and height $\sin x$ $V=\lim_{\delta x\to 0}\sum_{x=0}^{\frac{\pi}{2}}2\pi x\sin x\delta x=2\pi\int_0^{\frac{\pi}{2}}x\sin x\,dx$ $=2\pi\left(\left[x\cos x\right]_0^{\frac{\pi}{2}}\right)+\int_0^{\frac{\pi}{2}}\cos x\,dx$ $=2\pi\left[\sin x\right]_0^{\frac{\pi}{2}}=2\pi\left[1-0\right]=2\pi$	1 Mark: D
9	Use the substitution $u=\log_e x$ then $du=\frac{1}{x}dx$ $\int\frac{\sec^2(\log_e x)}{x}dx=\int\sec^2 u\,du=\tan u+c=\tan(\log_e x)+c$	1 Mark: A
10	$3x^2-3y^2\frac{dy}{dx}+3x\frac{dy}{dx}+3y=0$ $\frac{dy}{dx}(3x-3y^2)=-3x^2-3y$ $\frac{dy}{dx}=\frac{-3x^2-3y}{3x-3y^2}=\frac{x^2+y}{y^2-x}$	1 Mark: C

Section II

	Solution	Criteria
11(a) (i)	$\frac{x^2}{9}-\frac{y^2}{72}=1 \quad a^2=9$ and $b^2=72$ $b^2=a^2(e^2-1)$ $72=9\times(e^2-1)$ or $e=\sqrt{\frac{72}{9}+1}=\sqrt{\frac{81}{9}}=3$	1 Mark: Correct answer.
11(a) (ii)	Foci have coordinates $(\pm ae,0)=(\pm 9,0)$	1 Mark: Correct answer.
11(a) (iii)	Equations of the directrices are $x=\pm\frac{a}{e}=\pm\frac{3}{3}=\pm 1$	1 Mark: Correct answer.
11(b) (i)	$(1+ia)^4=1+4ia-6a^2-4ia^3+a^4$	1 Mark: Correct answer.
11(b) (ii)	$(1+ia)^4$ is real if $4a-4a^3=0$ Then $4a(1-a^2)=0$ $\therefore a=0,\pm 1$	2 Marks: Correct answer. 1 Mark: Finds an equation for a
11(c)	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}dx$ or $dt=\frac{1}{2}(1+t^2)dx$ or $dx=\frac{2}{1+t^2}dt$ When $x=0$ then $t=0$ and when $x=\frac{\pi}{2}$ then $t=1$ $3-\cos x-2\sin x=\frac{3(1+t^2)-(1-t^2)-4t}{1+t^2}=\frac{3+3t^2-1+t^2-4t}{1+t^2}$ $=\frac{2(2t^2-2t+1)}{1+t^2}=2\left[(t-\frac{1}{2})^2+\frac{1}{4}\right]\frac{2}{1+t^2}$ $\int_0^{\frac{\pi}{2}}\frac{1}{3-\cos x-2\sin x}dx=\int_0^1\frac{1}{2}\left[\frac{1}{(t-\frac{1}{2})^2+\frac{1}{4}}\right]\times\frac{1+t^2}{2}\times\frac{2}{1+t^2}dt$ $=\left[\tan^{-1}2(t-\frac{1}{2})\right]_0^1$ $=\tan^{-1}1-\tan^{-1}(-1)=\frac{\pi}{2}$	4 Marks: Correct answer 3 Marks: Correctly determines the primitive function. 2 Marks: Correctly expresses the integral in terms of t. 1 Mark: Correctly finds $d\theta$ in terms of dt and determines the new limits.
11(d)	$\|z-1\|\le 3$ represents a region with a centre is (1, 0) and radius less than or equal to 3. $\mathrm{Im}(z)\ge 3$ represents a region above the horizontal line $y=3$. The point (1,3) is where the two inequalities hold. 	3 Marks: Correct answer. 2 Marks: Correctly graphs one inequality. 1 Mark: Makes some progress

11(c)	Pythagoras theorem $zw=\sqrt{3^2+\sqrt{3}^2}=\sqrt{12}=2\sqrt{3}$ $zv=\sqrt{3^2+\sqrt{3}^2}=\sqrt{12}=2\sqrt{3}$ $wv=\sqrt{3}+\sqrt{3}=2\sqrt{3}$ $\therefore$ Equilateral triangle. 	2 Marks: Correct answer. 1 Mark: Plots the points or makes some progress.
12(a) (i)	$\frac{7x+4}{(x^2+1)(x+2)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$ $7x+4=(ax+b)(x+2)+c(x^2+1)$ Let $x=-2$ and $x=0$ $-10=5c$ $4=b(0+2)-2(0^2+1)$ $c=-2$ $b=3$ Equating the coefficients of x^2 $\quad 0=a-2$ or $a=2$ $\therefore a=2,\ b=3$ and $c=-2$	3 Marks: Correct answer. 2 Marks: Calculates two of the variables 1 Mark: Makes some progress in finding a, b or c.
12(a) (ii)	$\int\frac{7x+4}{(x^2+1)(x+2)}dx=\int(\frac{2x+3}{(x^2+1)}-\frac{2}{(x+2)})dx$ $=\int(\frac{2x}{(x^2+1)}+\frac{3}{(x^2+1)}-\frac{2}{(x+2)})dx$ $=\ln(x^2+1)+3\tan^{-1}x-2\ln\|x+2\|+c$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.
12(b)	$P(x)=x^3-6x^2+9x+c$ $P'(x)=3x^2-12x+9$ $=3(x-3)(x-1)$ Now $P'(x)=0$ and $P(x)=0$ for a double zero. $\therefore P(3)=3^3-6\times 3^2+9\times 3+c$ or $P(1)=1^3-6\times 1^2+9\times 1+c$ $c=0$ $c=-4$	2 Marks: Correct answer. 1 Mark: Solves $P'(x)=0$ to find possible double zeros.
12(c) (i)	Resolving the forces vertically and horizontally. $T\cos\theta-mg=0$ (1) $T\sin\theta=mr\omega^2$ (2) Equation (2) divided by Equation (1) $\frac{T\sin\theta}{T\cos\theta}=\frac{mr\omega^2}{mg}$ or $\tan\theta=\frac{r\omega^2}{g}$ However $\tan\theta=\frac{r}{h}$ Therefore $\frac{r\omega^2}{g}=\frac{r}{h}$ or $\omega^2=\frac{g}{h}$ or $\omega=\sqrt{\frac{g}{h}}$ 	3 Marks: Correct answer. 2 Marks: Solves the two equations of motion. 1 Mark: Correctly states the two equations of motion

12(c) (ii)	Period $=\frac{2\pi}{\omega}=\frac{2\pi}{\sqrt{\frac{g}{h}}}=\frac{2\pi\sqrt{h}}{\sqrt{g}}$	1 Mark: Correct answer.
12(d) (i)	x-intercepts at -1, 2 and 3. y-intercept $y=\left[(0+1)(0-2)(3-0)\right]^2$ $=[1\times -2\times 3]^2=36$ y values are always positive. $\lim_{x\to\pm\infty}\left[(x+1)(x-2)(3-x)\right]^2=\infty$ when $x=2.5$ $y=\left[(2.5+1)(2.5-2)(3-2.5)\right]^2$ $=[3.5\times 0.5\times 0.5]=0.765625$ 	2 Marks: Correct answer. 1 Mark: Determines the intercepts or shows some understanding.
12(d) (ii)	$\lim_{x\to\infty} e^{(x+1)(x-2)(3-x)}=0$ Horizontal asymptote $y=0$ When $x=-1$ then $y=e^0=1$ When $x=2$ then $y=e^0=1$ When $x=3$ then $y=e^0=1$ when $x=2.5$ $y=e^{(2.5+1)(2.5-2)(3-2.5)}=2.3988...$ 	2 Marks: Correct answer. 1 Mark: Shows some understanding of the graph.
13(a) (i)	$x=\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ $\alpha=\frac{1}{x}$ satisfies $x^3-3x^2-x+2=0$ $\left(\frac{1}{x}\right)^3-3\left(\frac{1}{x}\right)^2-\frac{1}{x}+2=0$ $1-3x-x^2+2x^3=0$ $2x^3-x^2-3x+1=0$ is the equation.	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.
13(a) (ii)	The equation $x^3-3x^2-x+2=0$ has roots α,β,γ. $\alpha+\beta+\gamma=3$ $x=2\alpha+\beta+\gamma=\alpha+3$ $x=\alpha+2\beta+\gamma=\beta+3$ $x=\alpha+\beta+2\gamma=\gamma+3$ $\alpha=x-3$ satisfies $x^3-3x^2-x+2=0$ $(x-3)^3-3(x-3)^2-(x-3)+2=0$ $x^3-12x^2+44x-49=0$ is the equation	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.

13(b) (i)	$z=1+i$ $=\sqrt{2}(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}})$ $=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$ 	1 Mark: Correct answer.
13(b) (ii)	$z^{10}=[\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})]^{10}$ $=\left(\sqrt{2}\right)^{10}\left[\cos\left(10\times\frac{\pi}{4}\right)+i\sin\left(10\times\frac{\pi}{4}\right)\right]$ $=32\text{cis}\frac{5\pi}{2}$ or $=32\text{cis}\frac{\pi}{2}$ $\lvert z^{10}\rvert=32$ and $\arg(z^{10})=\frac{\pi}{2}$	2 Marks: Correct answer. 1 Mark: Finds the modulus or the argument.
13(c)	Area of the slice is an annulus Inner radius is 1 and outer radius is $2+y^2$ and height y $A=\pi\left(R^2-r^2\right)$ $=\pi\left((2+y^2)^2-1^2\right)=\pi\left(4+4y^2+y^4-1\right)$ $=\pi\left(y^4+4y^2+3\right)$ $\delta V=\delta A.\delta y$ $V=\lim_{\delta y\to 0}\sum_{y=-1}^{1}\pi\left(y^4+4y^2+3\right)\delta y$ $=2\int_0^1\pi\left(y^4+4y^2+3\right)dy$ $=2\pi\left[\frac{y^5}{5}+\frac{4y^3}{3}+3y\right]_0^1=2\pi\left(\frac{1}{5}+\frac{4}{3}+3\right)=\frac{136\pi}{15}$ cubic units	3 Marks: Correct answer. 2 Marks: Correct integral for the volume of the solid. 1 Mark: Sets up the area of the slice.
13(d) (i)	$xy=c^2$ $x\frac{dy}{dx}+y=0$ or $\frac{dy}{dx}=-\frac{y}{x}$ At $P(cp,\frac{c}{p})$ $\frac{dy}{dx}=-\frac{\frac{c}{p}}{cp}=-\frac{1}{p^2}$	2 Marks: Correct answer. 1 Mark: Finds $\frac{dy}{dx}$
13(d) (ii)	Equation of the tangent at P $y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$ $p^2y-cp=-x+cp$ $x+p^2y-2cp=0$	1 Mark: Correct answer.

13(d) (iii)	Equation of tangent at Q is $x+q^2y-2cq=0$ Point T is the point of intersection of these tangents. Solve equations simultaneously $x+p^2y-2cp=0$ (1) $x+q^2y-2cq=0$ (2) Equation (1) – Equation (2) $p^2y-q^2y=2cp-2cq$ $y(p+q)(p-q)=2c(p-q)$ $y(p+q)=2c$ or $y=\frac{2c}{(p+q)}$ To find x substitute the value for y into equation (1) $x+p^2\frac{2c}{p+q}-2cp=0$ $x=2cp(1-\frac{p}{p+q})=2cp(\frac{p+q-p}{p+q})=\frac{2cpq}{p+q}$ Therefore the coordinates for T are $(\frac{2cpq}{p+q},\frac{2c}{p+q})$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the coordinates or finds correct x from incorrect y
14(a) (i)	$6!=720$	1 Mark: Correct answer.
14(a) (ii)	Numbers greater than 564,321 start with a 6. $1\times5!=120$	1 Mark: Correct answer.
14(a) (iii)	Numbers less than 564,321 do not include this number. $6!-5!-1=599$	1 Mark: Correct answer.
14(b)	Cylindrical shell – inner radius x, outer radius $x+\delta x$, height y. $\delta V=\pi\left[(x+\delta x)^2-x^2\right]y$ $=\pi\left[2x\delta x+\delta x^2\right]\left((4-x)-\frac{2}{x}\right)=\pi\frac{(2x+\delta x)\delta x}{x}(4x-x^2-2)$ $V=\lim_{\delta x\to0}\sum_{2-\sqrt{2}}^{2+\sqrt{2}}\pi\frac{(2x+\delta x)\delta x}{x}(4x-x^2-2)\delta x$ $=2\pi\int_{2-\sqrt{2}}^{2+\sqrt{2}}(4x-x^2-2)dx=2\pi\int_{2-\sqrt{2}}^{2+\sqrt{2}}\left(2-(x-2)^2\right)dx$ Make a substitution $u=x-2$, $du=dx$ $x=2+\sqrt{2}$ then $u=\sqrt{2}$ and $x=2-\sqrt{2}$ then $u=-\sqrt{2}$ $V=2\pi\int_{-\sqrt{2}}^{\sqrt{2}}(2-u^2)du=2\times2\pi\int_0^{\sqrt{2}}(2-u^2)du$ $=4\pi\left[2u-\frac{u^3}{3}\right]_0^{\sqrt{2}}=2\pi\left(2\sqrt{2}-\frac{1}{3}\left(\sqrt{2}\right)^3\right)=\frac{16\pi\sqrt{2}}{3}$ cubic units	4 Marks: Correct answer. 3 Marks: Correct integral for the volume of the solid. 2 Marks: Correct expression for δV. 1 Mark: Determines the radius or height of the cylindrical shell.

14(c) (i)	$\frac{x^2}{25}+\frac{y^2}{9}=1$ or $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$. Therefore $a=5$ and $b=3$ $b^2=a^2(1-e^2)$ or $9=25(1-e^2)$ or $e^2=\frac{16}{25}$ or $e=\frac{4}{5}$ Foci $(\pm ae,0)=(\pm5\times\frac{4}{5},0)=(\pm4,0)$ Directrices $x=\pm\frac{a}{e}=\pm\frac{5}{\frac{4}{5}}=\pm\frac{25}{4}$	2 Marks: Correct answer 1 Mark: Correct foci or directrices.
14(c) (ii)	$\frac{2x}{25}+\frac{2y}{9}\frac{dy}{dx}=0$ or $\frac{dy}{dx}=-\frac{2x}{25}\times\frac{9}{2y}=-\frac{9x}{25y}$ Gradient of the tangent at $P(x_0,y_0)$ is $-\frac{9x_0}{25y_0}$. Equation of the tangent at $P(x_0,y_0)$ $y-y_0=-\frac{9x_0}{25y_0}(x-x_0)$ $25y_0y-25y_0^2=-9x_0x+9x_0^2$ $9x_0x+25y_0y=9x_0^2+25y_0^2$ $\frac{1}{9\times25}\times\left(9x_0x+25y_0y\right)=\frac{1}{9\times25}\times\left(9x_0^2+25y_0^2\right)$ $\frac{x_0x}{25}+\frac{y_0y}{9}=\frac{x_0^2}{25}+\frac{y_0^2}{9}$ or $\frac{x_0x}{25}+\frac{y_0y}{9}=1$	2 Marks: Correct answer 1 Mark: Finds the gradient of the tangent or shows similar understanding.
14(c) (iii)	Q is the intersection of $\frac{x_0x}{25}+\frac{y_0y}{9}=1$ and $x=\frac{25}{4}$. $\frac{x_0\times\frac{25}{4}}{25}+\frac{y_0y}{9}=1$ or $\frac{y_0y}{9}=1-\frac{x_0\times\frac{25}{4}}{25}$ $y=\frac{9}{y_0}\times\left(1-\frac{x_0}{4}\right)=\frac{9(4-x_0)}{4y_0}$ Coordinates of Q is $\left(\frac{25}{4},\frac{9(4-x_0)}{4y_0}\right)$. Gradient of QS $m_1=\frac{\frac{9(4-x_0)}{4y_0}-0}{\frac{25}{4}-4}=\frac{9(4-x_0)}{4y_0}\times\frac{4}{9}=\frac{(4-x_0)}{y_0}$ Gradient of PS $m_2=\frac{y_0-0}{x_0-4}=\frac{y_0}{x_0-4}$ $m_1m_2=\frac{(4-x_0)}{y_0}\times\frac{y_0}{x_0-4}=-1$. Therefore ΔPSQ is a right angle.	2 Marks: Correct answer 1 Mark: Finds the coordinates of Q or the gradient of QS or the gradient of PS.

14(d)	Using the conjugate root theorem $1+i$ and $1-i$ are both roots of the equation $z^3+pz+q=0$. $(1+i)+(1-i)+\alpha=0$ (sum of the roots) $\alpha=-2$ $(1+i)\times(1-i)\times-2=-q$ (product of the roots) $(1+1)\times-2=-q$ $q=4$ $(1+i)(1-i)+(1-i)-2+(1+i)-2=p$ $p=-2$ Therefore $p=-2$ and $q=4$	2 Marks: Correct answer. 1 Mark: Recognises the conjugate root theorem.
15(a) (i)	$\text{LHS}=\sin x+\sin 3x$ $=\sin(2x-x)+\sin(2x+x)$ $=(\sin 2x\cos x-\cos 2x\sin x)+(\sin 2x\cos x+\cos 2x\sin x)$ $=2\sin 2x\cos x$ $=\text{RHS}$	1 Mark: Correct answer.
15(a) (ii)	$\sin x+\sin 2x+\sin 3x=0$ $2\sin 2x\cos x+\sin 2x=0$ $\sin 2x(2\cos x+1)=0$ $\therefore \sin 2x=0$ and $\cos x=-\frac{1}{2}$ $2x=0,\pi,2\pi,3\pi,4\pi$ $x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$ $x=\frac{2\pi}{3},\frac{4\pi}{3}$	2 Marks: Correct answer. 1 Mark: Using part (i) and factorising.
15(b)		

i) $ma = mg - kv$

$a = g - \frac{kv}{m}$ ✓1

$v\frac{dv}{dx} = g - \frac{kv}{m}$

$\frac{dv}{dx} = \frac{g}{v} - \frac{k}{m}$

$= \frac{mg-kv}{mv}$

$\frac{dx}{dv} = \frac{mv}{mg-kv}$

$\int dx = \int \frac{mv}{mg-kv}\,dv$

$mg-kv\,\overline{)\,mv}$ quotient $-\frac{m}{k}$; $mv-\frac{m^2g}{k}$; remainder $\frac{m^2g}{k}$

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$x = \int\left(-\frac{m}{k} + \frac{\frac{m^2g}{k}}{mg-kv}\right)dv$ ✓1

$= -\frac{mv}{k} - \frac{m^2g}{k^2}\ln(mg-kv) + c$

when $x=0,\ v=0$: $0 = -\frac{m^2g}{k^2}\ln(mg) + c$

$\therefore c = \frac{m^2g}{k^2}\ln(mg)$ ✓1

$\therefore x = -\frac{mv}{k} - \frac{m^2g}{k^2}\ln(mg-kv) + \frac{m^2g}{k^2}\ln(mg$

$= -\frac{mv}{k} + \frac{m^2g}{k^2}\ln\left(\frac{mg}{mg-kv}\right)$

$= -\frac{mv}{k} - \frac{m^2g}{k^2}\ln\left(\frac{mg-kv}{mg}\right)$

$= -\frac{mv}{k} - \frac{m^2g}{k^2}\ln\left(1-\frac{kv}{mg}\right)$ ✓1

ii) $ma = mg - kv$

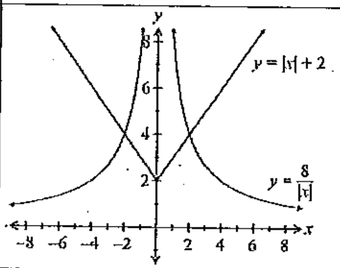
terminal velocity means $a=0$ ✓1

$0 = 70\times10 - 14\times v$

$v = \frac{700}{14}$

$= 50\ \text{m/s}$ ✓1

$\frac{50\times60\times60}{1000} = 180\ \text{km/h}$ ✓1

15(c) (i)	$\lvert x\rvert+2=\dfrac{8}{\lvert x\rvert}$ $(\lvert x\rvert)^2+2\lvert x\rvert-8=0$ $(\lvert x\rvert+4)(\lvert x\rvert-2)=0$ $\lvert x\rvert=-4$ (Not possible) or $\lvert x\rvert=2 \;\therefore\; x=\pm 2$	2 Marks: Correct answer. 1 Mark: Recognises a quadratic equation.
15(c) (ii)		2 Marks: Correct answer. 1 Mark: Correctly graphs $y=f(x)$ or $y=g(x)$
15(c) (iii)	$\lvert x\rvert+2<\dfrac{8}{\lvert x\rvert}$ From the graph $-2<x<0$ or $0<x<2$	1 Mark: Correct answer.
16(a) (i)	Consider ΔAXC and ΔAXE $AC=CE$ (Radii of the circle) $XC=XE$ (Tangents to a circle from an external point are equal) $XA=XA$ (Common side) $\Delta AXC\equiv\Delta AXE$ (SSS) $\angle CXA=\angle AXE$ (Matching angles in congruent triangles) AX bisects $\angle CXE$ AX bisects $\angle YXZ$ (same angle as $\angle CXE$)	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
16(a) (ii)	$\angle XCA=\angle ACD=\angle BDC=90^\circ$ (Angles between the tangent and the radius at the point of contact is a right angle) $\angle CXA=\dfrac{1}{2}\times\angle CXE$ (Using the result in part (i)) $\angle CXA=\theta$ $\angle CXA+\angle XCA+\angle XAC=180$ (Angles sum of a triangle is 180°) $\theta+90+\angle XAC=180$ $\angle XAC=90-\theta$ $\Delta BXD\equiv\Delta BXF$ (SSS) (Similar proof to part (i)) $\angle DXB=\dfrac{1}{2}\times\angle DXF$ (Using the result in part (i)) $\angle DXB=\theta$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.

	$\angle DXB+\angle XDB+\angle XBD=180$ (Angles sum of a triangle 180°) $\theta+90+\angle XBD=180$ $\angle XBD=90-\theta$ Consider quadrilateral $CDAB$ $\angle DCA+\angle CAB+\angle ABD+\angle CDB=360$ (Angle sum of a quadrilateral is 360°) $90+\angle CAB+(90-\theta)+90=360$ $\angle CAB=90+\theta$ $\angle XAC+\angle CAB=(90-\theta)+(90+\theta)$ $\angle XAC+\angle CAB=180^\circ$ Hence X, A and B are collinear	
16(b) (i)	$I_n=\int_0^{\pi}\cos^n x\,dx$ $=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$ Integration by parts $I_n=\int_0^{\frac{\pi}{2}}\cos^{n-1}t\cos t\,dt$ $=\left[\cos^{n-1}t\sin t\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\sin^2 t\,dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\sin^2 t\,dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t(1-\cos^2 t)dt$ $=(n-1)\int_0^{\frac{\pi}{2}}(\cos^{n-2}t-\cos^n t)dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt-(n-1)\int_0^{\frac{\pi}{2}}\cos^n t\,dt$ Using the original integral $\int_0^{\frac{\pi}{2}}\cos^n t\,dt=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt-n\int_0^{\frac{\pi}{2}}\cos^n t\,dt+\int_0^{\frac{\pi}{2}}\cos^n t\,dt$ $n\int_0^{\frac{\pi}{2}}\cos^n t\,dt=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt$ $\int_0^{\frac{\pi}{2}}\cos^n t\,dt=\dfrac{(n-1)}{n}\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt$ $I_n=\dfrac{(n-1)}{n}I_{n-2}$	2 Marks: Correct answer. 1 Mark: Correctly integrates by parts.

16(b) (ii)	$I_n=\frac{(n-1)}{n}I_{n-2}$ $I_4=\frac{(4-1)}{4}I_{4-2}$ $=\frac{3}{4}\int_0^{\frac{\pi}{2}}\cos^2 t\,dt$ $=\frac{3}{4}\int_0^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2t)dt$ $=\frac{3}{8}\left[(x+\frac{\sin 2t}{2})\right]_0^{\frac{\pi}{2}}$ $=\frac{3}{8}\left[(\frac{\pi}{2}+\frac{\sin 0}{2})-(0+\frac{\sin 0}{2}\right]$ $=\frac{3\pi}{16}$	2 Marks: Correct answer. 1 Mark: Using the result from (a)(i) to obtain the definite integral.
16(c) (i)	Step 1: To prove the statement true for $n=1$ $\sum_{i=1}^{1} i^2=\frac{1}{6}+\frac{1^2}{2}+\frac{1^3}{3}$ LHS = RHS (both 1) Result is true for $n=1$ Step 2: Assume the result true for $n=k$ $\sum_{i=1}^{k} i^2=\frac{k}{6}+\frac{k^2}{2}+\frac{k^3}{3}$ Step 3: To prove the result is true for $n=k+1$ $\sum_{i=1}^{k+1} i^2=\frac{k+1}{6}+\frac{(k+1)^2}{2}+\frac{(k+1)^3}{3}$ $\text{LHS}=\sum_{i=1}^{k+1} i^2=\sum_{i=1}^{k} i^2+(k+1)^2$ $=\frac{k}{6}+\frac{k^2}{2}+\frac{k^3}{3}+(k+1)^2$ $=\frac{k+3k^2+2k^3+6(k^2+2k+1)}{6}$ $=\frac{2k^3+9k^2+13k+6}{6}$ $\text{RHS}=\frac{k+1}{6}+\frac{(k+1)^2}{2}+\frac{(k+1)^3}{3}$ $=\frac{(k+1)(1+3(k+1)+2(k^2+2k+1))}{6}$ $=\frac{(k+1)(1+3k+3+2k^2+4k+2)}{6}$ $=\frac{(k+1)(2k^2+7k+6)}{6}$ $=\frac{2k^3+9k^2+13k+6}{6}$ LHS = RHS Result is true for $n=k+1$ if true for $n=k$ Step 4: Result true by principle of mathematical induction.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=1$ and attempts to use the result of $n=k$ to prove the result for $n=k+1$. 1 Mark: Proves the result true for $n=1$
16(c) (ii)	$\lim_{n\to\infty}\frac{1^2+2^2+3^2+...+n^3}{n^3}=\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^2}{n^3}$ $=\lim_{n\to\infty}\frac{1}{n^3}\left(\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}\right)$ $=\lim_{n\to\infty}\left(\frac{1}{6n^2}+\frac{1}{2n}+\frac{1}{3}\right)=\frac{1}{3}$	1 Mark: Correct answer.
16(d) (i)	$P(x)=(x^2+1)Q(x)+Ax+B$ for some polynomial $Q(x)$ $P(i)=(i^2+1)Q(i)+Ai+B$ $Ai+B=P(i)$ (1) $P(-i)=((-i)^2+1)Q(-i)+A(-i)+B$ $-Ai+B=P(-i)$ (2) Equation (1) − (2) $2Ai=P(i)-P(-i)$ $A=\frac{P(i)-P(-i)}{2i}$ Equation (1) + (2) $2B=P(i)+P(-i)$ $B=\frac{P(i)+P(-i)}{2}$	2 Marks: Correct answer. 1 Mark: Finds expressions for $P(i)$ or $P(-i)$
16(d) (ii)	If $P(x)$ is odd then $P(i)=-P(-i)$ Hence $A=\frac{2P(i)}{2i}=\frac{P(i)}{i}$ and $B=\frac{0}{2}=0$ from part (i) Remainder is $Ax+B=\frac{P(i)}{i}x$	1 Mark: Correct answer.