Student Number

St. Catherine's School Waverley
2017
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A Formula Reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Task Weighting 40%

Total marks - 70

Section 1 Pages 3-6

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section 2 Pages 7-12

60 marks

- Attempt Questions 11-14
- Allow about 1 hour and 45 min for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. The point P divides the interval AB externally in the ratio $2:3$.
A has coordinates $(-2,7)$ and B has coordinates $(3,10)$. Find the coordinates of P.

(A) $\left(0,\ 5\frac{2}{5}\right)$

(B) $(-12,1)$

(C) $(13,16)$

(D) $\left(2\frac{3}{5},\ 8\frac{4}{5}\right)$

2. A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. In how many ways can this committee be chosen if a particular women is included?

(A) 840

(B) 5040

(C) 560

(D) 14!

3. How many solutions are there to the equation $\sin 2x = \tan x$ where $0 \le x \le 2\pi$.

(A) 5

(B) 6

(C) 7

(D) 8

4. Find $\int \frac{7}{1+49x^2}\,dx$

(A) $(\tan^{-1} 7x) + C$

(B) $\frac{1}{7}(\tan^{-1} 49x) + C$

(C) $\left(\tan^{-1}\frac{x}{7}\right) + C$

(D) $7\left(\tan^{-1}\frac{x}{49}\right) + C$

5. A particle is moving in simple harmonic motion with displacement x given by

$$v^2 = 324 - 36x^2$$

What is the amplitude, A, and the period, T, of the motion?

(A) $A = 3$ and $T = \frac{\pi}{3}$

(B) $A = 3$ and $T = \frac{\pi}{18}$

(C) $A = 9$ and $T = \frac{\pi}{3}$

(D) $A = 9$ and $T = \frac{\pi}{18}$

6. If α, β and γ are the roots of the cubic equation $2x^3 + 8x^2 - x + 6 = 0$,

Find $\alpha^2 + \beta^2 + \gamma^2$

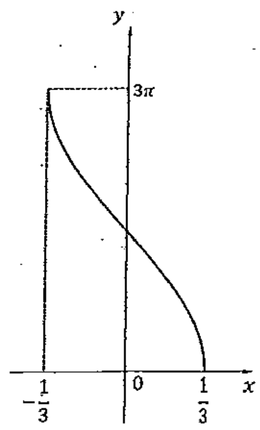
(A) 10

(B) 17

(C) $\frac{33}{2}$

(D) 16

7. Which function best describes the following graph?



(A) $y = \cos^{-1} 3x$

(B) $y = 3\cos^{-1} 3x$

(C) $y = 3\cos^{-1}\frac{x}{3}$

(D) $y = \cos^{-1}\frac{x}{3}$

8. When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$.

What is the remainder when $P(x)$ is divided by $x + 1$

(A) -1

(B) 2

(C) -4

(D) 0

9. Two out of every five St Catherine's students find work overseas after their HSC. A sample of 10 St Catherine's students were interviewed. What is the probability that at least 9 of these students will find work overseas?

(A) $\binom{5}{2}(0.9)^9(0.1)$

(B) $1-(0.4)^{10}$

(C) $9(0.4)^9$

(D) $\binom{10}{9}(0.4)^9(0.6)+(0.4)^{10}$

10. Find all values of x for which $2\cos\left(2x-\frac{\pi}{6}\right)=1$.

(A) $x=n\pi+\frac{\pi}{4}$ and $x=n\pi-\frac{\pi}{12}$

(B) $x=2n\pi\pm\dfrac{\pi}{3}$

(C) $x=2n\pi\pm\dfrac{\pi}{2}$

(D) $x=n\pi+\frac{\pi}{6}$ and $x=n\pi-\frac{\pi}{6}$

End of Section 1

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet — Marks

(a) Find $\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{5}}\right)$ — 1

(b) Find $\int\frac{1}{x-\sqrt{x}}\,dx$ by using the substitution $u=\sqrt{x}$ — 3

(c) Consider the word PARALLEL.

(i) How many ordered arrangements can be made from all the letters? — 1

(ii) If the letters are randomly selected and arranged in a straight line, what is the probability that the L's are all together? — 2

(d) Show that $\tan\left(2\tan^{-1}\frac{1}{3}\right)=\frac{3}{4}$ — 3

(e) Consider the binomial expansion of — 2

$$\left(3x^2+\frac{2}{x}\right)^{12}$$

Find the term independent of x in the expansion.

Question 11 Continued on the next page...

(f) The point $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$ such that PQ is a focal chord.

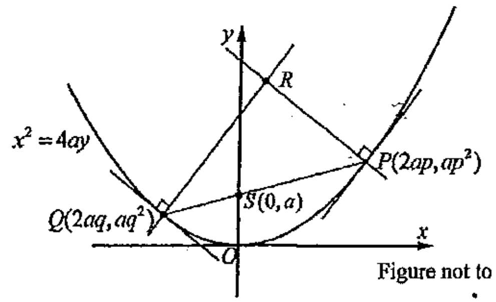


Figure not to scale

(i) Given that the equation of PQ is $2y - (p + q)x + 2pq = 0$. 1

Show that $pq = -1$.

(ii) The normals at P and Q intersect at 2

$R[-apq(p + q), a(p^2 + q^2 + pq + 2)]$. Do not prove this.

Show that the equation of the locus of R as P and Q move on the parabola is given by $x^2 = a(y - 3a)$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. Marks

(a) Consider the function $f(x) = x^3 - cosx$.

(i) Show that the equation $f(x) = 0$ has a root α such that $0 \leq \alpha \leq 1$. 1

(ii) Use one application of Newton's Method with an initial approximation of 0.8 to approximate α, giving the answer correct to 2 decimal places. 3

(b) Prove using mathematical induction that $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9 for all integers $n \geq 1$. 3

(c) Consider the equation $f(x) = 1 + ln\, x$.

(i) Show that the function $f(x)$ is increasing and is concave down for all values of x in the domain. 2

(ii) Show that the equation of the tangent to the curve $y = f(x)$ at $x = 1$ is $y = x$. 2

(iii) Find the equation of the inverse function $f^{-1}(x)$. 1

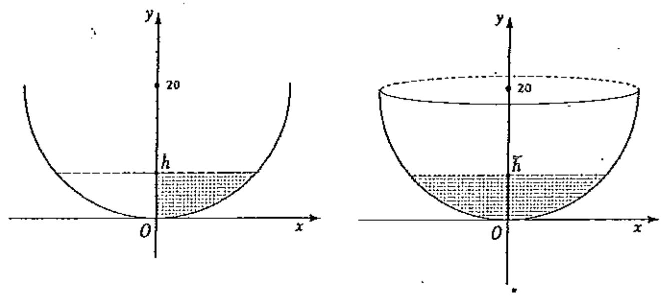
(iv) On the same diagram, sketch the graphs of the curves $y = f(x)$ and $y = f^{-1}(x)$. 3

Show clearly the asymptotes, coordinates of any points of intersection and any intercepts with the coordinate axes.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The left-hand diagram above shows the lower half of the circle $x^2 + (y - 20)^2 = 20^2$.

The shaded area in this diagram is bounded by the semicircle, the line $y = h$, and the y axis.

(i) A semicircle is rotated around the y axis to form a hemispherical bowl of radius 20 cm, as shown on the right-hand diagram. **3**

Show that the volume V formed when the shaded area is rotated around the y axis is given by

$$V = 20\pi h^2 - \frac{\pi h^3}{3}$$

(ii) The bowl is filled with water at a constant rate of 4 cm^3s^{-1}. Find the rate at which the water level is rising when the water level is 12 cm. **3**

(b)

(i) Show that $\dfrac{d}{dx}\left(x\sqrt{1-x^2} + \cos^{-1} x\right) = -\dfrac{2x^2}{\sqrt{1-x^2}}$ **2**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx$. **2**

Give the answer in simplest exact form.

Question 13 Continued on next page...

(c)

(i) Sketch the graph of the hyperbola $y = \dfrac{x-1}{2x-3}$ clearly indicating the vertical and horizontal asymptotes and any intercepts with the coordinate axes. **3**

(ii) Hence or otherwise, find the values of x for which $\frac{x-1}{2x-3} > -1$ **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Three consecutive coefficients ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in the ratio $14:7:2$. Find the value of n and r. 4

(b) A particle is moving in a straight line and performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by $x = 2\sin\left(2t - \frac{\pi}{4}\right)$, velocity v ms^{-1} and acceleration $\ddot{x}$ ms^{-2}.

(i) Show that $\ddot{x} = -4x$ 2

(ii) Sketch the graph of x as a function of t for $0 \le t \le \pi$ showing clearly the coordinates of the end points. 2

(iii) Show that it first returns to its starting point after $\frac{3\pi}{4}$ seconds. 2

(iv) By considering the length of its oscillation and the period, find the time taken to complete 120 metres, in exact form. 1

(c) Use the binomial expansion of $(1+x)^{2n}$,

(i) To show that 1

$$\sum_{r=0}^{2n} \binom{2n}{r} = 2^{2n}$$

(ii) Hence or otherwise show that 3

$$\sum_{r=0}^{n} \binom{2n}{r} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$

End of Paper

Section 1

1. B

$A(-8,7)$ $\quad$ $B(3,11)$

$3:-1$

$m:n$

let P have coordinates (x,y)

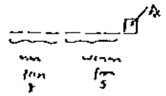
$x = \frac{nx_1 + mx_2}{m+n}$ $\quad$ $y = \frac{ny_1 + my_2}{m+n}$

$= \frac{(-1)(-8) + 3(3)}{3-1}$ $\quad$ $= \frac{(-1)(7) + 3(11)}{3-1}$

$= -12$ $\quad$ $= 1$

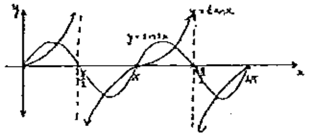
$\therefore P(-12, 1)$

2. C



$^8C_3 \times {}^5C_3 = 560$

3. C



Curves have 7 points of intersection.
$\therefore$ 7 solutions

OR

$\sin 2x = \tan x$ $\quad$ for $0 \le x \le 2\pi$

$2\sin x\cos x = \frac{\sin x}{\cos x}$

$2\sin x\cos^2 x = \sin x$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x(2\cos^2 x - 1) = 0$

$\sin x = 0$ $\quad$ or $\quad$ $\cos x = \pm\frac{1}{\sqrt{2}}$

$x = 0, \pi, 2\pi$ $\quad$ $x = \frac{\pi}{4}, \pi \pm \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

3 solutions $\quad$ 4 solutions

4. A

$\int \frac{7}{1+49x^2}\,dx = 7\int \frac{1}{49(\frac{1}{49}+x^2)}\,dx$

$= \frac{1}{7}\int \frac{1}{(\frac{1}{7})^2 + x^2}\,dx$

$= \frac{1}{7} \cdot \frac{1}{\frac{1}{7}} \tan^{-1}\frac{x}{\frac{1}{7}} + C$

$= \frac{7}{7}\tan^{-1} 7x + C$

$= \tan^{-1} 7x + C$

5. A

$v^2 = 324 - 36x^2$

$= 36(9 - x^2)$

$= 6^2(3^2 - x^2)$

$n = 6$ and [illegible]

$T = \frac{2\pi}{6}$

$= \frac{\pi}{3}$

6. B

$2x^3 + 8x^2 - x + 6 = 0$

$\alpha+\beta+\gamma = \frac{-b}{a}$ $\quad$ $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$= \frac{-8}{2}$ $\quad$ $= \frac{-1}{2}$

$= -4$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= (-4)^2 - 2(-\frac{1}{2})$

$= 16 + 1$

$= 17$

7. B

8. C

$P(x) = (x^2-1)Q(x) + 3x - 1$

$= (x-1)(x+1)Q(x) + 3x - 1$

$P(-1) = 0 + 3(-1) - 1$

$P(-1) = -4$

9. D

$P(\text{finds work}) = \frac{2}{5}$

$P(\text{not find work}) = \frac{3}{5}$

X = no. of students that find work

$P(X \ge 9) = P(X=9)$ or $P(X=10)$

$= {}^{10}C_9 (\frac{2}{5})^9 (\frac{3}{5})^1 + {}^{10}C_{10} (\frac{2}{5})^{10} (\frac{3}{5})^0$

$= \binom{10}{9}(0.4)^9(0.6) + (0.4)^{10}$

10. A

$2\cos(2x - \frac{\pi}{6}) = 1$

$\cos(2x - \frac{\pi}{6}) = \frac{1}{2}$

$2x - \frac{\pi}{6} = 2n\pi \pm \cos^{-1}\frac{1}{2}$

$= 2n\pi \pm \frac{\pi}{3}$

$2x = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6}$

$= 2n\pi + \frac{\pi}{3} + \frac{\pi}{6}$ $\quad$ or $\quad$ $2n\pi - \frac{\pi}{3} + \frac{\pi}{6}$

$= 2n\pi + \frac{\pi}{2}$ $\quad$ or $\quad$ $2n\pi - \frac{\pi}{6}$

$x = n\pi + \frac{\pi}{4}$ $\quad$ or $\quad$ $x = n\pi - \frac{\pi}{12}$

Section II

Question 11

a) $$\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{5}} = \lim_{x\to 0} \frac{5\sin\frac{x}{2}}{x}$$

$$= \frac{5}{2}\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$= \frac{5}{2}(1)$$

$$= \frac{5}{2}$$

1 mark for correct use of $\lim_{x\to 0} \frac{\sin x}{x} = 1$

b) $$\int \frac{1}{x-\sqrt{x}}\,dx$$

let $u = \sqrt{x}$

$u^2 = x$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$$= \int \frac{1}{u^2-u}\cdot 2u\,du$$

$$= 2\int \frac{\cancel{u}}{\cancel{u}(u-1)}\,du$$

$$= 2\left[\ln(u-1)\right] + C$$

$$= 2\ln(\sqrt{x}-1) + C$$

1 mark for correct substitution
1 mark for simplifying
1 mark for correct integration and expressing answer in terms of x.

c) PARALLEL

(i) $\frac{8!}{2!\,3!} = 3360$ A repeats twice
L repeats 3 times.

1 mark for correct answer

(ii) No. of arrangements that L are together.

LLL _ _ _ _ _

$\frac{6!}{2!} = 360$

$P(\text{L's are together}) = \frac{360}{3360}$

$= \frac{1}{}$

d.)

let $\theta = \tan^{-1}\frac{1}{3}$

$\tan\theta = \frac{1}{3}$

$$\tan(2\tan^{-1}\tfrac{1}{3}) = \tan 2\theta$$

$$= \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2(\frac{1}{3})}{1-(\frac{1}{3})^2}$$

$$= \frac{\frac{2}{3}}{\frac{8}{9}}$$

$$= \frac{3}{4} \text{ as required.}$$

2 mark for an correct expansion of $\tan 2\theta$ and ...
1 mark to simplify to $\frac{3}{4}$.

e.)

$$T_{k+1} = {}^{12}C_k (3x^2)^{12-k}\left(\frac{2}{x}\right)^k$$

$$= {}^{12}C_k\, 3^{12-k}\, x^{24-2k}\, 2^k\, x^{-k}$$

$$= {}^{12}C_k\, 3^{12-k}\, 2^k\, x^{24-3k}$$

for the term independent of x:

$24 - 3k = 0$
$-3k = -24$
$k = 8$

$\therefore$ the 9th term is independent of x.

$T_9 = {}^{12}C_8\, 3^{12-8}\, 2^8$

$= 10\,264\,320$

1 mark for expression of T_{k+1} term or an expansion of $(3x^2+\frac{2}{x})^{12}$
1 mark to solve for k and find the term independent of x.

f.) (i) Sub $S(0, a)$ into PQ

$2(a) - (p+q)0 + 2apq = 0$

$2apq = -2a$

$pq = \frac{-2a}{2a}$

$\therefore pq = -1$

1 mark to sub co-ordinates of focus into PQ and show $pq = -1$

(ii) $R\left(-apq(p+q),\ a(p^2+q^2+pq+2)\right)$

$x = -apq(p+q)$

using (i)

$= -a(-1)(p+q)$

$x = a(p+q)$ —— (1)

$\downarrow$

$\frac{x}{a} = p+q$ —— (1a)

$y = a(p^2+q^2+pq+2)$

using (i)

$= a(p^2+q^2-1+2)$

$y = a(p^2+q^2+1)$ —— (2)

$y = a\left[(p+q)^2 - 2pq + 1\right]$

using (i) and (1a)

$y = a\left(\left(\frac{x}{a}\right)^2 - 2(-1)+1\right)$

$y = a\left[\frac{x^2}{a^2} + 2 + 1\right]$

$y = \frac{x^2}{a} + 3a$

$ya = x^2 + 3a^2$

$x^2 = ya - 3a^2$

$\therefore x^2 = a(y-3a)$

1 mark to remove the parameters p and q by using appropriate substitutions.

1 mark to simplify and show.

Question 12

a.) $f(x) = x^2 - \cos x$

(i) $f(0) = 0 - \cos 0$

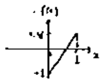
$= -1$

< 0

$f(1) = 1^2 - \cos 1$

$= 0.46$

> 0



$\therefore f(x)$ has a root between $x=0$ and $x=1$ since $f(0) < 0$ and $f(1) > 0$.

1 mark to test the end points with the function and show opposite sign.

(ii)

$x_0 = 0.8$

$f'(x) = 3x^2 + \sin x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0.8 - \frac{0.8^3 - \cos 0.8}{3(0.8)^2 + \sin 0.8}$

$\therefore x \approx 0.37$ (to 2 dp)

1 mark to differentiate $f(x)$
1 mark to substitute correctly into formula
1 mark to find root to 2 dp

b.) Prove $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for integers $n \geq 1$.

Step 1 Show true for $n=1$

$1^3 + (1+1)^3 + (1+2)^3 = 1 + 8 + 27$

$= 36$

Which is divisible by 9.

$\therefore$ true for $n=1$

Step 2 Assume true for $n=k$

ie $k^3 + (k+1)^3 + (k+2)^3 = 9M$ where M is an integer.

Step 3 Prove true for $n = k+1$

ie $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9N$ where N is an integer.

LHS $= (k+1)^3 + (k+2)^3 + (k+3)^3$

$(k+1)^3 + (k+3)^3 + \underbrace{9M - (k+1)^3 - k^3}_{\text{from assumption}}$

$= (k+3)^3 + 9M - k^3$

$= k^3 + 3k^2(3) + 3k(3)^2 + 3^3 + 9M - k^3$

$= 9k^2 + 27k + 27 + 9M$

$= 9(k^2 + 3k + 3 + M)$

$= 9N$ since k and M are positive integers.

$=$ RHS

$\therefore$ It's true for $n = k+1$ if it is true for $n=k$

$\therefore$ by mathematical induction the statement is true for $n \geq 1$.

1 mark for proving minimum condition

2 marks for correct assumption and proof for the $(k+1)$th term.

c.) $f(x) = 1 + \ln x$

(i) domain of $f(x)$: $x > 0$

$f'(x) = \frac{1}{x}$ $\qquad f''(x) = -\frac{1}{x^2}$

$\therefore f(x)$ is increasing since $f'(x) > 0$ for all $x > 0$.

$\therefore f(x)$ is concave down since $f''(x) < 0$ for all $x > 0$.

1 mark for finding $f'(x)$ and $f''(x)$

1 mark for stating $f'(x) > 0$ and $f''(x) < 0$ ...

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x-1)$

$y - 1 = x - 1$

$\therefore y = x$ equation of tangent at $x=1$.

1 mark for correct $f'(x)$ and finding $f'(1)$

1 mark for correct equation of tangent.

(iii) $y = 1 + \ln x$

$\downarrow$ Inverse

$x = 1 + \ln y$

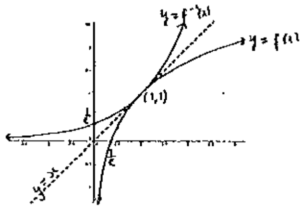
$x - 1 = \ln y$

$y = e^{x-1}$

$f^{-1}(x) = e^{x-1}$ Inverse function

1 mark for correct inverse function.

(iv)



when $x=1$ $f^{-1}(1) = e^{1-1} = e^0 = 1$

$\therefore$ pt of intersection $(1,1)$

x-intercept $f(x)$.

$0 = 1 + \ln x$

$-1 = \ln x$

$e^{-1} = x$

$x = \frac{1}{e}$

y-intercept $f^{-1}(x) = \frac{1}{e}$

1 mark for showing point of intersection.

1 mark for correct graph for $y = f(x)$ with intercept.

1 mark for correct graph for $y = f^{-1}(x)$ with intercept.

Question 13

$x^2 = 400 - (y-20)^2$

$= \pi \int_0^h 400 - (y-20)^2 \, dy$

$= \pi \int_0^h 400 - (y^2 - 40y + 400) \, dy$

$= \pi \int_0^h \cancel{400} - y^2 + 40y \cancel{- 400} \, dy$

$= \pi \left[-\frac{y^3}{3} + \frac{40y^2}{2} \right]_0^h$

$= \pi \left[-\frac{h^3}{3} + 20h^2 - 0 \right]$

$\therefore V = 20\pi h^2 - \frac{\pi h^3}{3}$

1 mark for correct substitution of x^2 into $\pi \int_0^h x^2 \, dy$.

1 mark for correct integration

1 mark for correct substitution of limits and showing volume as required.

(ii) $\frac{dV}{dt} = 4 \text{ cm}^3 \text{s}^{-1}$ $\frac{dh}{dt} = ?$

$\frac{dV}{dh} = 40\pi h - \pi h^2$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dh}{dt} = \frac{1}{40\pi h - \pi h^2} \times 4$

$\frac{dh}{dt} = \frac{4}{40\pi h - \pi h^2}$

when $h = 12$ cm

$\frac{dh}{dt} = \frac{4}{40\pi(12) - \pi(12)^2}$ cm/s

$= \frac{1}{84\pi}$ cm/s

1 mark for differentiating V with respect to h. ($\frac{dV}{dh}$)

1 mark for correct relationship between $\frac{dh}{dt}$, $\frac{dV}{dh}$ and $\frac{dV}{dt}$

1 mark to simplify and find $\frac{dh}{dt}$ when $h = 12$ cm

b) (i) $\frac{d}{dx}\left(x\sqrt{1-x^2} + \cos^{-1} x \right)$

$= \frac{d}{dx}\left(x(1-x^2)^{\frac{1}{2}} + \cos^{-1} x \right)$

$= x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (\cancel{2}x) + (1-x^2)^{\frac{1}{2}} + -\frac{1}{\quad}$

$$= \frac{-x^2}{(1-x^2)^{\frac{1}{2}}} + (1-x^2)^{\frac{1}{2}} - \frac{1}{(1-x^2)^{\frac{1}{2}}}$$

$$= \frac{-x^2 + (1-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} - 1}{(1-x^2)^{\frac{1}{2}}}$$

$$= \frac{-x^2 + 1 - x^2 - 1}{\sqrt{1-x^2}}$$

$$= \frac{-2x^2}{\sqrt{1-x^2}} \quad \text{as required.}$$

1 mark for differentiating correctly
1 mark for simplifying

(ii)

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x^2}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{1}{2}\left[x\sqrt{1-x^2} + \cos^{-1}x\right]_0^{\frac{1}{2}}$$

$$= -\frac{1}{2}\left[\frac{1}{2}\sqrt{1-(\tfrac{1}{2})^2} + \cos^{-1}\tfrac{1}{2} - (0 + \cos^{-1}0)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{2}\sqrt{\frac{3}{4}} + \frac{\pi}{3} - \frac{\pi}{2}\right]$$

$$= -\frac{1}{2}\left[\frac{\sqrt{3}}{4} - \frac{\pi}{6}\right]$$

$$= -\frac{\sqrt{3}}{8} + \frac{\pi}{12}$$

1 mark for manipulating integral to be able to use part (i).
1 mark for integrating and giving answer in exact form.

c.) (i) $y = \frac{x-1}{2x-3}$

$2x - 3 \neq 0$

$x \neq \frac{3}{2}$ (vertical asymptote)

$$\lim_{x\to\infty} \frac{x-1}{2x-3}$$

$$\lim_{x\to\infty} \frac{\frac{x}{x} - \frac{1}{x}}{\frac{2x}{x} - \frac{3}{x}}$$

$$\lim_{x\to\infty} \frac{1 - \cancel{\frac{1}{x}}^{\,0}}{2 - \cancel{\frac{3}{x}}^{\,0}}$$

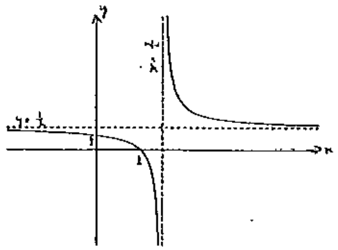
$= \frac{1}{2}$ (horizontal asymptote)

$\therefore x = 1$

y intercept sub $x = 0$

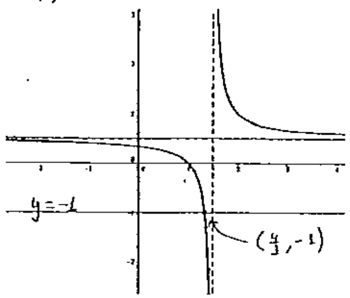
$y = \frac{-1}{-3}$

$y = \frac{1}{3}$



1 mark for finding vertical asymptotes
1 mark for finding horizontal asymptotes.
1 mark for finding intercepts and correct sketch.

(ii)



find point of intersection.

$\frac{x-1}{2x-3} = -1$

$x - 1 = -1(2x-3)$

$x - 1 = -2x + 3$

$3x = 4$

$x = \frac{4}{3}$

from graph:

$\frac{x-1}{2x-3} > -1$ for $\boxed{x < \frac{4}{3} \text{ and } x > \frac{3}{2}}$

1 mark to find point of intersection
1 mark for correct solution by inspection of the graph

Question 14

a)

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{14}{7} \qquad \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{7}{2}$$

$$\frac{n!}{\ldots} \qquad \frac{n!}{(n-r)!\,r!}$$

$(n-r)!\,r!$

$$\frac{(n-r)!\,r!}{(n-r+1)!\,(r-1)!} = 2$$

$$\frac{\cancel{(n-r)!}\,r\,\cancel{(r-1)!}}{(n-r+1)\cancel{(n-r)!}\,\cancel{(r-1)!}} = 2$$

$$\frac{r}{n-r+1} = 2$$

$$r = 2n-2r+2$$

$$2n-3r = -2 \quad \text{——— (1)}$$

$$\frac{(n-r-1)!\,(r+1)!}{(n-r)!\,r!} = \frac{7}{2}$$

$$\frac{\cancel{(n-r-1)!}\,(r+1)\cancel{r!}}{(n-r)\cancel{(n-r-1)!}\,\cancel{r!}} = \frac{7}{2}$$

$$\frac{r+1}{n-r} = \frac{7}{2}$$

$$2(r+1) = 7(n-r)$$

$$2r+2 = 7n-7r$$

$$7n-9r = 2 \quad \text{——— (2)}$$

$$2n-3r = -2 \quad \text{——— (1)}$$
$$7n-9r = 2 \quad \text{——— (2)}$$

(1) x 3

$$6n-9r = -6 \quad \text{——— (3)}$$
$$7n-9r = 2 \quad \text{——— (2)}$$

(2) − (3)

$$\boxed{\therefore \; n = 8}$$

sub n = 8 into (1)

$$2(8) - 3r = -2$$
$$16 - 3r = -2$$
$$-3r = -18$$
$$\boxed{\therefore \; r = 6}$$

2 marks for two simultaneous equations with n & r.

2 marks for correct solution for n & r.

b.) (i) $x = 2\sin\left(2t - \frac{\pi}{4}\right)$

$$\dot{x} = 2\cos\left(2t - \frac{\pi}{4}\right)\cdot 2$$
$$= 4\cos\left(2t - \frac{\pi}{4}\right)$$

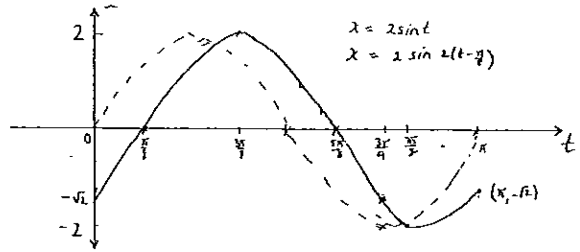
$$\ddot{x} = -4\sin\left(2t - \frac{\pi}{4}\right)\cdot 2$$
$$= -8\sin\left(2t - \frac{\pi}{4}\right)$$
$$= -4\left(2\sin\left(2t - \frac{\pi}{4}\right)\right)$$

$\therefore \ddot{x} = -4x$ as required

1 mark for finding $\dot{x}$ correctly

1 mark for finding $\ddot{x}$ correctly and showing link to x.

(ii) $x = 2\sin 2\left(t - \frac{\pi}{8}\right)$ ← affects period



endpoints: at $t = 0$

$$x = 2\sin 2\left(0 - \frac{\pi}{8}\right)$$
$$= 2\sin\left(-\frac{\pi}{4}\right)$$
$$= -2\sin\frac{\pi}{4}$$
$$= -2\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{-2\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$$
$$= -\sqrt{2}$$

at $t = \pi$

$$x = 2\sin 2\left(\pi - \frac{\pi}{8}\right)$$
$$= 2\sin\frac{14\pi}{8}$$
$$= 2\sin\frac{7\pi}{4}$$
$$= -2\sin\frac{\pi}{4}$$
$$= -\sqrt{2}$$

1 mark for correct coordinates of end points and max/min value in the domain.

1 mark for correct shape and intercepts

(iii) Starting point $x = -\sqrt{2}$

$$2\sin\left(2t - \frac{\pi}{4}\right) = -\sqrt{2}$$
$$\sin\left(2t - \frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$
$$\sin\left(2t - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$
$$2t - \frac{\pi}{4} = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$2t - \frac{\pi}{4} = \pi + \frac{\pi}{4}$$
$$2t = \frac{5\pi}{4} + \frac{\pi}{4}$$
$$2t = \frac{6\pi}{4}$$
$$t = \frac{6\pi}{8}$$
$$t = \frac{3\pi}{4}$$

$\therefore$ particle first returns to its starting point after $\frac{3\pi}{4}$ sec.

1 mark for identifying starting displacement and stating $2\sin\left(2t - \frac{\pi}{4}\right) = -\sqrt{2}$

1 mark to solve for t and show $t = \frac{3\pi}{4}$ sec.

(iv) one oscillation = 8m

$$T = \frac{2\pi}{n} = \frac{2\pi}{2}$$

$= 15$

$\therefore$ time for 15 oscillations $= 15\pi$ sec

1 mark for correct time in exact form.

c.) (i)

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \binom{2n}{3}x^3 + \dots + \binom{2n}{r}x^r + \dots \binom{2n}{2n}x^{2n}$$

let $x = 1$

$$2^{2n} = \binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \dots + \binom{2n}{r} + \dots \binom{2n}{2n}$$

$$\therefore 2^{2n} = \sum_{r=0}^{2n} \binom{2n}{r}$$

1 mark for correct expansion and substitution of $x=1$ to LHS and RHS.

(ii) from (i)

$$2^{2n} = \underbrace{\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \dots\dots + \binom{2n}{n-1}} + \binom{2n}{n} + \underbrace{\binom{2n}{n+1} + \binom{2n}{n+2} + \dots + \binom{2n}{2n-1} + \binom{2n}{2n}}$$

$2n$ is even, therefore there are $2n+1$ terms (odd)
and $\binom{2n}{n}$ remains unpaired. By symmetry of co-efficients:
ie $\binom{2n}{0} = \binom{2n}{2n}$, $\binom{2n}{1} = \binom{2n}{2n-1}$, $\binom{2n}{n-1} = \binom{2n}{n+1}$ etc.

$$2^{2n} = 2\left[\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \dots \binom{2n}{n-1}\right] + \binom{2n}{n}$$

$$2^{2n} + \binom{2n}{n} = 2\left[\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \binom{2n}{n-1} + \binom{2n}{n}\right]$$

$$2^{2n} + \binom{2n}{n} = 2\sum_{r=0}^{n} \binom{2n}{r}$$

$$2^{2n} + \frac{(2n)!}{(2n-n)!n!} = 2\sum_{r=0}^{n} \binom{2n}{r}$$

$$\frac{1}{2}\left[2^{2n} + \frac{(2n)!}{n!n!}\right] = \sum_{r=0}^{n} \binom{2n}{r}$$

$2^{2n-1} + \frac{(2n)!}{\dots}$ $\quad \sum^{n} \binom{2n}{\dots}$