St Catherine's School
Waverley

2017 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A Reference Sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Task Weighting – 40%

Total Marks – 100

Section I Pages 3 – 6

10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 14

90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 What is 3.086 96 correct to three significant figures?

(A) 3.08

(B) 3.09

(C) 3.086

(D) 3.087

2 What is the focus of $(x+3)^2 = 12y$?

(A) $(-3, 3)$

(B) $(-3, 0)$

(C) $(3, 3)$

(D) $(3, -3)$

3 What values of x is the curve $f(x) = x^3 + 5x^2 - 3x$ concave down?

(A) $x > \frac{5}{3}$

(B) $x < -\frac{3}{5}$

(C) $x < -\frac{5}{3}$

(D) $x > -\frac{5}{3}$

4 Find

$$\int \sin\frac{x}{4}\,dx$$

(A) $-\frac{1}{4}\cos\frac{x}{4}+c$

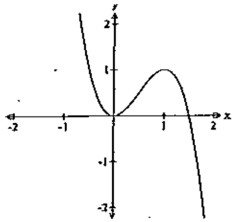
(B) $-4\cos\frac{x}{4}+c$

(C) $\frac{1}{4}\cos\frac{x}{4}+c$

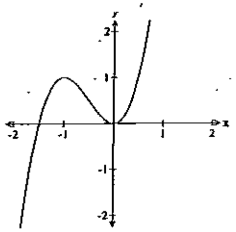
(D) $4\cos\frac{x}{4}+c$

5 Which of the following is the graph of $f(x) = -2x^3 - 3x^2$?

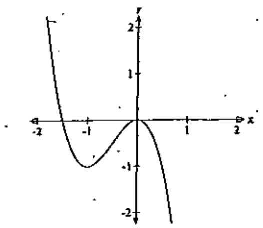
(A)



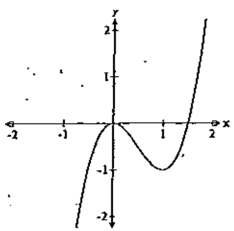
(B)



(C)

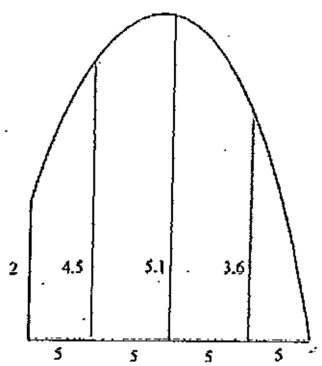


(D)



6 The diagram below shows a garden. All measurements are in metres

Not to scale



What is an approximate value for the area of the garden using the Simpson's Rule with four subintervals?

(A) $\frac{142}{3}\,m^2$

(B) $\frac{223}{3}\,m^2$

(C) $\frac{223}{2}\,m^2$

(D) $\frac{193}{3}\,m^2$

7 Evaluate

$$\sum_{k=2}^{40} 3k-8$$

(A) 110

(B) 2200

(C) 2090

(D) 2145

8 The quadratic equation $2x^2 - 10x + 9 = 0$ has roots α and β.
What is the value of $\alpha^2 + \beta^2$?

(A) 0

(B) 34

(C) 25

(D) 16

9 What is the derivative of $(1 + \log_e x^2)^4$?

(A) $\frac{8(1+\log_e x^2)^3}{x}$

(B) $\frac{4(1+\log_e x^2)^3}{x^2}$

(C) $\frac{4(1+\log_e 2x)^3}{x}$

(D) $4\left(1+\frac{2}{x}\right)^3$

10 What are the solutions to the equation $9^x - 9 \cdot 3^x + 18 = 0$?

(A) $x = 1$ or $x = \frac{\ln 3}{\ln 6}$

(B) $x = 0$ or $x = \frac{\ln 6}{\ln 3}$

(C) $x = 1$ or $x = \ln 2$

(D) $x = 1$ or $x = \frac{\ln 6}{\ln 3}$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Solve $2x^2 - x - 3 \geq 0$ **2**

(b) Evaluate $\lim_{x \to 3} \frac{x^3-27}{x-3}$. **2**

(c) The line $y = mx + b$ is a tangent to the curve $y = x^2 - 8x + 10$ at the point $(2, -2)$.

Find the value of m and b? **3**

(d) Solve $\sin 2\theta = \frac{\sqrt{3}}{2}$ for $0 \leq \theta \leq 2\pi$. **2**

(e) Find $\int_1^e \frac{x+1}{x^2}\, dx$ **3**

(f) Find the equation of normal to the curve $y = x \cos x$ at $x = \pi$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Differentiate the following:

(i) $y = \tan^2 x$ **2**

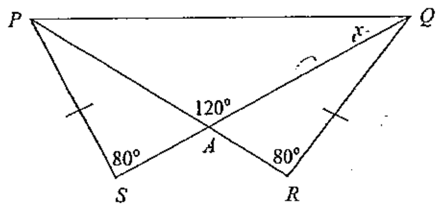
(ii) $f(x) = \frac{x}{e^x}$ **2**

(b) Find:

(i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \; dx$ **2**

(ii) $\int \left(\sqrt{x+1}\right)^3 dx$ **2**

(c) PR and QS are straight lines intersecting at point A. Also $PS = QR$, $\angle PSA = \angle QRA = 80°$, $\angle PAQ = 120°$ and $\angle PQA = x$.



NOT TO SCALE

Copy the diagram into your writing booklet.

(i) Prove that ΔPSA is congruent to ΔQRA. **2**

(ii) Hence, show that $x = 30°$. **2**

(d) Simplify $\frac{\sin(2\pi-\alpha)}{\sin\left(\frac{\pi}{2}-\alpha\right)}$ **2**

(e) For all x in the domain $x > 0$, a function $f(x)$ satisfies $f'(x) > 0$ and $f''(x) < 0$. Sketch a possible graph of $f(x)$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Find $\lim_{x\to 0} \frac{\sin 2x}{3x}$. Show working. **2**

(b) Solve $2\tan^2\theta - \sec^2\theta = 0$ for $0 \le \theta \le 2\pi$. **3**

(c) The third term of geometric series is $\frac{9}{2}$ and the sixth term is $\frac{243}{16}$.

(i) Find the first term and common ratio. **2**

(ii) How many terms of this geometric series must be taken for the sum to exceed 10 000? **2**

(d) A block of ice is removed from the fridge. The rate at which the ice melts is given by $\frac{dM}{dt} = -kM$, where M is amount of ice block remaining measured in grams and time t, in minutes.

(i) Show that $M = M_0 e^{-kt}$ satisfies $\frac{dM}{dt} = -kM$. **1**

(ii) After 35 minutes only half the ice remains. Find the value of k. Leave your answer correct to 4 decimal places. **2**

(iii) If, at a certain time, 95% of the ice melted, how long would it have been since the ice block was removed from the fridge?

Leave your answer in hours and minutes. **2**

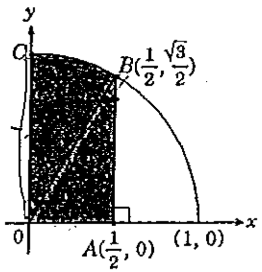
(iv) At what rate is the ice melting when 5 grams of the block remains. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) For what values of x will the series $1 + \ln x + (\ln x)^2 + (\ln x)^3 + \cdots$ have a limiting sum? **2**

(b) The diagram below shows the first quadrant of the circle $x^2 + y^2 = 1$. The point A and B have coordinates $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ respectively and AB is perpendicular to the x-axis.



(i) What is the exact value of $\angle COB$? **2**

(ii) Hence, show that the exact value of the shaded area $OABC$ is $\frac{2\pi+3\sqrt{3}}{24}$. **2**

(c) The velocity v, in m/s of an object moving along the x-axis is given by $v = 1 + 2\cos 2t$ for $0 \le t \le \pi$, where t is the time in seconds. Initially the particle is at the origin.

(i) At what time(s) is the object at rest between $t = 0$ and π? **2**

(ii) Find the maximum velocity of the object for the given period. **1**

(iii) Sketch the graph of v as a function of t, showing all important features for $0 \le t \le \pi$. **2**

(iv) Find the displacement function x in terms of t. **2**

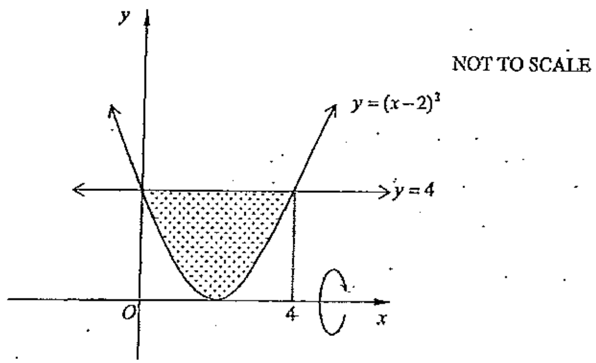
(v) Hence, or otherwise, show that the total distance travelled by the object in the given period is $\frac{\pi}{3} + 2\sqrt{3}$ metres. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) Solve $2\log_2 x - \log_2(3 - x) = 2$ **3**

(b) The shaded region bounded by the graph $y = (x-2)^2$ and the line $y = 4$ is rotated about the x-axis to form a solid of revolution as shown in the diagram.



(i) Show that the volume V of the solid is given by: **2**

$$V = 64\pi - 2\pi \int_0^2 (x-2)^4 \, dx$$

(ii) Find the volume of the solid formed, in exact form. **2**

(c) Ellena and Maddison play a table tennis match against each other. The probability in any set that Ellena wins is $\frac{2}{5}$. The first player to win 2 sets wins the match.

(i) Find the probability that Ellena wins the match after the second set? **1**

(ii) Find the probability that the match ends at the second set? **1**

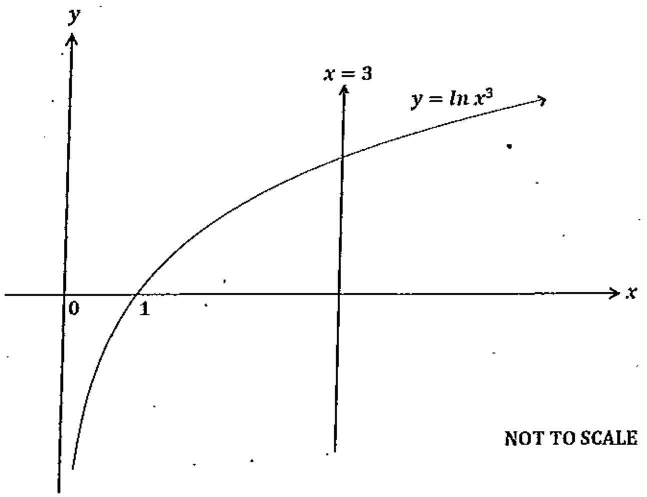
(iii) Find the probability that Ellena wins the match at the third set. **2**

Question 15 continues on page 12

Question 15 (continued)

(d) (i) Show that $\frac{d}{dx}(x\ln x) = \ln x + 1$. **1**

(ii) Hence, or otherwise, find the area bounded by the curve $y = \ln x^3$, line $x = 3$ and x-axis. **3**



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End of Question 15

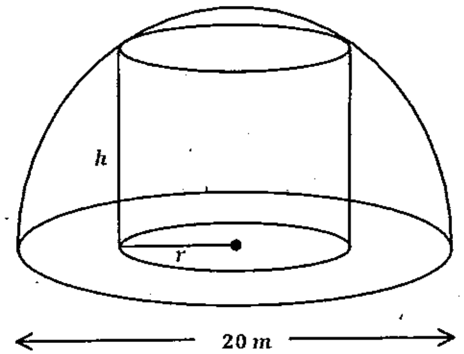
Question 16 (15 marks) Use a SEPARATE writing booklet

(a) For $y = 5 - 2\sin\left(3x - \frac{2\pi}{3}\right)$,

(i) What is the maximum value of y? **1**

(ii) Find the value of x for $0 \leq x \leq 2\pi$, when the maximum value occurs for the first time. **2**

(b) A glass dome is to be designed in the shape of hemisphere with a diameter 20 metres. This dome contains a cylindrical greenhouse with radius r and height h.



NOT TO SCALE

(i) Show that the area of the curved wall of the greenhouse is given by $A = 2\pi r\sqrt{100 - r^2}$. **1**

(ii) Find the radius, r, for which the area of the curved wall of the greenhouse is maximum. Leave the answer in exact form. **3**

Question 16 continues on page 14

Question 16 (continued)

(c) Consider the curve $f(x) = e^{\sin x}$ for $0 \le x \le 2\pi$,

(i) Show that the stationary points are at $\left(\frac{\pi}{2}, e\right)$ and $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$. **2**

(ii) Using the signs of the gradient function, or otherwise, determine the nature of the stationary points. **2**

(iii) Sketch the curve showing all important features including the coordinates of the stationary points in exact form and the end points. Label the points when $x = 0,\ \pi,\ 2\pi$. **3**

(iv) Find the number of solution(s) to the equation $e^{\sin x} - x = 0$ in the given domain $0 \le x \le 2\pi$. Justify your answer. **1**

End of paper

Student Number: SOLUTIONS

St Catherine's School
Waverley

2017 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics

Multiple Choice Answer Sheet

Completely fill the response circle representing the most correct answer

	A	B	C	D
1.	○	●	○	○
2.	●	○	○	○
3.	○	○	●	○
4.	○	●	○	○
5.	○	○	●	○
6.	○	●	○	○
7.	○	○	○	●
8.	○	○	○	●
9.	●	○	○	○
10.	○	○	○	●

2017 Mathematics Trial Examination SOLUTIONS

Section I – MULTIPLE CHOICE

(1) 3.09 (B)

(2) vertex $(-3,0)$, focal length $= 3$
Focus $(-3,3)$ (A)

(3) $f'(x) = 3x^2 + 10x - 3$
$f''(x) = 6x + 10$
Concave down $f''(x) < 0$
$6x + 10 < 0$
$x < -\frac{5}{3}$ (C)

(4) $\int \sin \frac{x}{4}\, dx = -4\cos\frac{x}{4} + c$ (B)

(5) $f(x) = -2x^3 - 3x^2$
$0 = -x^2(2x+3)$
$x = 0$ or $x = -\frac{3}{2}$ (C)

(6)

x	0	5	10	15	20
y	2	4.5	5.1	3.6	0

Area $\doteqdot \frac{h}{3}\left[y_0 + 4\Sigma_{odd} + 2\Sigma_{even} + y_n\right]$
$\doteqdot \frac{5}{3}\left[2 + 4(4.5+3.6) + 2(5.1) + 0\right]$
$\doteqdot \frac{223}{3}$ (B)

(7) $\sum_{k=2}^{40} 3k-8 = -2 + 1 + 4 + 7 + \dots + 112$
$= \frac{n}{2}[a + l]$
$= \frac{39}{2}(-2+112)$
$= 2145$ (D)

(8) $\alpha + \beta = -\frac{-10}{2} = 5$
$\alpha\beta = \frac{9}{2}$
$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= 5^2 - 2\left(\frac{9}{2}\right)$
$= 16$ (D)

(9) $\frac{d}{dx}(1+\log_e x^2)^4 = 4(1+\log_e x^2)^3 \cdot \frac{2x}{x^2}$
$= \frac{8(1+\log_e x^2)^3}{x}$ (A)

(10) $9^x - 9.3^x + 18 = 0$
$(3^x)^2 - 9(3^x) + 18 = 0$
let $u = 3^x$
$u^2 - 9u + 18 = 0$
$(u-3)(u-6) = 0$
$u = 3$ or $u = 6$
$3^x = 3$ $\quad$ $3^x = 6$
$x = 1$ $\quad$ $x\ln 3 = \ln 6$
$x = \frac{\ln 6}{\ln 3}$

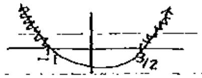
End of Section 1 Solutions

Section II

Question 11.

(a) $2x^2 - x - 3 \geq 0$

$(2x-3)(x+1) \geq 0$ ①



$\therefore x \leq -1$ or $x \geq \frac{3}{2}$ ①

(b) $\lim_{x \to 3} \frac{x^3 - 3^3}{x-3} = \frac{(x-3)(x^2+3x+9)}{x-3}$ ①

$= \lim_{x\to 3} x^2 + 3x + 9$

$= 3^2 + 3(3) + 9$

$= 27$ ①

(c) $y = x^2 - 8x + 10$

$y' = 2x - 8$ ①

$m_T = y'$

$= 2(2) - 8$ at $(2,-2)$

$= -4$ ①

then $y = -4x + b$

at $(2,-2)$: $-2 = -4(2) + b$

$b = 6$ ①

$\therefore m = -4$ and $b = 6$

(d) $\sin 2\theta = \frac{\sqrt{3}}{2}$

$2\theta = \frac{\pi}{3}, \pi - \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, (\pi - \frac{\pi}{3}) + 2\pi$

$= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$ $(\pi - \theta)$ ✓ S | A ✓ θ

T | C

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ $\frac{1}{2}$ each

(e) $\int_1^e \frac{x+1}{x^2}\,dx = \int_1^e \frac{x}{x^2} + \frac{1}{x^2}\,dx$

$= \int_1^e \frac{1}{x} + x^{-2}\,dx$ ①

$= \left[\ln x - x^{-1}\right]_1^e$ ①

$= \ln e - \frac{1}{e} - (\ln 1 - \frac{1}{1})$

$= 1 - \frac{1}{e} - 0 + 1$

$= 2 - \frac{1}{e}$ ①

(f) $y = x\cos x$

$y' = \cos x - x\sin x$ ①

at $x = \pi$, $y' = m_T$

$= \cos\pi - \pi\sin\pi$

$= -1 - \pi(0)$

$= -1$

$m_N = -\frac{1}{m_T}$

$= -\frac{1}{(-1)}$

$= 1$ ①

at $x = \pi$, $y = \pi\cos\pi$

$= \pi(-1)$

$= -\pi$

then equation of normal at $(\pi, -\pi)$ is

$y + \pi = 1(x - \pi)$

$\therefore y = x - 2\pi$ ①

End of Question 11 solutions

Question 12

(a) (i) $y = (\tan x)^2$

$y' = 2\tan x \sec^2 x$ ②

(ii) $f(x) = \dfrac{x}{e^x}$

$f'(x) = \dfrac{e^x - xe^x}{(e^x)^2}$ ①

$= \dfrac{e^x(1-x)}{(e^x)^2}$

$= \dfrac{1-x}{e^x}$ ①

(b) (i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \dfrac{\cos x}{\sin x} \, dx$

$= \left[\ln(\sin x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ ①

$= \ln(\sin\frac{\pi}{2}) - \ln(\sin\frac{\pi}{6})$

$= \ln 1 - \ln\frac{1}{2}$

$= 0 - \ln 2^{-1}$

$= \ln 2$ or $-\ln\frac{1}{2}$ ①

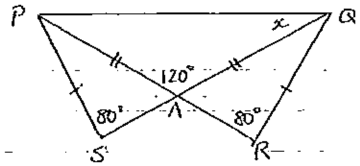
(ii) $\int (\sqrt{x+1})^3 \, dx$

$= \int (x+1)^{\frac{3}{2}} \, dx$

$= \frac{2}{5}(x+1)^{\frac{5}{2}} + c$ ① for $\frac{2}{5}$

① for $(x+1)^{\frac{5}{2}}$

(c)



(i) $\angle PSA = \angle QRA = 80°$ (given)

$PS = QR$ (given) ①

$\angle PAS = \angle QAR$ (vertically opposite ∠s are equal)

$\therefore \triangle PSA \equiv \triangle QRA$ (AAS) ①

(ii) $PA = QA$ (corresponding sides of congruent Δs are equal)

$\angle APQ = \angle AQP = x$ (angles opposite equal sides are equal)

$\angle APQ + \angle AQP + \angle PAQ = 180°$ (∠ sum of Δ) ①

$x + x + 120 = 180°$

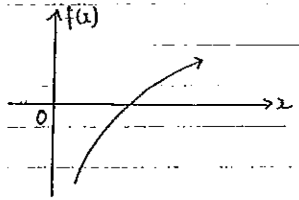
$2x = 60$

$\therefore x = 30$ ①

(d) $\dfrac{\sin(2\pi - \alpha)}{\sin(\frac{\pi}{2} - \alpha)} = \dfrac{-\sin\alpha}{\cos\alpha}$ ①

$= -\tan\alpha$ ①

(e)



①

End of Question 12 solutions

Question 13.

(a) $\lim_{x\to 0} \frac{\sin 2x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{\sin 2x}{x}$

$= \frac{2}{3} \lim_{x\to 0} \frac{\sin 2x}{2x}$ ①

$= \frac{2}{3}$ ①

(b) $2\tan^2\theta - \sec^2\theta = 0$

$2\tan^2\theta - (1+\tan^2\theta) = 0$

$2\tan^2\theta - 1 - \tan^2\theta = 0$

$\tan^2\theta - 1 = 0$

$(\tan\theta - 1)(\tan\theta + 1) = 0$

$\tan\theta = 1$ or $\tan\theta = -1$ ①

$\theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$ $\quad \theta = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$= \frac{\pi}{4}, \frac{5\pi}{4}$ $\quad = \frac{3\pi}{4}, \frac{7\pi}{4}$

$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ $\frac{1}{2}$ each

(c) $T_3 = ar^2 = \frac{9}{2}$ ①

$T_6 = ar^5 = \frac{243}{16}$ ②

(i) substitute ① into ②

$ar^2 \cdot r^3 = \frac{243}{16}$

$\frac{9}{2} r^3 = \frac{243}{16}$

$r^3 = \frac{27}{8}$

$r = \frac{3}{2}$ ①

substitute $r = \frac{3}{2}$ into ①

$a\left(\frac{3}{2}\right)^2 = \frac{9}{2}$

$\frac{9}{4} a = \frac{9}{2}$

$a = 2$ ①

$\therefore$ common ratio $r = \frac{3}{2}$ & first term $a = 2$

(ii) $S_n > 10\,000$

$S_n = \frac{a(r^n - 1)}{r-1}$ for $|r| > 1$

$= \frac{2\left(\left(\frac{3}{2}\right)^n - 1\right)}{\frac{3}{2} - 1}$

$= \frac{2\left(\left(\frac{3}{2}\right)^n - 1\right)}{\frac{1}{2}}$

$= 4\left(\left(\frac{3}{2}\right)^n - 1\right)$

then $4\left(\left(\frac{3}{2}\right)^n - 1\right) > 10\,000$

$\left(\frac{3}{2}\right)^n - 1 > 2500$

$\left(\frac{3}{2}\right)^n > 2501$ ①

$n \ln\frac{3}{2} > \ln 2501$

$n > \frac{\ln 2501}{\ln\frac{3}{2}}$

$n > 19.2974\ldots$ ①

$\therefore$ $n = 20$ since n is positive integer

$\therefore$ 20 terms needed

(d) (i) $M = M_0 e^{-kt}$

$\frac{dM}{dt} = -kM_0 e^{-kt}$

$= -kM$ as required ①

(ii) when $t = 35$, $M = 0.5M_0$

$0.5M_0 = M_0 e^{-35k}$

$0.5 = e^{-35k}$ ①

$-35k = \ln 0.5$

$k = -\frac{1}{35}\ln 0.5$

$= 0.01980\ldots$

$\therefore$ $k = 0.0198$ ①

(iii) 95% melted = 5% ice remaining

$= 0.05 M_0$

$0.05M_0 = M_0 e^{-0.0198t}$

$0.05 = e^{-0.0198t}$ ①

$-0.0198t = \ln 0.05$

$t = -\frac{1}{0.0198}\ln 0.05$

$= 151.2996\ldots$ mins

$=$ 2 hours 31 mins ①

∴ the ice block was removed 2 hours & 31 minutes from the fridge.

(iv) when $M = 5$

$\frac{dM}{dt} = -(0.0198)(5)$

$= -0.099$ grams/min ①

End of Question 13 solutions

Question 14

(a) $r = \ln x$

for limiting sum $|r| < 1$

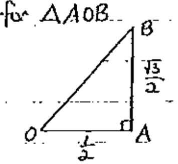
$|\ln x| < 1$

$-1 < \ln x < 1$ ①

$e^{-1} < x < e^{1}$

$\therefore \frac{1}{e} < x < e$ ①

(b) (i) for $\triangle AOB$



$\tan\angle BOA = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$\tan\angle BOA = \sqrt{3}$

$\angle BOA = \frac{\pi}{3}$ ①

then $\angle COB = \frac{\pi}{2} - \angle BOA$ (complementary $\angle$s)

$= \frac{\pi}{2} - \frac{\pi}{3}$

$= \frac{\pi}{6}$ ①

(ii) Area $OABC = A_{\text{sector } COB} + A_{\triangle AOB}$

$= \frac{1}{2}r^2\theta + \frac{1}{2}bh$

$= \frac{1}{2}\cdot OB^2\cdot\angle COB + \frac{1}{2}\cdot OA\cdot AB$

$= \frac{1}{2}\cdot 1^2\cdot\frac{\pi}{6} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}$ ① + ①

$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

$= \frac{2\pi + 3\sqrt{3}}{24}$ units^2

(c) (i) at rest $v = 0$

$0 = 1 + 2\cos 2t$

$\cos 2t = -\frac{1}{2}$ ①

$2t = \pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3},\ \frac{4\pi}{3}$

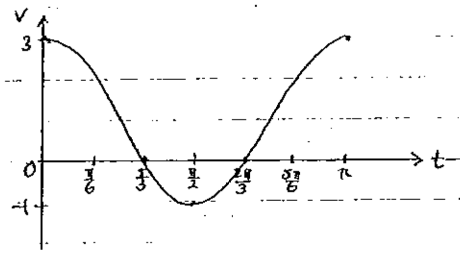
$t = \frac{\pi}{3},\ \frac{2\pi}{3}$ ①

(ii) $-2 \leqslant 2\cos 2t \leqslant 2$

$-1 \leqslant 1 + 2\cos 2t \leqslant 3$

$\therefore$ maximum velocity is 3 m/s ①

(iii)



① for endpoints

① for t intercepts

① for shape & minimum

(iv) $x = \int v\, dt$

$= \int 1 + 2\cos 2t\, dt$

$= t + \sin 2t + c$ ①

when $t = 0$, $x = 0$

$0 = 0 + \sin 2(0) + c$

$c = 0$ ①

$\therefore\ x = t + \sin 2t$

(v) the curve is symmetrical about $t = \frac{\pi}{2}$

so $\int_0^{\frac{\pi}{3}} v\, dt = \int_{\frac{2\pi}{3}}^{\pi} v\, dt$

and $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} v\, dt = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} v\, dt$

$\therefore$ Total distance $= 2\int_0^{\frac{\pi}{3}} v\, dt + 2\left|\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} v\, dt\right|$ ①

Total distance $= 2\left[t + \sin 2t\right]_0^{\frac{\pi}{3}} + 2\left|\left[t + \sin 2t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right|$

$= 2\left(\frac{\pi}{3} + \sin\frac{2\pi}{3} - 0\right) + 2\left|\frac{\pi}{2} + \sin\pi - \left(\frac{\pi}{3} + \sin\frac{2\pi}{3}\right)\right|$

$= 2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) + 2\left|\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{2}\right|$

$= \frac{2\pi}{3} + \sqrt{3} + 2\left|\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right|$

$= \frac{2\pi}{3} + \sqrt{3} + 2\left|-\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)\right|$

$= \frac{2\pi}{3} + \sqrt{3} + 2\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$

$= \frac{2\pi}{3} + \sqrt{3} + \sqrt{3} - \frac{\pi}{3}$

$= \frac{\pi}{3} + 2\sqrt{3}$ metres ①

$\therefore$ total distance travelled is $\frac{\pi}{3} + 2\sqrt{3}$ metres.

End of Question 14 solutions

Question 15

(a) $2\log_2 x - \log_2(3-x) = 2$, where $x > 0$

$\log_2 x^2 - \log_2(3-x) = 2$

$\log_2\left(\frac{x^2}{3-x}\right) = 2$ (1)

$\frac{x^2}{3-x} = 2^2$ (1)

$4(3-x) = x^2$

$0 = x^2 + 4x - 12$

$0 = (x+6)(x-2)$

$\therefore x = -6$ or $x = 2$

but $x = 2$ only since $x > 0$ (1)

(b) (i) $V = V_{cylinder} - V_{y=(x-2)^2 \text{ rotated}}$

$= \pi r^2 h - \pi\int_{x_1}^{x_2} y^2 \, dx$ (1)

$= \pi(4)^2(4) - \pi\int_0^4 (x-2)^4 \, dx$

$\therefore V = 64\pi - 2\pi\int_0^2 (x-2)^4 \, dx$ since $y=(x-2)^2$ is symmetrical about $x=2$ (1)

OR

$V = \pi\int_0^4 y_{top}^2 - y_{bottom}^2 \, dx$

$= \pi\int_0^4 4^2 - [(x-2)^2]^2 \, dx$ (1)

$= \pi\int_0^4 16 - (x-2)^4 \, dx$

$= 2\pi\int_0^2 16 - (x-2)^4 \, dx$ since symmetrical about $x=2$

$= 2\pi[16x]_0^2 - 2\pi\int_0^2 (x-2)^4 \, dx$

$= 2\pi(32) - 2\pi\int_0^2 (x-2)^4 \, dx$

$\therefore V = 64\pi - 2\pi\int_0^2 (x-2)^4 \, dx$ as required (1)

(ii) $V = 64\pi - 2\pi\int_0^2 (x-2)^4 \, dx$

$= 64\pi - \frac{2\pi}{5}\left[(x-2)^5\right]_0^2$ (1)

$= 64\pi - \frac{2\pi}{5}\left[(2-2)^5 - (0-2)^5\right]$

$= 64\pi - \frac{2\pi}{5}\left(0-(-2)^5\right)$

$= 64\pi - \frac{2\pi}{5}(32)$

$\therefore V = \frac{256\pi}{5}$ units^3 (1)

(c) $P(E) = \frac{2}{5}$ and $P(M) = \frac{3}{5}$

(i) $P(EE) = \frac{2}{5}\times\frac{2}{5}$ (1)

$= \frac{4}{25}$

(ii) P(Ellena wins or Maddison wins) = P(EE) + P(MM)

$= \frac{4}{25} + \frac{3}{5}\times\frac{3}{5}$

$= \frac{13}{25}$ (1)

(iii) P(EME or MEE) = P(EME) + P(MEE)

$= \frac{2}{5}\times\frac{3}{5}\times\frac{2}{5} + \frac{3}{5}\times\frac{2}{5}\times\frac{2}{5}$ (1) + (1)

$= \frac{24}{125}$

(d) (i) $\frac{d}{dx}(x\ln x) = \ln x + x\left(\frac{1}{x}\right)$

$= \ln x + 1$ (1)

(ii) $\frac{d}{dx}(x\ln x) = \ln x + 1$

$d(x\ln x) = (\ln x + 1)\,dx$

$\int d(x\ln x) = \int \ln x + 1 \, dx$

$x\ln x = \int \ln x \, dx + \int 1 \, dx$

$\therefore \int \ln x \, dx = x\ln x - \int 1 \, dx$ (1)

$\int_1^3 \ln x \, dx = [x\ln x]_1^3 - \int_1^3 1 \, dx$

$= 3\ln 3 - 1\ln 1 - [x]_1^3$

$= 3\ln 3 - 0 - (3 - 1)$

$= 3\ln 3 - 2$ (1)

then Area $= \int_1^3 \ln x^3 \, dx$

$= 3\int_1^3 \ln x \, dx$

$= 3(3\ln 3 - 2)$ using answer above

$\therefore$ Area $= 9\ln 3 - 6$ units^2 (1)

End of Question 15 solutions

Question 16

(a) $y = 5 - 2\sin(3x - \frac{2\pi}{3})$

(i) $-1 \leq \sin(3x - \frac{2\pi}{3}) \leq 1$

$-2 \leq -2\sin(3x - \frac{2\pi}{3}) \leq 2$

$3 \leq 5 - 2\sin(3x - \frac{2\pi}{3}) \leq 7$

$\therefore$ maximum value of $y = 7$ (1)

(ii) for maximum value of y we need

$\sin(3x - \frac{2\pi}{3}) = -1$

$3x - \frac{2\pi}{3} = -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ (1)

$3x = \frac{\pi}{6}, \frac{13\pi}{6}, \ldots$

$x = \frac{\pi}{18}, \frac{13\pi}{18}, \ldots$ (1)

$\therefore$ the maximum occurs for the first time when $x = \frac{\pi}{18}$.

(b) (i) using Pythagoras' theorem

$10^2 = h^2 + r^2$

$h = \sqrt{100 - r^2}$

then Area of curved wall $= 2\pi r h$

$\therefore A = 2\pi r\sqrt{100 - r^2}$ as required (1)

(ii) $\frac{dA}{dr} = 2\pi\sqrt{100 - r^2} + \cancel{2}\pi r(\cancel{\tfrac{1}{2}})(100 - r^2)^{-\frac{1}{2}}(-2r)$

$= 2\pi\sqrt{100 - r^2} - \frac{2\pi r^2}{\sqrt{100 - r^2}}$ (1)

for turning point $\frac{dA}{dr} = 0$

$0 = 2\pi\sqrt{100 - r^2} - \frac{2\pi r^2}{\sqrt{100 - r^2}}$

$2\pi\sqrt{100 - r^2} = \frac{2\pi r^2}{\sqrt{100 - r^2}}$

$\cancel{2\pi} r^2 = \cancel{2\pi}(100 - r^2)$

$r^2 = 100 - r^2$

$2r^2 = 100$

$r^2 = 50$

$r = 5\sqrt{2}$ (1)

Sign test:

when $r = 7 < 5\sqrt{2}$, $\frac{dA}{dr} = 2\pi\sqrt{100-7^2} - \frac{2\pi(7)^2}{\sqrt{100-7^2}}$

$= 1.759\ldots > 0$

when $r = 8 < 5\sqrt{2}$, $\frac{dA}{dt} = 2\pi\sqrt{100-8^2} - \frac{2\pi(8)^2}{\sqrt{100-8^2}}$

$= -29.32\ldots < 0$

r	7	$5\sqrt{2}$	8
$\frac{dA}{dr}$	+	0	−
	/	—	\

$\therefore$ maximum turning point

$\therefore$ maximum area when $r = 5\sqrt{2}$ m (1)

(c) $f(x) = e^{\sin x}$

(i) $f'(x) = \cos x\, e^{\sin x}$

for stationary points $f'(x) = 0$

$0 = \cos x\, e^{\sin x}$

$\cos x = 0$ or $e^{\sin x} = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ no solution (1)

when $x = \frac{\pi}{2}$, $y = e^{\sin\frac{\pi}{2}}$

$= e$

when $x = \frac{3\pi}{2}$, $y = e^{\sin\frac{3\pi}{2}}$

$= e^{-1}$

$= \frac{1}{e}$ (1)

$\therefore$ stationary points are $(\frac{\pi}{2}, e)$ and $(\frac{3\pi}{2}, \frac{1}{e})$

(ii) $f'(x) = \cos x\, e^{\sin x}$

at $x = \frac{\pi}{3}$, $f'(\frac{\pi}{3}) = \cos\frac{\pi}{3}\, e^{\sin\frac{\pi}{3}}$

$= \frac{1}{2} e^{\frac{\sqrt{3}}{2}} > 0$

at $x = \pi$, $f'(\pi) = \cos\pi\, e^{\sin\pi}$

$= (-1)e^0$

$= -1 < 0$

at $x = \frac{5\pi}{3}$, $f'(\frac{5\pi}{3}) = \cos\frac{5\pi}{3}\, e^{\sin\frac{5\pi}{3}}$

$= \frac{1}{2} e^{-\frac{\sqrt{3}}{2}} > 0$

x	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$f'(x)$	+	0	−	0	+
	/	—	\	—	/
		max.		min.	

$\therefore$ $(\frac{\pi}{2}, e)$ is a maximum stationary point and (1)

$(\frac{3\pi}{2}, \frac{1}{e})$ is a minimum stationary point. (1)

(iii) at $x = 0$, $f(0) = e^{\sin 0}$

$= e^0$

$= 1$ $\therefore (0, 1)$

at $x = \pi$, $f(\pi) = e^{\sin\pi}$

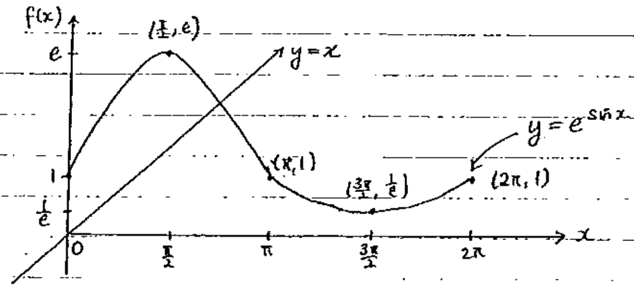
$= e^0$

$= 1$ $\therefore (\pi, 1)$

at $x = 2\pi$, $f(2\pi) = e^{\sin 2\pi}$

$= e^0$

$= 1$ $\therefore (2\pi, 1)$



(½) for each 5 points

(½) for correct shape.

(iv) $e^{\sin x} - x = 0$

$e^{\sin x} = x$

i.e. finding point of intersection(s) of $y = e^{\sin x}$ and $y = x$

from the graph in part (iii), it can be seen that $y = x$ line intersects the curve $y = e^{\sin x}$ <u>once</u> for $0 \leq t \leq 2\pi$

$\therefore$ there is <u>one solution</u> to the equation $e^{\sin x} - x = 0$ in the given domain. (1)

End of Paper solutions