Sydney Girls High School
2017

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 3 – 6

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 8 – 19

90 Marks

- Attempt Questions 11 – 16
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 2 hours and 45 minutes for this section

Name: .... Teacher:	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2017 HSC Examination Paper in this subject.

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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(1) What is the double root of the equation $x^3 - 5x^2 + 8x - 4 = 0$?

(A) $x = -2$

(B) $x = -1$

(C) $x = 1$

(D) $x = 2$

(2) A small car of mass 1200 kg is rounding a curve of radius 500 metres on a level road at 84km/h.

What force of friction is necessary between the wheels and the ground?

(A) 3·36 N

(B) 52·27 N

(C) 1306·67 N

(D) 16 934.4 N

(3) Which of the following parametric equations represent the hyperbola $x^2 - y^2 = 4$?

(A) $x = 2\tan\theta$ and $y = 2\sec\theta$

(B) $x = 4\tan\theta$ and $y = 4\sec\theta$

(C) $x = 2\sec\theta$ and $y = 2\tan\theta$

(D) $x = 4\sec\theta$ and $y = 4\tan\theta$

(4) Which of the following is the modulus-argument form of $2-2i$?

(A) $2\sqrt{2}cis\left(\frac{\pi}{4}\right)$

(B) $2\sqrt{2}cis\left(-\frac{\pi}{4}\right)$

(C) $2cis\left(\frac{7\pi}{4}\right)$

(D) $2cis\left(-\frac{7\pi}{4}\right)$

(5) The graph of $y=\frac{x^2}{x^2-4}$ has:

(A) a single vertical asymptote, two horizontal asymptotes and no turning points

(B) a single horizontal asymptote, two vertical asymptotes and no turning points

(C) a single vertical asymptote, two horizontal asymptotes and one turning point

(D) a single horizontal asymptote, two vertical asymptotes and one turning point

(6) The point P on the Argand diagram represents the complex number z.

The point P moves such that $|z|^2+|z+2i|^2=10$.

Which of the following best describes the path traced out by P?

(A) An ellipse

(B) A hyperbola

(C) A circle

(D) A straight line

(7) A committee of 5 people is to be chosen from a group of 6 girls and 4 boys.

How many different committees could be formed that have at least one boy.

(A) $^{10}C_5 - 1$

(B) $^{4}C_1 + {}^{6}C_4$

C) $^{4}C_1 \times {}^{6}C_4$

(D) $^{10}C_5 - 6$

(8) The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

Which of the following is the expression for $\frac{dy}{dx}$?

(A) $\dfrac{y^2 - x}{x^2 + y}$

(B) $\dfrac{y^2 + x}{x^2 - y}$

(C) $\dfrac{x^2 + y}{y^2 - x}$

(D) $\dfrac{x^2 - y}{y^2 + x}$

(9) At time t seconds, $t \geq 0$, the velocity v m/s of a particle moving in a straight line is given by

$v = \sqrt{3}\cos(t) + \sin(t) - 2$. For what value of t does the particle first attain its maximum speed of

4 m/s ?

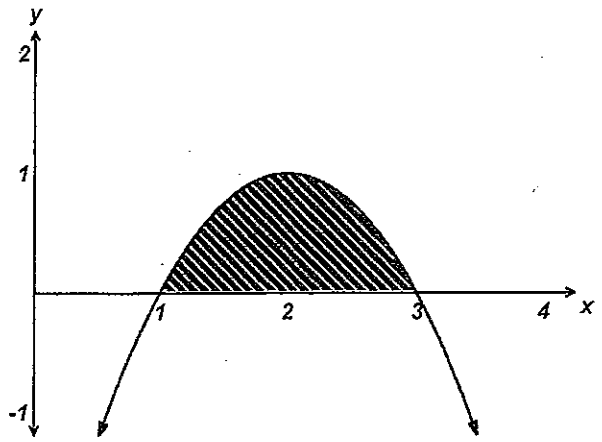
(A) $t = \dfrac{\pi}{6}$

(B) $t = \dfrac{7\pi}{6}$

(C) $x = \dfrac{4\pi}{3}$

(D) The particle never attains a speed of 4 m/s .

(10)



The diagram above shows the graph $y = 4x - x^2 - 3$.

The shaded region bounded by the graph and the x-axis is rotated around the y-axis to form a solid.

Which of the integrals below gives the volume of the solid?

(A) $8\pi \int_0^1 \sqrt{1-y}\, dy$

(B) $8\pi \int_0^1 2 + \sqrt{1-y}\, dy$

(C) $\pi \int_0^1 1 + \sqrt{1-y}\, dy$

(D) $\pi \int_0^1 \sqrt{1-y}\, dy$

Section II

90 marks

Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer on the blank lined paper provided. Begin a new page for each question
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks)

Use a NEW sheet of paper.

(a) Find:

i) $\int e^x (1+e^x)^5 dx.$ [1]

ii) $\int \frac{dt}{\sqrt{7+6t-t^2}}.$ [2]

(b) Let $\alpha = -\sqrt{3}+i$ and $\beta = 1-i$.

i) Express $\bar{\alpha}$ and β in modulus-argument form. [2]

ii) Find $\bar{\alpha}\beta$ in modulus-argument form. [1]

iii) Hence, or otherwise, find the exact value of $\tan\frac{11\pi}{12}$. [2]

Express your answer in its simplest form.

Question 11 continues on the next page

Question 11 (Continued)

(c) For the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

i) Find the eccentricity. [1]

ii) Find the coordinates of the foci S and S'. [1]

iii) Find the equations of the directrices. [1]

iv) Show that the coordinates of any point P on the ellipse can be represented by $(5\cos\theta, 4\sin\theta)$. [2]

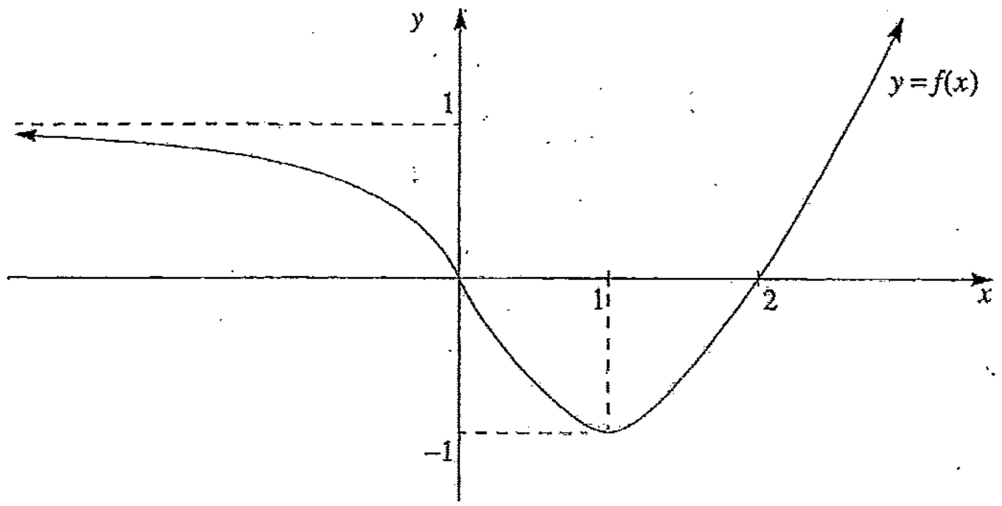
v) Show that $PS+PS'$ is a constant. [2]

End of Question 11

Question 12 (15 Marks)

Use a NEW sheet of paper.

(a)



Given the function $y = f(x)$ in the diagram above, sketch on separate diagrams, showing all intercepts, turning points and asymptotes:

i) $y = f(|x|)$ [1]

ii) $|y| = f(x)$ [2]

iii) $y = f(2x)$ [2]

iv) $y = \dfrac{1}{f(x)}$ [2]

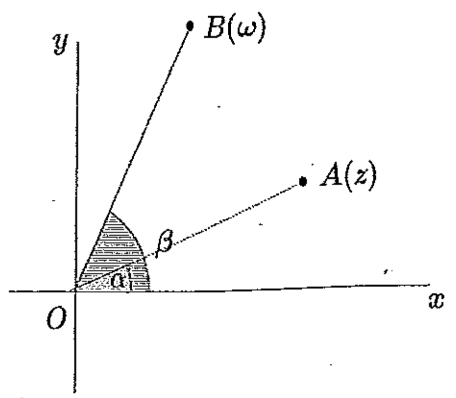
v) $y = e^{f(x)}$ [2]

Question 12 continues on the next page

Question 12 (Continued)

(b) Using the substitution $t = \tan\frac{\theta}{2}$, find $\int \frac{2}{4+3\sin\theta}\,d\theta$. [3]

(c) [3]



The points A and B on the Argand diagram above represent the complex numbers z and ω respectively and $|z| = |\omega| = 2$.

If $\arg z = \alpha$ and $\arg \omega = \beta$ show that $|z+\omega| = 4\cos\left(\frac{\beta-\alpha}{2}\right)$.

End of Question 12

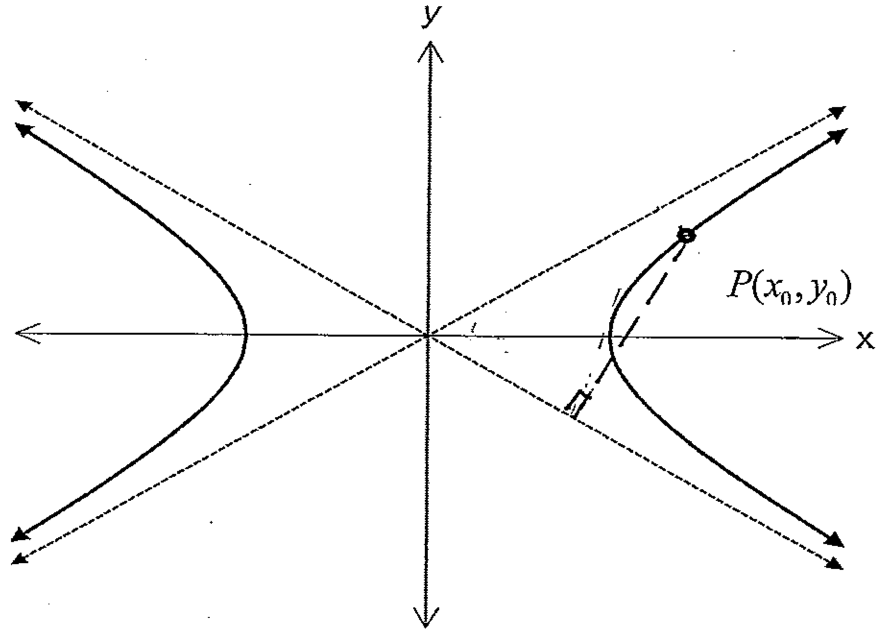
Question 13 (15 Marks)

Use a NEW sheet of paper.

(a) The roots of the equation $x^3 - 9x^2 + 31x + m = 0$ are in an arithmetic sequence. [3]

Find the roots of the equation and hence the value of m.

(b) The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$.



i) Write down the equations of the two asymptotes of the hyperbola. [1]

ii) Show that the acute angle α between the two asymptotes satisfies [2]

$\tan\alpha = \frac{2ab}{a^2 - b^2}$.

iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP \times NP = \frac{a^2b^2}{a^2 + b^2}$. [3]

iv) Hence find the area of ΔPMN in terms of a and b. [2]

Question 13 continues on the next page

(c)

i) Find the rational values of A, B and C given: [2]

$$\frac{y^2+8}{(y-2)(y^2+2y+4)} \equiv \frac{A}{y-2} + \frac{By+C}{y^2+2y+4}.$$

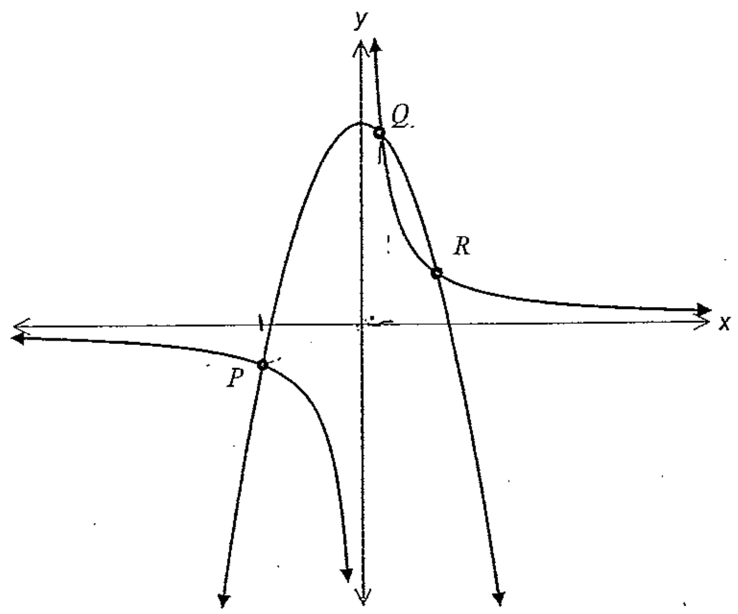
ii) Hence find $\int \frac{y^5-7y^2+8}{y^3-8}\,dy$. [2]

End of Question 13

Question 14 (15 Marks)

Use a NEW sheet of paper.

(a)



The curves $y=\frac{1}{x}$ and $y=k-x^2$, for some real number k, intersect at the points P, Q and R where the x-coordinates are $x=\alpha$, $x=\beta$ and $x=\gamma$ respectively.

i) Show that the monic cubic equation with coefficients in terms of k whose roots are α^2, β^2 and γ^2 is given by $x^3-2kx^2+k^2x-1=0$. [3]

ii) Find the monic cubic equation with coefficients in terms of k whose roots are: $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [1]

iii) Hence find in simplest form $OP^2+OQ^2+OR^2$ in terms of k, where O is the origin. [2]

Question 14 continues on the next page

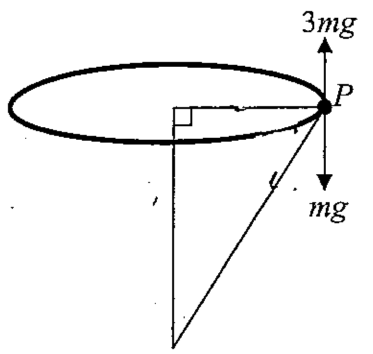
Question 14 (Continued)

(b)

i) Show that a reduction formula for $I_n = \int (\ln x)^{\frac{n}{2}} dx$, where n is a positive integer, is $I_n = x(\ln x)^{\frac{n}{2}} - \frac{n}{2} I_{n-2}$. [2]

ii) Hence, or otherwise, evaluate $\int_1^e (\ln x)^4 dx$. [2]

(c)



A model aircraft P, of mass $m = 8$ kg is attached to the end of a 10 m long inelastic wire, with the other end fixed to the ground.
The model flies in a horizontal circle so that the wire makes an angle of 30° with the ground. The uplift created by the wings of the aircraft is a vertical force $3mg$. (take $g = 10ms^{-1}$)

i) By resolving the forces at P, calculate the tension in the wire. [3]

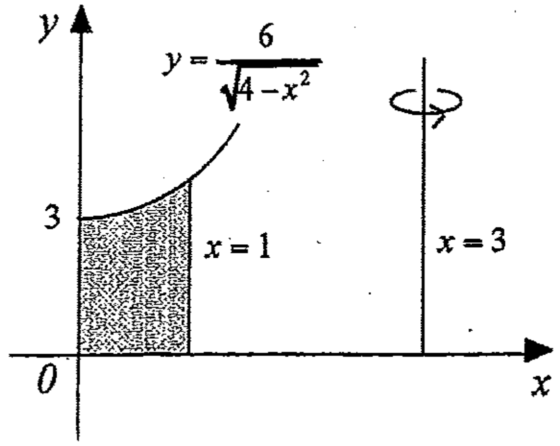
ii) Calculate the angular velocity about the centre of the horizontal circle. [2]

End of Question 14

Question 15 (15 Marks)

Use a NEW sheet of paper.

(a)



A mould for a section of concrete piping is made by rotating the region bounded by the curve $y = \frac{6}{\sqrt{4-x^2}}$ and the x axis between the lines $x = 0$ and $x = 1$ through one complete revolution about the line $x = 3$. All measurements are in metres.

i) By considering strips of width δx parallel to the axis of rotation, show that the volume V m^3 of the concrete used in the piping is given by [2]

$$V = 12\pi \int_0^1 \frac{3-x}{\sqrt{4-x^2}}\, dx.$$

ii) Hence find the volume of the concrete used in the piping, giving your answer correct to the nearest cubic metre. [2]

Question 15 continues on the next page

b)

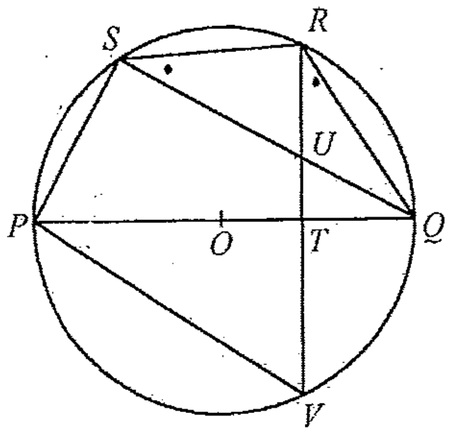
i) Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. [2]

ii) Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots [2]

$x = \cos\frac{\pi}{12}$, $x = -\cos\frac{\pi}{12}$, $x = \cos\frac{5\pi}{12}$ and $x = -\cos\frac{5\pi}{12}$.

iii) By considering this equation as a quadratic equation in x^2, find the exact value of $\cos\frac{5\pi}{12}$. [3]

c) In the diagram, PQ is the diameter of the circle with centre O.



RV intersects SQ and PQ at U and T respectively.
If $\angle QRT = \angle RSQ$, prove that:

i) $\angle TPV = \angle RSQ$. [1]

ii) $\angle RTQ$ is a right angle. [2]

iii) PU is a diameter of the circle passing through P, T, U and S. [1]

End of Question 15

Question 16

Use a NEW sheet of paper. (15 Marks)

(a) Use the letters of the word $STRETCH$ to answer the following.

i) How many two-letter arrangements can be made? [1]

ii) If the letters are selected at random to create a two-letter arrangement, what is the probability that the two-letter arrangement will be "TT"? [1]

iii) The creation of two-letter arrangements from the word $STRETCH$ is repeated. [2]

How many two-letter arrangements need to be created to ensure that the probability of obtaining the arrangement "TT" at least once, exceeds 90%?

(b)

i) Show that $\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$, where r is a positive integer. [1]

ii) Hence show that for $n \geq 1$ [2]

$$\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2}\left\{\sin(2n+1)\theta - \sin\theta\right\}.$$

iii) Hence evaluate $\sum_{r=1}^{100}\cos^2\left(\frac{r\pi}{100}\right)$. [3]

(c)

i) Use the principle of mathematical induction to prove that $3^n > n^3$ for all integers $n \geq 4$. [4]

ii) Hence or otherwise show that $\sqrt[3]{3} > \sqrt[n]{n}$ for all integers $n \geq 4$. [1]

End of paper

Sydney Girls High School
Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics Extension 2

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

A ☒ B ☒ (correct) C ○ D ○

Student Number: ANSWERS

Completely fill the response oval representing the most correct answer.

1.	A ○	B ○	C ○	D ●
2.	A ○	B ○	C ●	D ○
3.	A ○	B ○	C ●	D ○
4.	A ○	B ●	C ○	D ○
5.	A ○	B ○	C ○	D ●
6.	A ○	B ○	C ●	D ○
7.	A ○	B ○	C ○	D ●
8.	A ○	B ○	C ●	D ○
9.	A ○	B ●	C ○	D ○
10.	A ●	B ○	C ○	D ○

2017 THSC EXT2

QUESTION 11

(a) (i) $\int e^x (1+e^x)^5 \, dx$ Let $u = 1+e^x$

$du = e^x dx$

$= \int u^5 du$

$= \frac{u^6}{6} + C$

$= \frac{(1+e^x)^6}{6} + C$

(ii) $-(t^2 - 6t - 7) = -(t-3)^2 + 9 + 7$

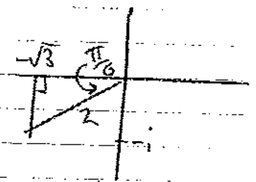
$= 16 - (t-3)^2$

$\int \frac{dt}{\sqrt{7+6t-t^2}} = \int \frac{dt}{\sqrt{16-(t-3)^2}}$

$= \sin^{-1}\left(\frac{t-3}{4}\right) + C$

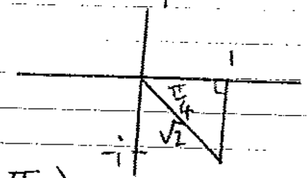
(b) $\alpha = -\sqrt{3} + i$ $\beta = 1 - i$

(i) $\bar{\alpha} = -\sqrt{3} - i$



$\bar{\alpha} = 2\,\text{cis}\left(-\frac{5\pi}{6}\right)$

$\beta = 1 - i$



$\beta = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

(ii) $\bar{\alpha}\beta = 2\sqrt{2}\,\text{cis}\left(-\frac{5\pi}{6} - \frac{\pi}{4}\right)$

$\bar{\alpha}\beta = 2\sqrt{2}\,\text{cis}\left(-\frac{13\pi}{12}\right)$

$\bar{\alpha}\beta = 2\sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right)$

(iii) $\bar{z}\beta = 2\sqrt{2}\cos\left(\frac{11\pi}{12}\right) + i\,2\sqrt{2}\sin\left(\frac{11\pi}{12}\right)$

$\bar{z}\beta = (-\sqrt{3}-i)(1-i)$

$= -\sqrt{3} + i\sqrt{3} - i - 1$

$= (-\sqrt{3}-1) + (\sqrt{3}-1)i$

So $2\sqrt{2}\sin\left(\frac{11\pi}{12}\right) = \sqrt{3}-1$

and $2\sqrt{2}\cos\left(\frac{11\pi}{12}\right) = -\sqrt{3}-1$

Thus $\tan\left(\frac{11\pi}{12}\right) = \frac{\sqrt{3}-1}{-\sqrt{3}-1}$

that is $\tan\left(\frac{11\pi}{12}\right) = \sqrt{3}-2$

(c)(i) $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$\frac{16}{25} = 1-e^2$

$e^2 = \frac{25}{25} - \frac{16}{25}$

$e^2 = \frac{9}{25}$

$e = \frac{3}{5}$

(ii) $S(ae, 0)$ $S'(-ae, 0)$

$S(5\times\frac{3}{5}, 0)$

$S(3, 0)$ $S'(-3, 0)$

(iii) $x = \pm\frac{a}{e}$

$x = \pm\frac{5}{\frac{3}{5}}$

$x = \pm\frac{25}{3}$

(iv) LHS $= \frac{x^2}{25} + \frac{y^2}{16}$

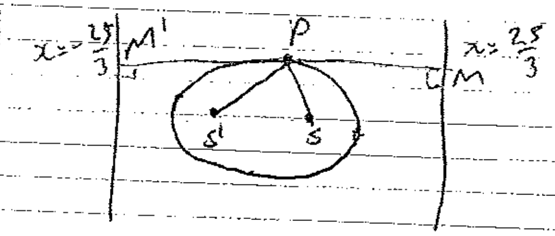
$= \frac{5^2\cos^2\theta}{25} + \frac{4^2\sin^2\theta}{16}$

$= \cos^2\theta + \sin^2\theta$

$= 1$

$=$ RHS.

(v) Let M and M' be the feet of the perpendiculars to the corresponding directries from the point P.



Thus $M\left(\frac{25}{3}, 4\sin\theta\right)$

$M'\left(-\frac{25}{3}, 4\sin\theta\right)$.

By the locus definition of an ellipse $\frac{PS}{PM} = e \qquad \frac{PS'}{PM'} = e$.

So $PS + PS' = ePM + ePM'$

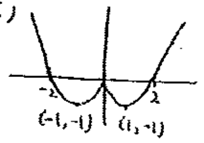
$$= \frac{3}{5}\left(\frac{25}{3} - 4\sin\theta\right) + \frac{3}{5}\left(4\sin\theta + \frac{25}{3}\right)$$

$$= 5 - \frac{12}{5}\sin\theta + \frac{12}{5}\sin\theta + 5$$

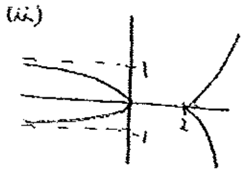
$$= 10.$$

Which is a constant.

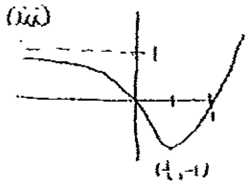
12 (a) (i)



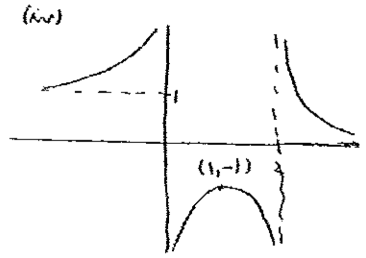
(ii)



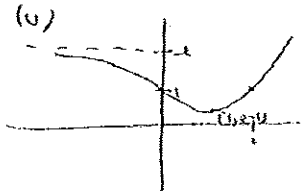
(iii)



(iv)



(v)



(b) $\frac{dt}{d\theta} = \frac{\sec^2\frac{\theta}{2}}{2}$

$= \frac{1+t^2}{2}$

$\frac{2\,dt}{1+t^2} = d\theta$

$$\int \frac{2}{4+3\frac{2t}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$

$$= \int \frac{4\,dt}{4+4t^2+6t}$$

$$= \int \frac{dt}{t^2+\frac{3}{2}t+1}$$

$$= \int \frac{dt}{(t+\frac{3}{4})^2 + \frac{7}{16}}$$

$$= \frac{4}{\sqrt{7}} \tan^{-1}\left\{\frac{4(\tan\frac{\theta}{2}+\frac{3}{4})}{\sqrt{7}}\right\}$$

Many made errors in completing the square

(c) Let C represent $z+w$

OABC is a rhombus ($|z| = |w|$)

$C\hat{O}A = \frac{\beta - \alpha}{2}$ (diagonals of a rhombus bisect the ∠s through which they pass)

$|w|^2 = |z|^2 + |z+w|^2 - 2|z|.|z+w|\cos\left(\frac{\beta-\alpha}{2}\right)$

$2^2 = 2^2 + |z+w|^2 - 2\times 2\times |z+w|\cos\left(\frac{\beta-\alpha}{2}\right)$

$4 = 4 + |z+w|^2 - 4|z+w|\cos\left(\frac{\beta-\alpha}{2}\right)$

$|z+w|\left(|z+w| - 4\cos\left(\frac{\beta-\alpha}{2}\right)\right)$

$\therefore |z+w| = 4\cos\left(\frac{\beta-\alpha}{2}\right)$

Many students lost marks for insufficient reasons.

Question 13

Use a NEW sheet of paper. (15 Marks)

(a) The roots of the equation $x^3-9x^2+31x+m=0$ are in an arithmetic sequence. [3]

Find the roots of the equation and hence the value of m.

Let the roots be: $\alpha-d,\ \alpha,\ \alpha+d$.

Sum (1 at a time): $\alpha-d+\alpha+\alpha+d = \frac{-b}{a}$

$3\alpha = \frac{-(-9)}{1}$

$\therefore \alpha = 3.$ ✓

Sum (2 at a time): $\alpha^2-\alpha d+\alpha^2-d^2+\alpha^2+\alpha d = \frac{c}{a}$

$3\alpha^2-d^2 = \frac{31}{1}$

$3(3)^2-d^2 = 31$

$-d^2 = 4$

$d = \pm 2i$ ✓

$\therefore$ Roots are: $3-2i,\ 3,\ 3+2i$

Product: $(\alpha^2-d^2)\alpha = \frac{-d}{a}$

$[3^2-(-4)]\times 3 = \frac{-m}{1}$

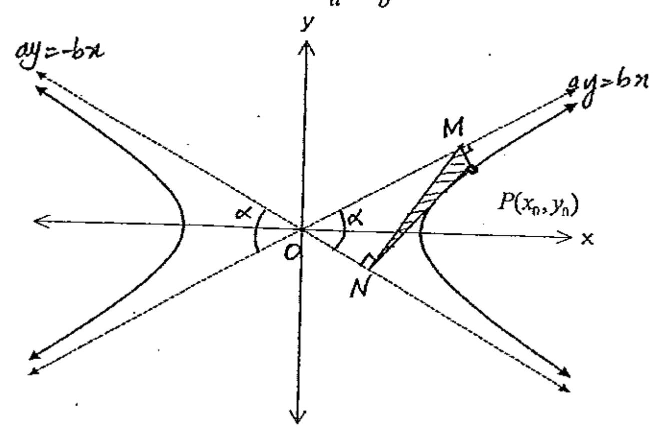
$(9+4)\times 3 = -m$

$-39 = m$

$\therefore m = -39.$ ✓

★ Students did well in this part.

(b) The point $P(x_0,y_0)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b>0$.



i) Write down the equations of the two asymptotes of the hyperbola. [1]

ii) Show that the acute angle α between the two asymptotes satisfies [2]

$\tan\alpha = \frac{2ab}{a^2-b^2}$.

i) asymptotes: $y = \frac{bx}{a}$, $y = -\frac{bx}{a}$ ✓

ii) $\tan\alpha = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{(b/a)-(-b/a)}{1+(b/a)(-b/a)}\right|$ ✓

$= \left|\frac{2b/a}{1-b^2/a^2}\right|$

$= \left|\frac{2b}{a}\times\frac{a^2}{a^2-b^2}\right|$ ✓ $a>b>0$.

$\therefore \tan\alpha = \frac{2ab}{a^2-b^2}$, as required.

★ Absolute value signs should be included and then a reference to $a>b>0$

iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP \times NP = \frac{a^2b^2}{a^2+b^2}$. [3]

iv) Hence find the area of ΔPMN in terms of a and b. [2]

iii) MP = distance from $P(x_0, y_0)$ to $bx - ay = 0$.
NP = distance from $P(x_0, y_0)$ to $bx + ay = 0$.

$\therefore MP = \frac{|bx_0 - ay_0|}{\sqrt{a^2+b^2}}$ $\quad NP = \frac{|bx_0 + ay_0|}{\sqrt{a^2+b^2}}$ ✓

$\therefore MP \times NP = \left|\frac{bx_0 - ay_0}{\sqrt{a^2+b^2}}\right| \times \left|\frac{bx_0 + ay_0}{\sqrt{a^2+b^2}}\right|$

$= \left|\frac{b^2x_0^2 - a^2y_0^2}{a^2+b^2}\right|$ ✓

*(The simplest approach was to use the perpendicular distance formula)

Now $P(x_0, y_0)$ lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

*Reference had to made at this step

$\therefore \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$ ✓

$\therefore b^2x_0^2 - a^2y_0^2 = a^2b^2$

$\therefore MP \times NP = \left|\frac{a^2b^2}{a^2+b^2}\right| = \frac{a^2b^2}{a^2+b^2}$, as required

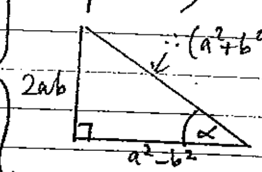
iv) Area ΔPMN

$= \frac{1}{2} MP \cdot NP \sin \angle MPN$

*(Students who assumed that $\angle MPN = 90°$ did not receive any marks)

$= \frac{1}{2} \cdot MP \cdot NP \sin(180° - \alpha)$ ← (MPNO is a cyclic quad)

$= \frac{1}{2} \cdot \frac{a^2b^2}{a^2+b^2} \cdot \sin\alpha$ ← (From part ii)



$= \frac{1}{2} \cdot \frac{a^2b^2}{a^2+b^2} \cdot \frac{2ab}{a^2+b^2}$

$= \frac{a^3b^3}{(a^2+b^2)^2}$ ✓✓

Question 13 (Continued)

(c)

i) Find the rational values of A, B and C given: [2]

$$\frac{y^2+8}{(y-2)(y^2+2y+4)} \equiv \frac{A}{y-2} + \frac{By+C}{y^2+2y+4}.$$

ii) Hence find $\int \frac{y^5 - 7y^2 + 8}{y^3 - 8}\, dy$. [2]

i) $y^2 + 8 \equiv A(y^2+2y+4) + (By+C)(y-2)$
$\equiv Ay^2 + 2Ay + 4A + By^2 - 2By + Cy - 2C$
$\equiv (A+B)y^2 + (2A - 2B + C)y + (4A - 2C)$

$\therefore A + B = 1$, $2A - 2B + C = 0$, $4A - 2C = 8$ $\Rightarrow$ $A = 1$, $B = 0$, $C = -2$ ✓✓

ii) $\frac{y^5 - 7y^2 + 8}{y^3 - 8} = \frac{y^5 - 8y^2 + y^2 + 8}{(y-2)(y^2+2y+4)}$

$= \frac{y^2(y^3-8) + (y^2+8)}{(y-2)(y^2+2y+4)}$

$= y^2 + \frac{y^2+8}{(y-2)(y^2+2y+4)}$ ✓

$\therefore \int \frac{y^5 - 7y^2 + 8}{y^3 - 8}\, dy = \int \left[y^2 + \frac{1}{y-2} - \frac{2}{y^2+2y+4}\right] dy$

*(Some students did not read the question carefully, and integrated the answer directly from part i)

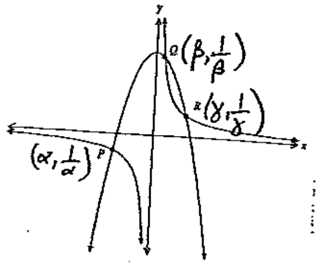
$= \int \left[y^2 + \frac{1}{y-2} - \frac{2}{(\sqrt{3})^2 + (y+1)^2}\right] dy$

$= \frac{y^3}{3} + \ln|y-2| - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{y+1}{\sqrt{3}}\right) + C$ ✓

Question 14

Use a NEW sheet of paper.

(a)



The curves $y=\frac{1}{x}$ and $y=k-x^2$, for some real number k, intersect at the points P, Q and R where the x-coordinates are $x=\alpha$, $x=\beta$ and $x=\gamma$ respectively.

[3] i) Show that the monic cubic equation with coefficients in terms of k whose roots are α^2, β^2 and γ^2 is given by $x^3-2kx^2+k^2x-1=0$.

[1] ii) Find the monic cubic equation with coefficients in terms of k whose roots are: $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$.

[2] iii) Hence find in simplest form $OP^2+OQ^2+OR^2$ in terms of k, where O is the origin.

i) α, β, γ are the roots of $k-x^2=\frac{1}{x}$

$\Rightarrow x^3-kx+1=0$. ✓

$\alpha^2, \beta^2, \gamma^2$ are the roots of the equation:

$(\sqrt{x})^3-k(\sqrt{x})+1=0$. (replace x by $\sqrt{x}$)

$x\sqrt{x}-k\sqrt{x}+1=0$.

$(x-k)\sqrt{x}=-1$

$(x-k)^2(\sqrt{x})^2=(-1)^2$ ✓ ✱ (This step had to be shown)

$(x^2-2kx+k^2)x=1$ ✓

$\therefore x^3-2kx^2+k^2x-1=0$, as required.

ii) Replace x by $\frac{1}{x}$:

$\left(\frac{1}{x}\right)^3-2k\left(\frac{1}{x}\right)^2+k^2\left(\frac{1}{x}\right)-1=0$

$\frac{1}{x^3}-\frac{2k}{x^2}+\frac{k^2}{x}-1=0$.

$\Rightarrow 1-2kx+k^2x^2-x^3=0$

$\therefore x^3-k^2x^2+2kx-1=0$. ✓

iii) $OP^2+OQ^2+OR^2$

$=\left(\alpha^2+\frac{1}{\alpha^2}\right)+\left(\beta^2+\frac{1}{\beta^2}\right)+\left(\gamma^2+\frac{1}{\gamma^2}\right)$

$=\left(\alpha^2+\beta^2+\gamma^2\right)+\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)$. ✓

From i) $\alpha^2+\beta^2+\gamma^2=2k$

From ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=k^2$ ✓

$\therefore OP^2+OQ^2+OR^2=k^2+2k$.

(b)

i) Show that a reduction formula for $I_n=\int(\ln x)^{\frac{n}{2}}dx$, where n is a positive integer, is $I_n=x(\ln x)^{\frac{n}{2}}-\frac{n}{2}I_{n-2}$. [2]

$I_n=\int(\ln x)^{\frac{n}{2}}dx$

$=x(\ln x)^{\frac{n}{2}}-\int x.\frac{n}{2}(\ln x)^{\frac{n}{2}-1}.\frac{1}{x}dx$

$=x(\ln x)^{\frac{n}{2}}-\frac{n}{2}\int(\ln x)^{\frac{n}{2}-1}dx$

$u=(\ln x)^{n/2}$

$du=\frac{n}{2}(\ln x)^{\frac{n}{2}-1}.\frac{1}{x}dx$

$dv=dx$

$v=x$ ✓

$\therefore I_n=x(\ln x)^{\frac{n}{2}}-\frac{n}{2}\int(\ln x)^{\frac{n-2}{2}}dx$.

$=x(\ln x)^{\frac{n}{2}}-\frac{n}{2}I_{n-2}$. ✓

✱ (This was achieved by most students.)

ii) Hence, or otherwise, evaluate $\int_1^e (\ln x)^4 dx$. [2]

$\frac{n}{2} = 4$

$n = 8$

$\therefore I_8 = \left[x(\ln x)^4\right]_1^e - 4I_6$

$= e - 4\left(\left[x(\ln x)^3\right]_1^e - 3I_4\right)$

$= e - 4e + 12I_4$

$= -3e + 12\left(\left[x(\ln x)^2\right]_1^e - 2I_2\right)$

$= -3e + 12e - 24I_2$

$= 9e - 24\left(\left[x(\ln x)\right]_1^e - I_0\right)$ ✓

$= 9e - 24e + 24I_0$

$= 9e - 24e + 24\left(\left[x(\ln x)^0\right]_1^e\right)$

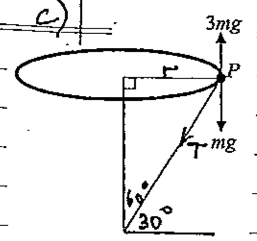
$= 9e - \cancel{24e} + \cancel{24e} - 24$

$= 9e - 24$ ✓

☆ Most common error was finding $I_4 = e-2$ instead of I_8. Loss of one mark.

c) A model aircraft P, of mass $m = 8$kg is attached to the end of a 10 m long inelastic wire, with the other end fixed to the ground.

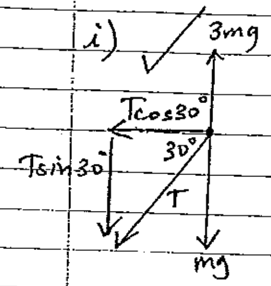
The model flies in a horizontal circle so that the wire makes an angle of 30° with the ground. The uplift created by the wings of the aircraft is a vertical force $3mg$. (take $g = 10ms^{-1}$)



[3] i) By resolving the forces at P, calculate the tension in the wire.

[2] ii) Calculate the angular velocity about the centre of the horizontal circle.

i) ✓



vertically: $T\sin 30° + mg = 3mg$ ✓

$T\sin 30° = 2mg$ ①

horizontally: $T\cos 30° = mr\omega^2$ ②

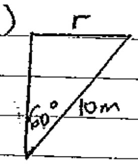
Tension: $T\sin 30° = 2 \times 8 \times 10$

$\frac{1}{2}T = 160$ ✓

$\therefore T = 320$ Newtons.

☆ Most common error was labelling the angle made with the ground.

ii)



$\frac{r}{10} = \sin 60°$

$r = 10\sin 60°$

$r = \frac{10\sqrt{3}}{2}$ ✓

$r = 5\sqrt{3}$ cm.

From ②:

$T\cos 30° = mr\omega^2$

$320 \times \frac{\sqrt{3}}{2} = 8 \times 5\sqrt{3} \times \omega^2$

$160 = 40\omega^2$

$\omega^2 = 4$ ✓

$\therefore \omega = 2$ rad/sec $(\omega > 0)$.

15 (a)(i) $V_{shell} = \pi(R^2 - r^2)h$

$= \pi\{(3-x)^2 - (3-x-\delta x)^2\}y$

$= 2\pi(3-x)y\,\delta x$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi(3-x)y\,\delta x$

$= 2\pi \int_0^1 (3-x)y\,dx$

$= 2\pi \int_0^1 (3-x) \times \frac{6}{\sqrt{4-x^2}}\,dx$

$= 12\pi \int_0^1 \frac{3-x}{\sqrt{4-x^2}}\,dx$ Q.E.D.

ii) $V = 12\pi\left(\int_0^1 \left(\frac{3}{\sqrt{4-x^2}} - \frac{x}{\sqrt{4-x^2}}\right)dx\right)$

$= 12\pi\left[3\sin^{-1}\frac{x}{2}\right]_0^1 - 12\pi\int_0^1 \frac{x}{\sqrt{4-u}} \times \frac{du}{2x}$ $u = x^2$

$= 12\pi \times 3 \times \frac{\pi}{6} - 12\pi\left[-(4-u)^{\frac{1}{2}}\right]_0^1$ $\frac{du}{dx} = 2x$

$= 6\pi^2 + 12\pi \times \sqrt{3} - 24\pi$ $\frac{du}{2x} = dx$

$= 6\pi^2 + 12\sqrt{3}\pi - 24\pi$ ~~A calculator answer.~~

(b)(i) $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$ Q.E.D.

(ii) let $x = \cos\theta$

$16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$2(8\cos^4\theta - 8\cos^2\theta + 1) - 1 = 0$

$2\cos 4\theta - 1 = 0$

$\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

$x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$

$x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, -\cos\frac{5\pi}{12}, -\cos\frac{\pi}{12}$

(iii) $x = \frac{16 \pm \sqrt{16^2 - 4 \times 16 \times 1}}{2 \times 16}$

$= \frac{16 \pm \sqrt{192}}{32}$

$= \frac{16 \pm 8\sqrt{3}}{32}$ $\therefore \cos\frac{5\pi}{12} = \frac{2-\sqrt{3}}{4}$

$= \frac{2 \pm \sqrt{3}}{4}$ [because $\cos\frac{5\pi}{12} < \cos\frac{\pi}{12}$]

(c)(i) $T\hat{P}V = G\hat{R}T$ (∠ in same segment)

$G\hat{R}T = R\hat{S}Q$ (given) → Many students forgot to mention this.

$\therefore T\hat{P}V = R\hat{S}Q$ Q.E.D.

(ii) In ΔPGV and ΔVGT

$P\hat{G}V$ is common

$T\hat{V}Q = R\hat{S}Q$ (∠s in same segment)

$= T\hat{P}V$ (from i)

$\therefore \Delta PQV \,|||\, \Delta VGT$ (equiangular)

$\therefore G\hat{T}V = P\hat{V}Q$ (corresponding ∠ in similar Δ)

$P\hat{V}G = 90°$ (∠ in a semi circle)

$\therefore G\hat{T}V = 90°$

$R\hat{T}G + G\hat{T}V = 180°$ (straight ∠)

$\therefore R\hat{T}G = 90°$ (Q.E.D.)

(iii) $P\hat{S}G = 90°$ (∠ in a semi circle)

$\therefore$ PU is a diameter of a circle through P, S and V.

$\therefore R\hat{T}G = P\hat{S}G$

$\therefore$ PTUS is a cyclic quadrilateral (exterior ∠ equals interior opposite ∠)

$\therefore$ PU is a diameter of the circle passing through P, T, U and S.

Q.E.D.

Both elements are required

2017 THSC Ext 2

Question 16

(a)(i) Method 1

Case 0, No Ts: $5 \times 4 = 20$

Case 1, One T: $5 \times 2! = 10$

Case 2, TT: 1

Total = 31

Method 2

Distinct letters: ${}^6C_2 \times 2! = {}^6P_2 = 30$

Same letter (TT): 1

Total = 31

> Most students had $\frac{{}^7P_2}{2!} + 1 = 21$ but you only need to divide by 2! if you have the two TTs.

(ii) $P(TT) = \frac{1}{31}$

(iii) $P(TT) = 1 - P(not\ TT)$

$$1 - \underbrace{\frac{30}{31} \times \frac{30}{31} \times \cdots \times \frac{30}{31}}_{n} > 0.9$$

$$0.1 > \left(\frac{30}{31}\right)^n$$

> This part was answered poorly by most students.

$$\log\left(\frac{30}{31}\right)^n < \log(0.1)$$

$$n > \frac{\log 0.1}{\log\left(\frac{30}{31}\right)} \quad \text{Since } \log\left(\tfrac{30}{31}\right) < 0$$

$$n > 70.22$$

That is $n = 71$

You would need to create 71 arrangements to have over 90% chance of getting at least one TT.

(b)(i) $LHS = \sin(2r\theta + \theta) - \sin(2r\theta - \theta)$

$= \sin(2r\theta)\cos\theta + \sin\theta\cos(2r\theta) - \sin(2r\theta)\cos\theta + \sin\theta\cos(2r\theta)$

$= 2\sin\theta\cos(2r\theta)$

$= RHS$

(ii) $2\sin\theta\sum_{r=1}^{n}\cos 2r\theta$

$= 2\sin\theta\cos 2\theta + 2\sin\theta\cos 4\theta + 2\sin\theta\cos 6\theta + \cdots$

$\quad + 2\sin\theta\cos 2(n-1)\theta + 2\sin\theta\cos 2n\theta$

$= \sin 3\theta - \sin\theta + \sin 5\theta - \sin 3\theta + \sin 7\theta - \sin 5\theta + \cdots$

$\quad + \sin(2n-1)\theta - \sin(2n-3)\theta + \sin(2n+1)\theta - \sin(2n-1)\theta$

$= -\sin\theta + \sin(2n+1)\theta$

So $2\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \sin(2n+1)\theta - \sin\theta$

That is $\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2}\{\sin(2n+1)\theta - \sin\theta\}$

(iii) $\sum_{r=1}^{100}\cos^2\left(\frac{r\pi}{100}\right)$

> Some students could not see the connection between parts (ii) and (iii).

$= \sum_{r=1}^{100}\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2r\pi}{100}\right)\right)$

$= \frac{1}{2}\sum_{r=1}^{100} 1 + \frac{1}{2}\sum_{r=1}^{100}\cos\left(2r\frac{\pi}{100}\right)$

$= \frac{1}{2} \times 100 + \frac{1}{2}\left\{\frac{1}{2\sin\left(\frac{\pi}{100}\right)}\left[\sin\left(201\frac{\pi}{100}\right) - \sin\left(\frac{\pi}{100}\right)\right]\right\}$ from part (ii)

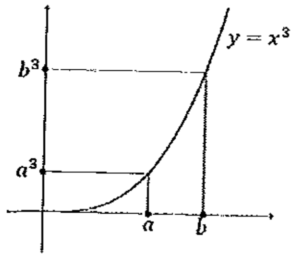
$= 50 + \frac{1}{4\sin\left(\frac{\pi}{100}\right)}\left[\sin\left(2\pi + \frac{\pi}{100}\right) - \sin\left(\frac{\pi}{100}\right)\right]$

$= 50 + \frac{1}{4\sin\left(\frac{\pi}{100}\right)}\left[\sin\left(\frac{\pi}{100}\right) - \sin\left(\frac{\pi}{100}\right)\right]$

$= 50$

(c)(i) Method 1

Cubing positive numbers preserves order, since $y = x^3$ is monotonically increasing for all $x > 0$.



That is, if $0 < a < b$ then $a^3 < b^3$.

Call this result (*)

Given that $k \geq 4$

$\Rightarrow k > 2.27$

$\Rightarrow k > \dfrac{1}{\sqrt[3]{3} - 1}$

$\Rightarrow \sqrt[3]{3}k - k > 1$

> Students tended to do these side calculations as part of the induction structure and got themselves very confused.

$\Rightarrow \sqrt[3]{3}k > k + 1$ use result (*)

$\Rightarrow 3k^3 > (k+1)^3$ Call this result (**)

Prove $3^n > n^3$ for $n = 4$.

$RHS = 4^3 = 64$

$LHS = 3^4 = 81 > RHS$

Assume for $n = k$.

$3^k > k^3$

Prove for $n = k + 1$.

> Required to prove
> $3^{k+1} > (k+1)^3$

$3^{k+1} = 3 \times 3^k$

$> 3k^3$ by assumption

$> (k+1)^3$ by result (**)

Therefore by the principle of mathematical induction $3^n > n^3$ for all $n \geq 4$.

(c)(i) Method 2

> Most students that gained full marks for this question used this method.

Let $f(k) = 2k^3 - 3k^2 - 3k - 1$

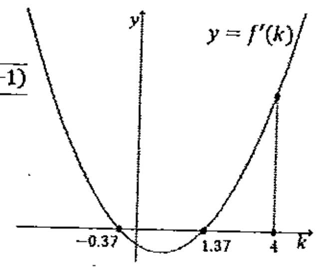
then $f'(k) = 6k^2 - 6k - 3$

$y = 3(2k^2 - 2k - 1)$

$k = \dfrac{2 \pm \sqrt{4 - 4 \times 2 \times (-1)}}{4}$

$k = \dfrac{1 \pm \sqrt{3}}{2}$

$k \approx -0.37, 1.37$



So for $k \geq 4$, $f'(k) > 0$.

That is $f(k)$ is an increasing function for $k \geq 4$.

$f(4) = 2(4)^3 - 3(4)^2 - 3(4) - 1$

$f(4) = 67$

At $k = 4$, $f(k)$ is positive and $f(k)$ is an increasing function for $k \geq 4$.

So $f(k) > 0$ for $k \geq 4$. Call this result (†)

Prove $3^n > n^3$ for $n = 4$.

$RHS = 4^3 = 64$

$LHS = 3^4 = 81 > RHS$

Assume for $n = k$.

$3^k > k^3$

Prove for $n = k + 1$.

> Required to prove
> $3^{k+1} > (k+1)^3$

$3^{k+1} - (k+1)^3 = 3 \times 3^k - (k^3 + 3k^2 + 3k + 1)$

$> 3k^3 - k^3 - 3k^2 - 3k - 1$ by assumption

$= 2k^3 - 3k^2 - 3k - 1$

$= f(k)$

> 0 for $k \geq 4$ by result (†)

So $3^{k+1} - (k+1)^3 > 0$

That is $3^{k+1} > (k+1)^3$

Therefore by the principle of mathematical induction $3^n > n^3$ for all $n \geq 4$.

(c)(i) Method 3

Prove $3^n > n^3$ for $n = 4$.

> Nobody used this method correctly, you must break the algebra down to obviously true statements.

$RHS = 4^3$
$= 64$

$LHS = 3^4$
$= 81$
$> RHS$

Assume for $n = k$.

$3^k > k^3$

That is $3^k - k^3 > 0$

Prove for $n = k + 1$.

> Required to prove
> $3^{k+1} > (k+1)^3$

$$3^{k+1} - (k+1)^3 = 3 \times 3^k - (k^3 + 3k^2 + 3k + 1)$$
$$= 3(3^k - k^3) + 2k^3 - 3k^2 - 3k - 1$$
$$= 3(3^k - k^3) + (k^3 - 3k^2 + 3k - 1) + (k^3 - 6k)$$
$$= 3(3^k - k^3) + (k-1)^3 + k(k^2 - 6)$$

Now $3(3^k - k^3)$ is positive by assumption,
and $(k-1)^3$ is positive since $k \geq 4$,
and $k(k^2 - 6)$ is positive since $k \geq 4$,

So $3^{k+1} - (k+1)^3 > 0$
That is $3^{k+1} > (k+1)^3$

Therefore by the principle of mathematical induction $3^n > n^3$ for all $n \geq 4$.

(ii) Method 1

$$3^n > n^3$$
$$(3^n)^{\frac{1}{3n}} > (n^3)^{\frac{1}{3n}}$$
$$3^{\frac{1}{3}} > n^{\frac{1}{n}}$$
$$\sqrt[3]{3} > \sqrt[n]{n}$$

Method 2

$$3^n > n^3$$
$$\log(3^n) > \log(n^3)$$
$$n\log(3) > 3\log(n)$$
$$\frac{1}{3}\log(3) > \frac{1}{n}\log(n)$$
$$\log(\sqrt[3]{3}) > \log(\sqrt[n]{n})$$
$$\sqrt[3]{3} > \sqrt[n]{n}$$