2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension I

General Instructions

- Reading time – 5 minutes
- Working time –2 hours
- Write using black or blue pen
  Black pen is preferred
- Board- approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-14

Total marks – 70

- Section I Pages 2-3
  10 Marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Section II Pages 4-6
  60 Marks
- Attempt Questions 11-14
- Allow about 1 hours 45 minutes for this section

Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of past HSC papers. The questions have been adapted from various sources, in an attempt to provide students with exposure to a broad range of questions.

However, there is no guarantee whatsoever that the HSC examination will have similar content, style or format. This paper is intended only as a trial for the HSC examination or as revision leading to the examination.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

Question 1

The polynomial $P(x)=3x^3-5x+a$ is divisible by $x+1$. The value of a is

(A) -1 (B) 1 (C) 2 (D) -2

Question 2

The size of the acute angle between the lines $y=x-4$ and $y=3x+4$

(A) $26^0 33'$ (B) $26^0 34'$ (C) $63^0 26'$ (D) $63^0 27'$

Question 3

Which statement is correct for $f(x)=2\cos^{-1}\left(\frac{x}{5}\right)$

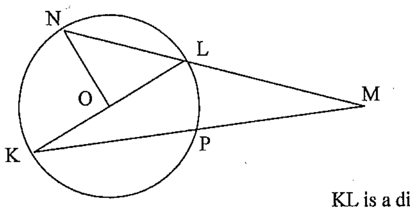
(A) Domain is $\frac{-1}{5}\le x\le\frac{1}{5}$ and the range is $0\le f(x)\le 2\pi$

(B) Domain is $-5\le x\le 5$ and the range is $0\le f(x)\le \pi$

(C) Domain is $-1\le x\le 1$ and the range is $0\le f(x)\le 2\pi$

(D) Domain is $-5\le x\le 5$ and the range is $0\le f(x)\le 2\pi$

Question 4



NOT TO SCALE

KL is a diameter of the circle centre O.
KL = LM

If $\angle NOK=120^0$, calculate the size of $\angle LMP$.

(A) 30^0 (B) 60^0 (C) 120^0 (D) 150^0

Question 5

A particle moves in SHM on a horizontal line and its acceleration is $\frac{d^2x}{dt^2}=25-5x$, where x is the displacement after t seconds. Find the centre of its motion.

(A) $x=0$

(B) $x=5$

(C) $x=10$

(D) $x=25$

Question 6

The velocity of a particle moving in a straight line at position x is given by: $v=2e^{-2x}$

Initially the particle is at the origin. What is the acceleration of the particle at position x?

(A) $a=-4e^{-2x}$ (B) $a=-16e^{-2x}$ (C) $a=-4e^{-4x}$ (D) $a=-8e^{-4x}$

Question 7

In how many ways can a soccer team of eleven be chosen from 15 players?

(A) 1365 (B) 165 (C) 5.4×10^{10} (D) 15

Question 8

$\frac{d}{dx}\left(\sin^{-1}2x\right)=$

(A) $\frac{1}{\sqrt{1+4x^2}}$ (B) $\frac{1}{\sqrt{1-4x^2}}$ (C) $\frac{2}{\sqrt{1-4x^2}}$ (D) $\frac{1}{\sqrt{4-x^2}}$

Question 9

If $\sin\theta=\frac{4}{5}$ and $\frac{\pi}{2}\le\theta\le\pi$ then $\sin 2\theta$ is

(A) $-\frac{25}{24}$ (B) $\frac{8}{5}$ (C) $-\frac{24}{25}$ (D) $\frac{24}{25}$

Question 10

Twelve people sit at a round table. The number of arrangements possible if two particular persons are seated together is?

(A) 24 (B) 3 628 800 (C) 725 760 (D) 7 257 600

Section II

60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) Use a separate page/booklet **Marks**

(a) Solve for x $\frac{1}{x-2} > 1$ **3**

(b) $\int \frac{dx}{4+9x^2}$ **3**

(c) What is the exact value of $\sin\frac{\pi}{12}$? **3**

(d) The points $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ lie on the parabola $x^2 = 8y$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - 2p^2$ **2**

(ii) The tangent at P and the line passing through Q parallel to the y axis intersect at T.

Show that the coordinates of T are $(4q, 4pq - 2p^2)$ **2**

(iii) Find the coordinates of M, the midpoint of PT. **1**

(iv) Find the Cartesian equation of the locus of M when $pq = -1$ **1**

Question 12 (15 marks) Use a separate page/booklet **Marks**

(a) Consider the function $f(x) = (x-1)^2 + 3$

(i) What is the greatest positive domain for $f(x)$ such that the inverse of $f(x)$ is also a function? **1**

(ii) What is the domain of the inverse function $f^{-1}(x)$? **1**

(iii) Find the equation of the inverse function $f^{-1}(x)$? **2**

(b) Using the principle of Mathematical Induction, show that $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n greater than or equal to 1. **4**

(c) Find the term independent of x in $\left(4x^8 - \frac{2}{x^3}\right)^{11}$ **4**

(d) Evaluate: $\int \cos^2 4x\,dx$ **3**

Question 13 (15 marks) Use a separate page/booklet **Marks**

(a) (i) Show that $\frac{1}{9\cos^2 x + 25\sin^2 x} = \frac{\sec^2 x}{9 + 25\tan^2 x}$ **2**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x}$, using the substitution $u = \tan x$ **3**

(b) Newton's law of cooling states that for an object placed in surroundings at constant temperature, the rate of change of the temperature of the object is proportional to the difference between the temperature of the object and the surroundings i.e.

$$\frac{dT}{dt} = k(T - A)$$

where A is the temperature of the surroundings, T is the temperature of the object at any time.

(i) Show that $T = A + Ce^{kt}$ satisfies Newton's law of cooling. Note: C and k are constants. **1**

(ii) A solid drops in temperature from 60^0C to 30^0C in 25 minutes. The room in which the solid has been placed has a constant temperature of 12^0C.

(α) Find the values of C and k. **2**

(β) How long will it take the solid to reach a temperature of 15^0C? **2**

(c) (i) Show that the function defined by $f(x) = 2x^3 + 2x - 5$ has a zero between $x = 0$ and $x = 2$ **1**

(ii) Explain why this is the only zero. **1**

(iii) Taking $x = 1$ as a first approximation to the solution of $2x^3 + 2x - 5 = 0$, use Newton's method once to find a closer approximation. **1**

(d) An arrow is fired horizontally at 50 m/s from the top of a 180m building, ($g = 10\text{m/s}^2$). You may assume that the horizontal and vertical components of the arrow are given by

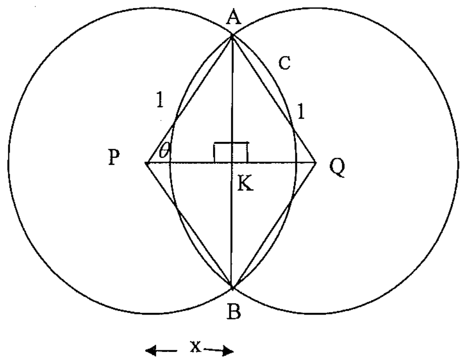
$x = 50t$ and $\frac{dx}{dt} = 50$

$y = -5t^2 + 180$ and $\frac{dy}{dt} = -10t$

Find the angle the arrow hits the ground. **2**

Question 14 (15 marks) Use a separate page/booklet **Marks**

(a) Two circles of unit radius intersect at A and B. $\angle AKP = \angle AKQ = 90^0$ and $\angle APK = \theta$. If their centres P and Q are $2x$ units apart, show that the area (A) common to the two circles is given by $A = 2\left(\cos^{-1} x - x\sqrt{1-x^2}\right)$



4

(b) A committee of three is to be chosen at random from 4 women and n men $(n \geq 2)$. Find the number of possible committees containing exactly

(i) one woman **2**

(ii) two women **1**

(iii) show that the probability p of the committee containing either one or two women is $p = \frac{12n}{(n+4)(n+3)}$ **2**

(iv) Show that the turning points of the curve $y = \frac{x}{(x+4)(x+3)}$ occur when $x = \pm 2\sqrt{3}$ **2**

(v) Sketch the curve, for $x \geq 0$. **2**

(vi) Deduce that there are two values of n which maximize p and state the maximum

2013 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension I

- **Solutions including marking scale**
- **Mapping grid**

We have endeavored to ensure that the solutions are free of errors and follow the spirit of the syllabus in the methods used to solve the problems. However, individual teachers may opt for alternate solutions and/or may choose a different marking system.

1.D 2. B 3. D 4 A 5.B 6.D 7A 8.C 9.C 10.D

1. $P(-1)=0=3(-1)^3-5(-1)+a \therefore a=-2$

2. $\tan\theta=\left|\frac{1-3}{1+1\times3}\right|=\frac{1}{2}\therefore\theta=26.565...^0=26^0 34'$

3.Domain $-5\le x\le 5$ range $0\le f(x)\le 2\pi$

4.
$\angle NLK=60^0$ (when standing on the same arc, angle at centre is double the angle at the cir.)
$\angle LKM=\angle LMK$ (base $\angle$'s of iss triangle), $\angle LMK=30^0$ (Ext.$\angle$ of a triangle thm)

5. $\frac{d^2x}{dt^2}=25-5x=5(5-x)\therefore$ centre of motion is $x=5$

6. $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{d}{dx}\left(\frac{1}{2}\times 4e^{-4x}\right)=-8e^{-4x}$

7. ${}^{15}C_{11}=1365$

8. $y=\sin^{-1}u$ where $u=2x\therefore\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{\sqrt{1-u^2}}\times 2=\frac{2}{\sqrt{1-4x^2}}$

9. $\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{4}{5}\times\frac{-3}{5}=-\frac{24}{25}$

10. $2\times 10!=7257600$

Marking Guidelines: Mathematics Extension I – Solutions

Section II

Question 11

Criteria
(a) One for multiplying by $(x-2)^2$, one for forming quadratic inequality, one for simplification. (b) One for line (A), one for line (B), one for simplification. (c) One for expansion, one for exact values, one for simplification. (d) (i) One for *dy/dx*, one for final answer. (ii) One for *x* coordinate, one for *y* coordinate. (iii) One for the correct answer. (iv) One for the correct answer.

Answers:

(a)

$\frac{1}{x-2}>1$

$\frac{1}{x-2}\times(x-2)^2>(x-2)^2$

$x-2>(x-2)^2$

$0>(x-2)^2-(x-2)$

$0>(x-2)[x-3]$

$2<x<3$

(b)

$\int\frac{dx}{4+9x^2}$

$=\int\frac{dx}{9\left(\frac{4}{9}+x^2\right)}$ (A)

$=\frac{1}{9}\int\frac{dx}{\left(\frac{4}{9}+x^2\right)}$

$=\frac{1}{9}\times\frac{3}{2}\tan^{-1}\left(\frac{3x}{2}\right)$ (B)

$=\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+c$

(c)

$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$

$=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}$

$=\frac{\sqrt{6}-\sqrt{2}}{4}$

(d) (i)

$x^2=8y$

$y=\frac{x^2}{8}\therefore\frac{dy}{dx}=\frac{x}{4}$

$\text{at } x=4p\therefore\frac{dy}{dx}=\frac{4p}{4}=p$

$y-2p^2=p(x-4p)$

$y-2p^2=px-4p^2$

$y=px-2p^2$

(ii)

Line through Q parallel to the *y* axis is

$x=4q\rightarrow$ sub into $y=px-2p^2$

$y=4pq-2p^2$

$\therefore T(4q,4pq-2p^2)$

(iii)

$m=\left(\frac{4p+4q}{2},\frac{2p^2+4pq-2p^2}{2}\right)$

$=(2p+2q,2pq)$

(iv)

When $pq=-1\rightarrow y=2pq=-2$

$\therefore$ locus is $y=-2$

Question 12

Criteria
(a) (i) One for correct answer. (ii) One for correct answer (iii) One for interchanging *x* and *y*, one for simplification. (b) One for steps 1 and 2, one for $3^{3(k+1)}+2^{k+3}=5p$, one for $27(5m-2^{k+2})+2\times2^{k+2}$), one for simplification with conclusion. (c) One for $T_{r+1}={}^{11}c_r(4x^8)^{11-r}\left(-\frac{2}{x^3}\right)^r$ one for $T_{r+1}=Ax^{88-11r}$, one for $88-11r=0$ ie $r=8$ one for simplification. (d) One mark for each line.

Answers

(a) (i) $x\geq1$

(ii) $x\geq3$

(iii)

$y=(x-1)^2+3$

inverse function is

$x=(y-1)^2+3$

$(y-1)^2=x-3$

$y-1=\pm\sqrt{x-3}$

$f^{-1}(x)=1+\sqrt{x-3}$

(b) Step 1 Prove true for $n=1$

$3^3+2^3=35=5\times7\therefore$ true

Step 2 Assume true for $n=k$

i.e. $3^{3k}+2^{k+2}=5m$ where m is an integer

Step 3 Show true for $n=k+1$

$3^{3(k+1)}+2^{k+3}=5p$ where p is an integer

Now $3^{3k+3}+2^{k+3}=3^{3k}\times3^3+2^{k+2}\times2$

$=27(5m-2^{k+2})+2\times2^{k+2}$ from assumption

$=27(5m)-27(2^{k+2})+2(2^{k+2})$

$=27(5m)-25(2^{k+2})$

$=5[27m-5(2^{k+2})]$

$=5p$

Step 4 Conclusion

By assuming the result is true for $n=k$, we have proved it true for $n=k+1$, since it is true for $n=1\therefore$ it is true for $n=1+1=2, n=2+1=3$ and so on. Hence true for all integral values.

(c) $(4x^8-\frac{2}{x^3})^{11}$

$T_{r+1}={}^{11}c_r(4x^8)^{11-r}\left(-\frac{2}{x^3}\right)^r$

$=Ax^{88-8r-3r}$ where A is a constant

$=Ax^{88-11r}$

For the term independent of x

$88-11r=0$ ie $r=8$

$\therefore T_9={}^{11}C_8 4^3(-2)^8$

$=2703360$

(d)

$\cos8x=2\cos^2 4x-1$

$\cos^2 4x=\frac{\cos8x+1}{2}$

$\int\cos^2 4x\,dx=\frac{1}{2}\left(\frac{\sin8x}{8}+x\right)+c$

Question 13

Criteria

(a) (i) One mark for dividing by $\cos^2 x$, one for simplification. (ii) One for changing the limits, one for finding the integral, one for simplification. (b) (i) One mark for correct answer. (ii) (α) One mark each for C and k. (ii) (β) One mark for substitution, one mark for evaluation. (c) (i) (ii) (iii) One mark for the correct answer. (d) One for $t = 6$ and one for simplification

Answers

(a) (i)

$$\frac{1}{9\cos^2 x + 25\sin^2 x} = \frac{\frac{1}{\cos^2 x}}{\frac{9\cos^2 x}{\cos^2 x} + \frac{25\sin^2 x}{\cos^2 x}}$$

$$= \frac{\sec^2 x}{9 + 25\tan^2 x}$$

(ii)

$$\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x dx}{9 + 25\tan^2 x}$$

$u = \tan x \therefore du = \sec^2 x dx$

when $x = 0, u = 0$

when $x = \frac{\pi}{4}, u = 1$

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x dx}{9 + 25\tan^2 x} = \int_0^1 \frac{du}{9 + 25u^2}$$

$$= \frac{1}{25}\int_0^1 \frac{du}{u^2 + \frac{9}{25}}$$

$$= \frac{1}{25} \times \frac{5}{3}\left[\tan^{-1}\frac{5u}{3}\right]_0^1$$

$$= \frac{1}{15}\tan^{-1}\left(\frac{5}{3}\right)$$

$= 0.068691788$

(b) (i) $T = A + Ce^{kt}$,

$$\frac{dT}{dt} = kCe^{kt} = k(T - A)$$

$\therefore T = A + Ce^{kt}$ satisfies Newton's law of cooling.

(ii) (α) $A = 12^0$; at $t = 0, T = 60^0 \Rightarrow 60^0 = 12^0 + Ce^0$

$\therefore C = 48^0 \Rightarrow T = 12^0 + 48^0 e^{kt}$, $T = 30^0$

when $t = 25$ mins

$$30^0 = 12^0 + 48^0 e^{25k} \Rightarrow \frac{18}{48} = e^{25k}$$

$$\therefore \frac{1}{25}\log_e \frac{3}{8} = k = -0.03923317$$

(β)

$$15^0 = 12^0 + 48^0 e^{-0.03923317t} \Rightarrow t = -\frac{1}{0.03923317}\log_e \frac{3}{48}$$

$= 70.669\text{min}$

$= 71\text{min}$

(c) (i)

$f(x) = 2x^3 + 2x - 5$

$f(0) = -5$ and $f(2) = 15$

$\therefore$ a root lies between $x = 0$ and $x = 2$

(ii)

$f'(x) = 6x^2 + 2$

$6x^2 + 2 > 0$ for all x

$\therefore f(x)$ is an increasing function $\therefore$ only one zero can exist

(iii)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{-1}{8} = 1\frac{1}{8}$$

(d)

$0 = -5t^2 + 180$

$5t^2 = 180$

$t = 6\text{sec}$

$\dot{y} = -60, \dot{x} = 50$

$$\tan\theta = \frac{\dot{y}}{\dot{x}} = \frac{-60}{50}$$

$\theta = 129^0 48'$

Question 14

Criteria

(a) (i) One for proving the congruent triangles, one for Area of sector $ACB = \frac{1}{2} \times 1^2 \times 2\theta$, one for finding A in terms of θ and one for the conclusion. (b) (i) One for ${}^4C_1 \times {}^nC_2$ and one for simplification (ii) one for the correct answer. (iii) one for $\frac{\frac{2n(n-1)+6n}{(n+4)(n+3)(n+2)}}{6}$ and one for simplification. (iv) one for the differentiation and one for the turning points.

(v) one for the shape and one for proving it is a max.and finding the limit. (vi) one for $y = \frac{1}{14}$ and one for the conclusion.

Answers

(a)

$PK = PK$ (common)

$AP = PB$ (radii)

$\angle AKP = \angle BKP = 90^0$

$\Delta APK \equiv \Delta BPK$ (RHS)

$\therefore \angle APQ = \angle BPQ$ (corresponding sides of congruent triangles)

$\angle APK = \theta = \angle BPQ$

Area of sector $ACB = \frac{1}{2} \times 1^2 \times 2\theta$

Area of $\Delta APB = \frac{1}{2} \times 1^2 \times \sin 2\theta$

Area of segment $ACB = \frac{1}{2}(2\theta - \sin 2\theta)$

total area $A = 2 \times \frac{1}{2}(2\theta - \sin 2\theta)$

$= 2\theta - \sin 2\theta$

from ΔAPK

$\cos\theta = \frac{x}{1} = x \therefore \theta = \cos^{-1}x$

$A = 2\theta - \sin 2\theta$

$= 2\theta - 2\sin\theta\cos\theta$

$= 2(\theta - \sin\theta\cos\theta)$

$= 2\left(\cos^{-1}x - x\sqrt{1 - x^2}\right)$

(b)

(i)

$${}^4C_1 \times {}^nC_2 = \frac{4n!}{(n-2)!2!} = \frac{4n(n-1)}{2} = 2n(n-1)$$

(ii)

$${}^4C_2 \times {}^nC_1 = \frac{4!}{2!2!} \times n = 6n$$

(iii)

Three has to be chosen from (n+4) people. $\therefore$ the total number of committees is

$${}^{4+n}C_3 = \frac{(n+4)!}{(n+4-3)!3!}$$

$$= \frac{(n+4)(n+3)(n+2)}{6}$$

The probability of containing one or two women is

$$\frac{2n(n-1)+6n}{\frac{(n+4)(n+3)(n+2)}{6}} = \frac{12n(n+2)}{(n+4)(n+3)(n+2)}$$

$$= \frac{12n}{(n+4)(n+3)}$$

(iv)

$$y = \frac{x}{(x+4)(x+3)} = \frac{x}{x^2 + 7x + 12}$$

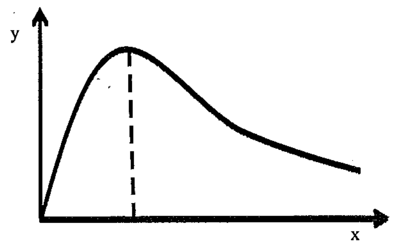
$$\frac{dy}{dx} = \frac{(x^2 + 7x + 12) - x(2x + 7)}{(x^2 + 7x + 12)^2}$$

$$0 = \frac{x^2 + 7x + 12 - 2x^2 - 7x}{(x^2 + 7x + 12)^2}$$

$0 = 12 - x^2$

$x^2 = 12$

$x = \pm\sqrt{12} = \pm 2\sqrt{3}$

	(v) x 3 $2\sqrt{3}$ 4 $\frac{dy}{dx}$ − 0 + $\therefore$ max. at $x = 2\sqrt{3}$ $\lim_{x\to\infty} \frac{x}{(x+4)(x+3)} = 0$  $2\sqrt{3} = 3.46$ (vi) From the sketch the integer values of x which will give the largest values of y are 3 and 4 $x = 3 \therefore y = \frac{1}{14}$ and $x = 4 \therefore y = \frac{1}{14}$ In the context of p and n, n = 3 and n = 4 maximize p $\therefore$ the maximum value is $\frac{6}{7}$