Caringbah High School

2015

Trial HSC Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen (Black pen is preferred)
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–15, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2 – 4

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 5 – 10

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x,\ x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax},\ a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax,\ a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax,\ a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax,\ a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax,\ a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

Question 1 - 10 (1 mark each) Answer on page provided.

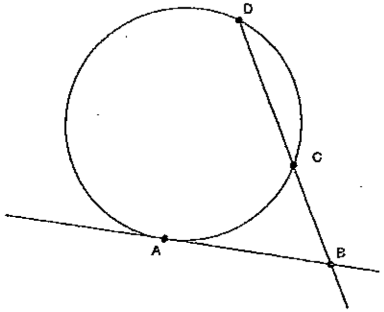
1) The exact value of $\tan\frac{\pi}{12}$ is:

A) $\frac{1}{2\sqrt{3}}$ B) $\left(\sqrt{3}-1\right)^2$ C) $2+\sqrt{3}$ D) $2-\sqrt{3}$

2) The polynomial $p(x)=2x^3-x^2-6x+k$ has a factor $(x+2)$. What is the value of k?

A) 8 B) 0 C) -24 D) 32

3) AB is a tangent at A

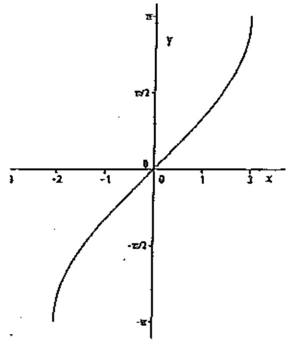


Which of the following is true?

A) $AB = BC.BD$ B) $AB = BC.CD$

C) $AB^2 = BC.CD$ D) $AB^2 = BC.BD$

4) The diagram shows the graph of a function. Which function does the graph represent?



A) $y=2\cos^{-1}(2x)$ B) $y=2\sin^{-1}(2x)$

C) $y=2\sin^{-1}\left(\frac{x}{2}\right)$ D) $y=2\cos^{-1}\left(\frac{x}{2}\right)$

5) Given the parametric equations $x=2(\theta-\sin\theta)$ and $y=2(1-\cos\theta)$ which of the following represent $\frac{dy}{dx}$ in terms of θ ?

A) $\frac{2\sin\theta}{2-\cos\theta}$ B) $\frac{1-\cos\theta}{\sin\theta}$

C) $\frac{\sin\theta}{1-\cos\theta}$ D) $\frac{\sin\theta}{1+\cos\theta}$

6) $\lim_{x\to\infty}\left[\frac{x+2}{1-x}\right]=$

A) 1 B) -1 C) 2 D) -2

7) A particle is moving in simple harmonic motion and the acceleration is given by $\ddot{x} = -4x + 8$. The centre of the motion is:

A) -8 B) 8 C) 2 D) -2

8) Using the substitution $x = 1 - u^2$ then $\int \frac{x\,dx}{\sqrt{1-x}} =$

A) $-2\int 1-u^2 du$ B) $-2\int u^2 - 1\, du$

C) $\frac{1}{2}\int u^2 - 1\, du$ D) $\frac{1}{2}\int 1-u^2\, du$

9) $A(1,-3)$ and $B(x,y)$ are 2 points, $P(-1,-1)$ divides these points A and B externally in the ratio $(2,3)$. The co-ordinates of B are:

A) $(-2,-5)$ B) $(2,-4)$ C) $\left(\frac{2}{3},\frac{-7}{3}\right)$ D) $(-2,4)$

10) $y = f(x)$ is a linear function with slope $\frac{1}{3}$, the slope of $y = f^{-1}(x)$ is

A) 3 B) $\frac{1}{3}$

C) -3 D) $-\frac{1}{3}$

END OF MULTIPLE CHOICE QUESTIONS

Section II

60 marks
Attempt all questions 11–14Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

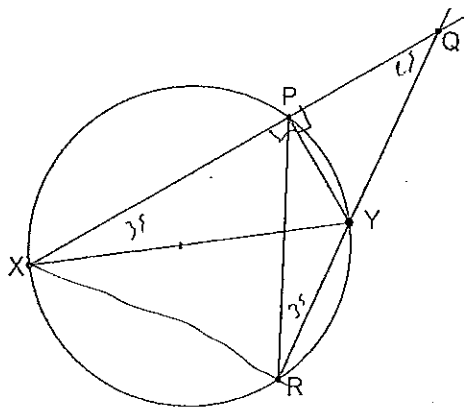
Question 11 (15 marks) Start a NEW booklet. **Marks**

a) Find $\frac{d^2y}{dx^2}$ if $y = \ln(e^x + 1)$ 2

b) i) Show that $\frac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$ 2

ii) Hence find the value of $\tan 22\frac{1}{2}^\circ$ in simplest exact form 2

c)



XY is the diameter in the circle. 3

Given that $\angle PXY = 35^\circ$ and $\angle PQY = 25^\circ$,
Find the size of $\angle YPR$ giving reasons.

Question 11.continues on the next page

Question 11 continued. Marks

d) $P(x)=x^3+3x^2+6x-5$

i) Show that the equation $P(x)=0$ has a root α such that $0<\alpha<1$ 2

ii) Use one application of Newton's method with a starting value of $x=0.5$ to find an approximation for α. Answer to 2 decimal places. 2

e) Find the exact value of 2

$$\int_{\frac{1}{6}}^{\frac{\sqrt{3}}{6}} \frac{3}{\sqrt{1-9x^2}}\,dx$$

End of Question 11

Question 12 (15 marks) Start a NEW booklet. **Marks**

a) Find $\lim\limits_{x\to 0}\dfrac{\sin 2x+\tan x}{4x}$. 2

b) Use the substitution $u=\tan^{-1}x$ to evaluate the following .Leave your answer in exact form 3

$$\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$$

c) The function $f(x)=\dfrac{1}{1+3e^{-x}}$ is defined for all real x and $e^x>0$

i) Sketch the curve $y=f(x)$ mark in any asymptotes, x, y intercepts 3

ii) Explain why an inverse function exists for $y=f(x)$ 1

iii) Find the inverse function $y=f^{-1}(x)$ 2

d) The volume, V of a spherical balloon of radius r mm is increasing at a constant rate of 400mm^3 per second.

i) Find $\dfrac{dr}{dt}$ in terms of r 2

ii) Find the rate of increase of the surface area S of the balloon when the radius is 25mm 2

End of Question 12.

Question 13 (15 marks) Start a NEW booklet. **Marks**

a) i) Sketch the graph of $y = 2\cos^{-1} 2x$, show any intercepts with axes, and the domain and range. 2

ii) The region in the first quadrant in the above graph is rotated about the y axis.

α) Show that $x^2 = \frac{1}{4}\cos^2 \frac{y}{2}$ 1

β) Find the volume of the solid formed (Answer in terms of π) 3

b) Find $\int 2x^2 e^{4x^3+2} dx$ 2

c) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x is the displacement from the origin. Initially the object is at the origin with velocity (v) $2ms^{-1}$

i) Prove that $V = 2e^{\frac{-x}{2}}$ 2

ii) What happens to v as x increases without bound ? 1

d) Use Mathematical Induction to show that $\cos(x+n\pi) = (-1)^n \cos x$ for all positive integers $n \geq 1$ 4

End of Question 13.

Question 14 (15 marks) Start a NEW booklet. **Marks**

a) The acceleration $\ddot{x}$ m/s^2 at time, t seconds, of a particle moving in a straight line is given by

$$\ddot{x} = -4\cos 2t - 8\sin 2t$$

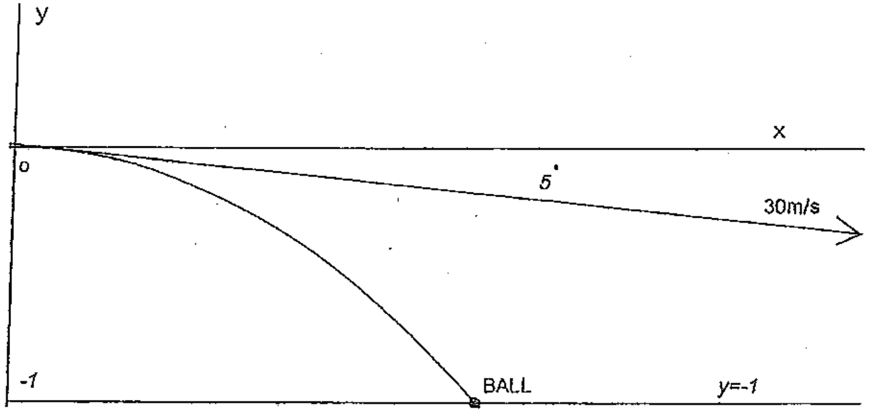
The particle is at a distance of x metres from the origin at time t and initially it is at $x = 1$ with a velocity of 4m/s

i) Show that $\ddot{x} = -4x$ 3

ii) Show that the position of the particle after $\frac{\pi}{4}$ seconds is 2 metres to the right of the origin and the magnitude of its velocity is 2m/s at this time. 2

iii) Is the speed of the particle increasing or decreasing when $t = \frac{\pi}{4}$. Justify your answer. 2

Question 14 continued. Marks



b) A batman hits a cricket ball which leaves the bat 1 metre above the ground with an initial speed of 30ms^{-1} at an angle of $5°$ in a downward direction. The equations of motion for the ball are $\ddot{x}=0$ and $\ddot{y}=-10$

i) Taking the origin to be the point where the ball leaves the bat, prove by using calculus that the ball has co-ordinates at time t given by 4

$$x = 30t\cos 5^0 \quad \text{and}$$

$$y = -30t\sin 5^0 - 5t^2$$

ii) Find the time which elapses for the ball to strike the ground. (3dp) 2

iii) Calculate the angle at which the ball strikes the ground. (nearest degree) 2

END OF EXAM

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Mathematics 2015

Multiple Choice

1, D 2, A 3, D 4, C 5, C 6, B 7, C 8, A 9, B 10, A

Question 11

a) $y = \ln(e^x + 1)$

$y' = \dfrac{e^x}{e^x + 1}$

$y'' = \dfrac{(e^x+1).e^x - e^x.e^x}{(e^x+1)^2}$

$= \dfrac{e^x}{(e^x+1)^2}$

b) (i) $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$

L.H.S

$\dfrac{1 - (\cos^2 x - \sin^2 x)}{1 + (\cos^2 x - \sin^2 x)}$

$= \dfrac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)}$

$= \dfrac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x$

(ii) $\tan^2 22\tfrac{1}{2}^\circ = \dfrac{1 - \cos 45}{1 + \cos 45}$

$\tan^2 22\tfrac{1}{2}^\circ = (\sqrt{2} - 1)^2$

$\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1$

c) Join PY and XR

$\angle PXY = 35^\circ$ } given
$\angle PQY = 25^\circ$ }

$\angle XRY = 90$ (angle in semicircle given XY is diameter)

$\angle YXR = 30^\circ$ (angle sum $\triangle QXR$)

$\angle YXR = \angle YPR = 30^\circ$ (angle at circ standing on same chord or arc)

Q11 cont'd

(i) $P(x) = x^3 + 3x^2 + 6x - 5$

d) Since $P(0) = -5 < 0$ & $P(1) = 5 > 0$ and the curve is continuous ∴ there is a root α between 0 and 1

(ii) $f(x) = x^3 + 3x^2 + 6x - 5$

$f(0.5) = -1.125$

$f'(0.5) = 9.75$

$x_2 = 0.5 - \dfrac{(-1.125)}{9.75}$

$x_2 \doteqdot 0.62$ (2dp)

e) $\displaystyle\int_{1/6}^{\sqrt{3}/6} \frac{3}{\sqrt{1 - 9x^2}}\,dx$

$= 3\displaystyle\int_{1/6}^{\sqrt{3}/6} \frac{1}{\sqrt{9(1/9 - x^2)}}\,dx$

$= \displaystyle\int_{1/6}^{\sqrt{3}/6} \frac{1}{\sqrt{(1/3)^2 - x^2}}\,dx$

$= \Big[\sin^{-1} 3x\Big]_{1/6}^{\sqrt{3}/6}$

$= \sin^{-1}\sqrt{3}/2 - \sin^{-1} 1/2$

$= \pi/3 - \pi/6$

$= \pi/6.$

Question 12.

(a) $\displaystyle\lim_{x \to 0} \frac{\sin 2x + \tan x}{4x}$

$= \dfrac{\sin 2x}{4x} + \dfrac{\tan x}{4x}$

$= \dfrac{1}{2}\cdot\dfrac{\sin 2x}{2x} + \dfrac{1}{4}\cdot\dfrac{\tan x}{x}$

$= 1/2 + 1/4$

$= 3/4$

b) $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1 + x^2}\,dx$

$u = \tan^{-1} x$

$du = \dfrac{1}{1+x^2}\,dx$

$x = 1,\ u = \pi/4$

$x = 0,\ u = 0$

$\displaystyle\int_0^{\pi/4} u\,.\,du$

$= \left[\dfrac{u^2}{2}\right]_0^{\pi/4}$

$= \dfrac{1}{2}\left[\dfrac{\pi^2}{16} - 0\right]$

$= \dfrac{\pi^2}{32}.$

c) $f(x) = \dfrac{1}{1 + 3e^{-x}}$

$= \dfrac{1}{1 + \frac{3}{e^x}}$

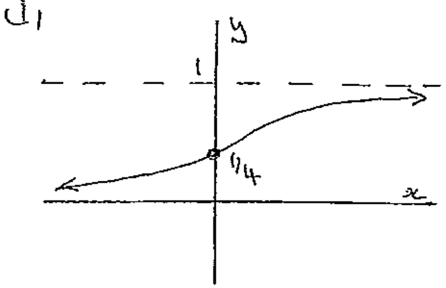
when $x = 0$, $y = 1/4$

$x \to \infty$, $f(x) \to 1$

$x \to -\infty$, $f(x) \to 0$

(c) cont'd

(i)



(ii) By horizontal line test only 1 intercept

(iii) $y = \dfrac{1}{1 + 3e^{-x}}$

$x = \dfrac{1}{1 + 3e^{-y}}$

$1 + 3e^{-y} = \dfrac{1}{x}$

$3e^{-y} = \dfrac{1}{x} - 1$

$\dfrac{3}{e^y} = \dfrac{1-x}{x}$

$\dfrac{e^y}{3} = \dfrac{x}{1-x}$

$e^y = \dfrac{3x}{1-x}$

$\ln e^y = \ln\left[\dfrac{3x}{1-x}\right]$

$y = \ln\left[\dfrac{3x}{1-x}\right]$

d) $\dfrac{dV}{dt} = 400\,mm^3/s$, $V = \frac{4}{3}\pi r^3$

(i) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$

$400 = 4\pi r^2 \times \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{100}{\pi r^2}$

(ii) $\dfrac{dS}{dt} = \dfrac{dS}{dr} \times \dfrac{dr}{dt}$

$= 8\pi(25).\dfrac{100}{\pi(25)^2}$

$= 32\,mm^2/s$

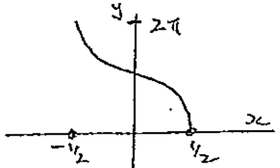
Question 13

(a) $y = 2\cos^{-1}2x$

(i) $D: -1 \le 2x \le 1$

$-\tfrac{1}{2} \le x \le \tfrac{1}{2}$

$R: 0 \le y \le 2\pi$



(ii) $y = 2\cos^{-1}2x$

(α) $\tfrac{1}{2}\cos\tfrac{y}{2} = x$

$x^2 = \tfrac{1}{4}\cos^2\tfrac{y}{2}$

(β) $V = \pi\int x^2\,dy$

$V = \pi\int_0^{\pi} \tfrac{1}{4}\cos^2\tfrac{y}{2}\,dy$

$V = \tfrac{\pi}{4}\int_0^{\pi} \tfrac{1}{2}(1+\cos y)\,dy$

$V = \tfrac{\pi}{8}\left[y + \sin y\right]_0^{\pi}$

$V = \tfrac{\pi}{8}\left[(\pi + \sin\pi) - 0\right]$

$V = \dfrac{\pi^2}{8}\ u^3.$

(b) $\int 2x^2 e^{4x^3+2}\,dx$

$= \tfrac{1}{6}\int 12x^2 e^{4x^3+2}\,dx$

$= \tfrac{1}{6}e^{4x^3+2} + C$

(c) $\ddot{x} = -2e^{-x}$

(i) $\dfrac{d}{dx}\tfrac{1}{2}v^2 = -2e^{-x}$

$\tfrac{1}{2}v^2 = 2e^{-x} + c$

when $x = 0$, $v = 2$

$\tfrac{1}{2}(2)^2 = 2e^0 + c$

$2 = 2 + c$

$c = 0$

$\therefore \tfrac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}$

$V = (4e^{-x})^{1/2}$

$V = 2e^{-x/2}$

(ii) $x \to \infty$

$V = \dfrac{2}{e^{x/2}}$

$e^{x/2} \to \infty \quad \therefore V \to 0$

(d) $\cos(x + n\pi) = (-1)^n\cos x,\ n \ge 1$

Prove true for $n = 1$

$\cos(x+\pi) = (-1)^1\cos x$

$-\cos x = -\cos x$

Assume true for $n = k$

$\cos(x + k\pi) = (-1)^k\cos x$

Prove true for $n = k+1$

L.H.S. $\cos[x + (k+1)\pi]$

$= \cos[(x + k\pi) + \pi]$

$= \cos(x+k\pi)\cos\pi - \sin(x+k\pi)\sin\pi$

$= -1\left[\cos(x+k\pi)\right]$

$= -1\left[(-1)^k\cos x\right]$

$= (-1)^{k+1}\cos x$

plus M.I. statement

Question 14

(a) $\ddot{x} = -4\cos 2t - 8\sin 2t$

(i) $\dot{x} = -2\sin 2t + 4\cos 2t + c$

when $t = 0$ $\dot{x} = 4$ $\therefore c = 0$

$\dot{x} = -2\sin 2t + 4\cos 2t$

$x = \cos 2t + 2\sin 2t + c$

$t = 0,\ x = 1,\ \therefore c = 0$

$\therefore x = \cos 2t + 2\sin 2t$

$-4x = -4\cos 2t - 8\sin 2t$

$\therefore \ddot{x} = -4x$

(ii) $x = \cos 2t + 2\sin 2t$

when $t = \pi/4$

$x = \cos\pi/2 + 2\sin\pi/2$

$x = 0 + 2(1)$

$x = 2.$

when $t = \pi/4$

$\dot{x} = -2\sin\pi/2 + 4\cos\pi/2$

$\dot{x} = -2(1) + 4(0)$

$\dot{x} = -2.$

(iii) when $t = \pi/4$, $x = 2$, $\dot{x} = -2$ m/s.

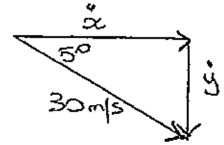
$\ddot{x} = -4x$

$= -4(2)$

$= -8\ m/s^2$

$\therefore$ Speed increasing as $\ddot{x} < 0,\ \dot{x} < 0$

(b) Initial velocity diagram



Vert $\sin 5^\circ = -\dfrac{\dot{y}}{30}$

$-30\sin 5^\circ = \dot{y}$

Hor $\cos 5^\circ = \dfrac{\dot{x}}{30}$

$30\cos 5^\circ = \dot{x}$

Equations of motion

$\ddot{x} = 0$

$\dot{x} = C$

$\dot{x} = 30\cos 5^\circ$

$x = 30t\cos 5^\circ + C$

$t = 0,\ x = 0,\ c = 0$

$x = 30t\cos 5^\circ$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$t = 0,\ \dot{y} = -30\sin 5^\circ$

$\therefore c = -30\sin 5^\circ$

$\dot{y} = -10t - 30\sin 5^\circ$

$y = -5t^2 - 30t\sin 5^\circ + C$

$t = 0,\ y = 0,\ C = 0$

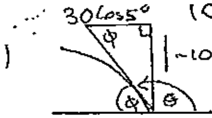
$y = -5t^2 - 30t\sin 5^\circ$

(ii) $-1 = -5t^2 - 30t\sin 5^\circ$

$5t^2 + 30t\sin 5^\circ - 1 = 0$

$t = \dfrac{-30\sin 5^\circ \pm \sqrt{(30\sin 5^\circ)^2 + 20}}{10}$ $\therefore t \doteqdot 0.2566$

(iii)



$-10(0.2566) - 30\sin 5^\circ = 5.18067228$

$\tan\phi = \dfrac{5.180672282}{30\cos 5}$

$\phi = 9^\circ 50' 3.91''$

Ball strikes ground at $\theta = 170^\circ 9' 56.09''$