Student Number:	
Class:	

JAMES RUSE AHS

GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2013

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11 - 16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

1 Let $z = 1 + i$. What is the value of z^{12}?

(A) 64

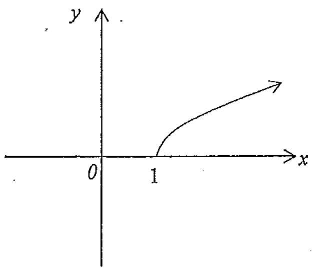
(B) -64

(C) $64i$

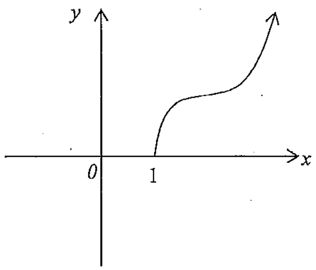
(D) $-64i$

2 Given $f(x) = x^2(x-1)$. Which of the following best represents the graph of $y = \sqrt{f(x)}$?

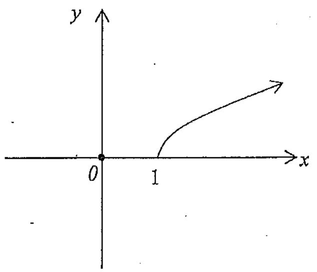
(A)



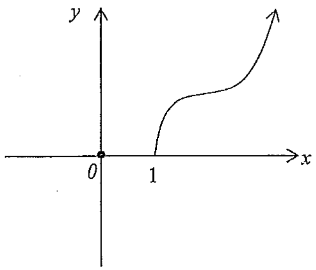
(B)



(C)



(D)



3 Given $2x^2 + xy + 2y^2 = 30$, what are the coordinates of one of the vertical tangents?

(A) (-1, 4)

(B) (4, -1)

(C) (-1, -4)

(D) (1, -4)

4 What is the equation of the chord of contact of tangents from (2, 1) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$?

(A) $\frac{2x}{9} - \frac{y}{4} = 1$

(B) $\frac{2x}{9} + \frac{y}{4} = 1$

(C) $\frac{x}{9} - \frac{y}{2} = 1$

(D) $\frac{x}{9} + \frac{y}{4} = 1$

5 Given $3x^3 - 2x + 5 = 0$ has roots α, β and γ, what is the equation with roots $\alpha + 1, \beta + 1$ and $\gamma + 1$?

(A) $3x^3 - 9x^2 + 7x + 6 = 0$

(B) $3x^3 + 9x^2 + 7x + 6 = 0$

(C) $3x^3 - 9x^2 + 7x + 4 = 0$

(D) $3x^3 + 9x^2 + 7x + 4 = 0$

6 Which of the following is the correct expression for the integral $\int \frac{dx}{4+\sin^2 x}$?

(A) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{5}{4}\tan x\right)+C$

(B) $2\sqrt{5}\tan^{-1}\left(\frac{5}{4}\tan x\right)+C$

(C) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}}{2}\tan x\right)+C$

(D) $2\sqrt{5}\tan^{-1}\left(\frac{\sqrt{5}}{2}\tan x\right)+C$

7 Given $3x^3+6x-5=0$ has roots α, β and γ, what is the value of $\alpha^3+\beta^3+\gamma^3$?

(A) 5

(B) 9

(C) 15

(D) -1

8 The equation of motion of a particle falling with velocity v m/s is given by $\ddot{x}=10-\frac{v}{2}$. Which of the following is the value of the terminal velocity?

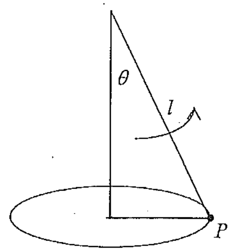
(A) 5

(B) 15

(C) 20

(D) $\sqrt{20}$

9 A bob P of mass m kg is suspended from a fixed point A by a string of length l metres, and acceleration due to gravity g. P describes a horizontal circle with uniform angular velocity ω rad/s.



Which of the following expressions represents the tension in the string?

(A) $ml\omega$

(B) $ml\omega^2$

(C) $mgl\omega$

(D) $mgl\omega^2$

10 Which of the following is the correct expression for the integral $\int e^{\alpha x}\sin\beta x\,dx$?

(A) $\frac{e^{\alpha x}}{\alpha^2+\beta^2}[\beta\sin\beta x+\alpha\cos\beta x]+C$

(B) $\frac{e^{\alpha x}}{\alpha^2+\beta^2}[\beta\sin\beta x-\alpha\cos\beta x]+C$

(C) $\frac{e^{\alpha x}}{\alpha^2+\beta^2}[\alpha\sin\beta x+\beta\cos\beta x]+C$

(D) $\frac{e^{\alpha x}}{\alpha^2+\beta^2}[\alpha\sin\beta x-\beta\cos\beta x]+C$.

Section II

90 marks
Attempt Questions 11–16.
Allow about 2 hours and 45 minutes for this section.
In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW page.

(a) $|z|<1$ and $z=\cos\theta+i\sin\theta$, where $-\pi<\theta\le\pi$..

(i) Show $1+z=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$. 2

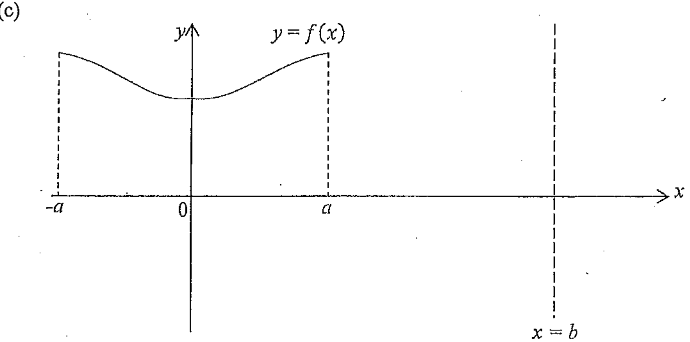
(ii) z_1 and z_2 are complex numbers such that $|z_1|=|z_2|=1$. If z_1 and z_2 have arguments α and β respectively, where $-\pi<\alpha\le\pi$ and $-\pi<\beta\le\pi$, show that $\frac{z_1+z_1z_2}{z_1+1}$ has modulus $\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ and Argument $\frac{\alpha+\beta}{2}$. 2

(iii) If $|z_1|=|z_2|=1$ and $\frac{z_1+z_1z_2}{z_1+1}=2i$ find z_1 and z_2 in the form $x+iy$ where x and y are real rational numbers. 4

(b) Shade the region $-\frac{\pi}{4}\le\operatorname{Arg} z\le\frac{\pi}{4}$ and $|z|\le 3$. 2

Question 11 (c) is continued over the page.

(c)

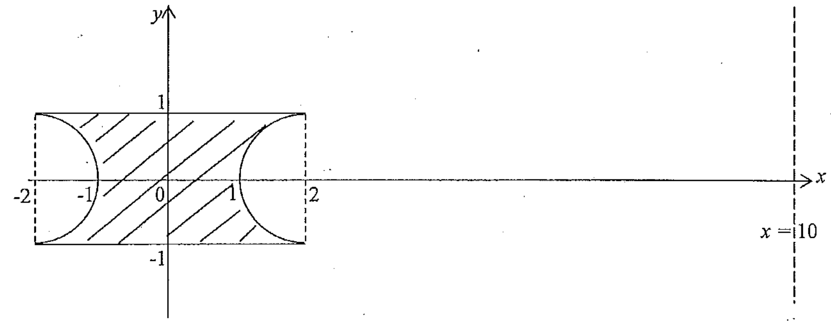


$f(x)$ is an even function such that $f(x)\ge 0$ for $-a\le x\le a$.

The region bounded by $y=f(x)$, the x-axis, and the ordinates $x=-a$ and $x=a$ has area A. The region is rotated about the line $x=b$ where $b>a>0$.

(i) Using the method of cylindrical shells show that the volume V of rotation is $2\pi bA$. 3

(ii)



The region shown with circular ends is rotated about $x=10$ to form a circular sealing ring. Find the volume of revolution. 2

End of Question 11.

Question 12 (15 marks) Start a NEW page.

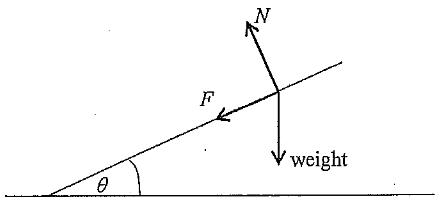
(a) Graph $y=\dfrac{x}{(x+4)(x+2)}$ showing all intercepts with the coordinate axes and all asymptotes. 3

(b) The region bounded by $y=\dfrac{x}{(x+4)(x+2)}$, the x-axis and $x=1$ is rotated around the y-axis.

(i) Find the values A, B and C such that $\dfrac{x^2}{(x+4)(x+2)}=A+\dfrac{B}{x+2}+\dfrac{C}{x+4}$. 4

(ii) Using the method of cylindrical shells show that the volume V of revolution is given by $V=2\pi\int_0^1\dfrac{x^2dx}{(x+4)(x+2)}$, hence find the exact value of the volume of revolution. 4

(c)



A car of mass 2000 kg travels around a curve of radius 150 m at a speed of 110km/h. The car experiences a lateral resistance force F of 0.22 × normal force, N, as shown. 4

By resolving the forces vertically and horizontally find the ~~minimum~~ angle θ ~~(to the nearest minute)~~ for the car to negotiate the curve. (Assume acceleration due to gravity of 10 m/s^2).

End of Question 12.

Question 13 (15 marks) Start a NEW page.

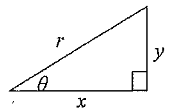
(a) (i) Show $\int_{-a}^{0}f(x)dx=\int_0^a f(-x)dx$ 1

(ii) Deduce $\int_{-a}^{a}f(x)dx=\int_0^a\left[f(x)+f(-x)\right]dx$ 1

(iii) Hence evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+\sin x)^2}$ 4

(b) A shape is defined as $r=\dfrac{9}{5+4\cos\theta}$ where r is the distance from origin and θ is the angle anticlockwise from the positive x-axis.

(i) Using the notation 3



find the equivalent Cartesian equation and show that the shape is an ellipse translated.

(ii) State the minor axis, major axis and location of the foci. 4

(iii) The area A enclosed by the shape is given by $A=\int_0^{2\pi}\frac{1}{2}r^2d\theta$. 2

Using (b)(i) and(b)(ii) evaluate $\int_0^{2\pi}\dfrac{d\theta}{(5+4\cos\theta)^2}$.

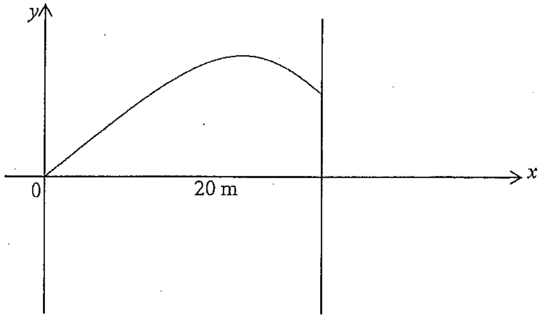
End of Question 13.

Question 14 (15 marks) Start a NEW page.

(a) (i) Find the coordinates of the intersection of the curves $y^2 = 8x$ and $x^2 = 8y$. 1

(ii) The base of a solid is in the region bounded by the curves $y^2 = 8x$ and $x^2 = 8y$, and its cross sections by planes perpendicular to the x-axis are semicircles. Find the volume of the solid. 3

(b)



A liquid particle of mass m kg is projected from the ground and hits a vertical wall 20m from the point of projection as shown.

(i) The equations of motion before the particle hits the wall are 3

$x = 4t$ and $y = 30t - 5t^2$

where t is time in seconds. Show that the particle hits the wall 25 m above the ground with a downwards velocity of 20 m/s.

(ii) After hitting the wall the particle slides down the wall with a resistance force equal to $0.04mv^2$.

(α) If acceleration due to gravity is 10 m/s^2 show that the velocity on return to the ground is approximately 16.44 m/s. 4

(β) Find the total time for the particle to return to the ground. Give your answer to two decimal places. 4

End of Question 14.

Question 15 (15 marks) Start a NEW page.

The hyperbola $xy = c^2$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ where $t_1 > t_2 > 0$. Tangents to the hyperbola at P and Q meet at T, while tangents to the ellipse at P and Q meet at V.

(i) Show the above information on a sketch. 1

(ii) Show that the parameter of point $\left(ct, \frac{c}{t}\right)$ which lies on the intersection of $xy = c^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ satisfies the equation $b^2c^2t^4 - a^2b^2t^2 + a^2c^2 = 0$. 2

(iii) Given the equation of the tangent to the hyperbola at $\left(ct, \frac{c}{t}\right)$ is $x + t^2y = 2ct$, show that the coordinates of T are $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$. 2

(iv) Given that the equation of the tangent to the ellipse at (x_1, y_1) is $b^2x_1x + a^2y_1y = a^2b^2$, show that the coordinates of V are $\left(\frac{a^2}{c(t_1+t_2)}, \frac{b^2t_1t_2}{c(t_1+t_2)}\right)$. 2

(v) Show that the line TV passes through the origin. 3

(vi) Point V lies at a focus of the hyperbola.

(α) Show that the ellipse is a circle. 2

(β) Find the radius of the circle in terms of c. 3

End of Question 15.

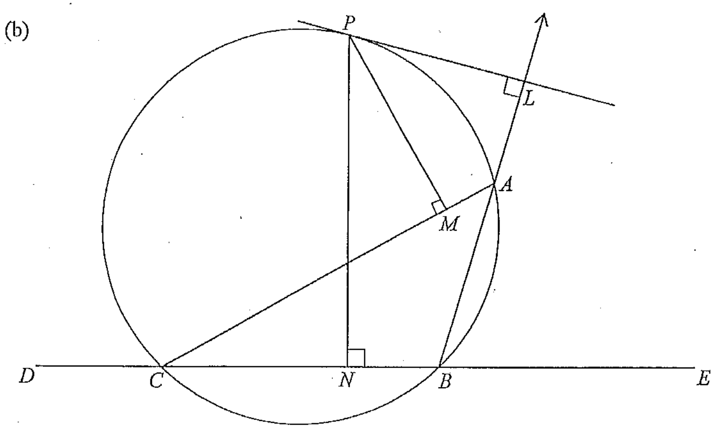
Question 16 (15 marks) Start a NEW page.

(a) $I_n = \int_0^{2\pi} (1+\cos\theta)^n \, d\theta$ for n ≥ 0.

(i) Show $I_{n+1} = \dfrac{2n+1}{n+1} I_n$. 4

(ii) Find I_3. 1

(b)



ABC is a triangle inscribed in a circle. L, M and N are the feet of the perpendiculars from P to AB, AC and BC respectively.

(i) Copy the diagram. 1

(ii) Show P, M, A and L are concyclic points. 2

(iii) Show P, C, N and M are concyclic points. 2

(iv) Show that L, M and N are collinear. 5

End of paper.

» Section I

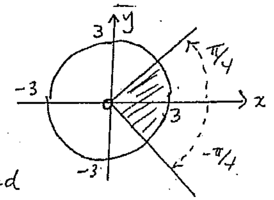
1 mk for each question.

1. A
2. D
3. B
4. A
5. C
6. C
7. A
8. C
9. B
10. D

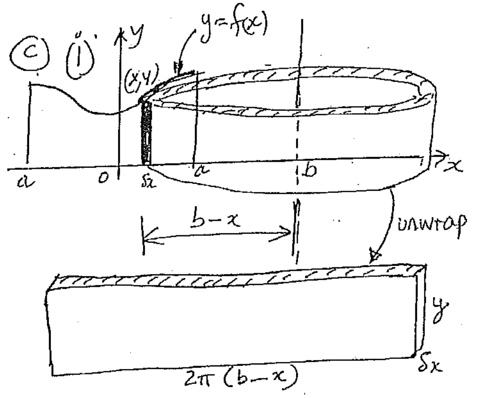
2013 TRIAL **MATHEMATICS Extension 2: Question...** 11

Suggested Solutions	Marks	Marker's Comments
(a) $1+z=(1+\cos\theta)+i(\sin\theta)$ (i) $=(1+\cos 2\times\frac{\theta}{2})+i(\sin 2\times\frac{\theta}{2})$ $=2\cos^2\frac{\theta}{2}+i(2\sin\frac{\theta}{2}\cos\frac{\theta}{2})$ $=2\cos\frac{\theta}{2}(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})$ or $2\cos\frac{\theta}{2}\operatorname{cis}\frac{\theta}{2}$	1 1	This part was well done by most students
(ii) $z_1=2\cos\frac{\alpha}{2}\operatorname{cis}\frac{\alpha}{2}$ $z_2=2\cos\frac{\beta}{2}\operatorname{cis}\frac{\beta}{2}$ $\left\|\frac{z_1(1+z_2)}{1+z_1}\right\|=\frac{\|z\|\,\|1+z_2\|}{\|1+z_1\|}$ $=\frac{(1)(2\cos\frac{\beta}{2})}{(2\cos\frac{\alpha}{2})}$ $=\frac{\cos\beta/2}{\cos\alpha/2}$ NOTE This is positive since $-\pi<\alpha,\beta<\pi$ $\arg\left(\frac{z_1(1+z_2)}{1+z_1}\right)=\arg z_1+\arg(1+z_2)-\arg(1+z_1)$ $=\alpha+\beta/2-\alpha/2$ $=\frac{\alpha+\beta}{2}$	1 1	Quite a few failed to factorise $z_1+z_1z_2$ and hence did not use a(i) which made the question more difficult. A significant number of students confused $\cos\frac{\alpha}{2}$ and $\operatorname{cis}\frac{\alpha}{2}$

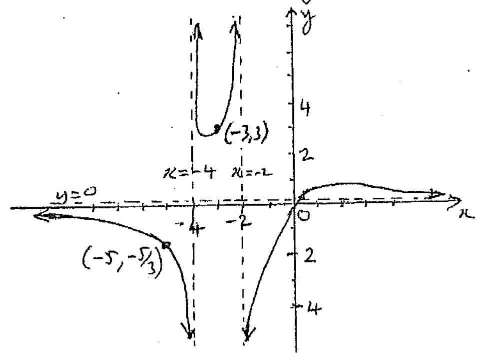
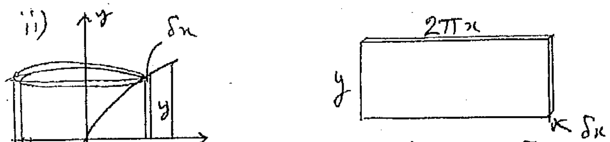
MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments
(iii) $\arg(2i) = \frac{\pi}{2}$		Some thought that the $\arg 2i = \pi$
$\therefore \frac{\alpha+\beta}{2} = \frac{\pi}{2}$		
$\alpha+\beta = \pi$	1	
$\lvert 2i \rvert = 2$		
$\frac{\cos \beta/2}{\cos \alpha/2} = 2$		
$\frac{\cos(\pi/2 - \alpha/2)}{\cos \alpha/2} = \frac{\sin \frac{\alpha}{2}}{\cos \alpha/2} = 2$		Many missed the fact that $\cos(\frac{\pi}{2}-\frac{\alpha}{2}) = \sin \frac{\alpha}{2}$
$t = \tan \alpha/2 = 2$	1	This mark for $\tan \alpha/2 = 2$ or $\tan \beta/2 = \frac{1}{2}$
$\cos\alpha = \frac{1-t^2}{1+t^2} = \frac{1-2^2}{1+2^2} = -\frac{3}{5}$		
$\sin\alpha = \frac{2t}{1+t^2} = \frac{2\times 2}{1+2^2} = \frac{4}{5}$		Many made arithmetic mistakes or assumed things like $z_1 = -z_2$ or $z_1 = z_2$
$\therefore z_1 = -\frac{3}{5} + \frac{4}{5} i$	1	
Similarly $\tan \beta/2 = \frac{1}{2}$		
$\cos\beta = \frac{3}{5}$ $\sin\beta = \frac{4}{5}$		
$z_2 = \frac{3}{5} + \frac{4}{5} i$	1	Full marks for z_1 & z_2 correctly obtained.
(b)  NOTE $(0,0)$ excluded	1	Some students did not note the $\pi/4$ angles.
	1	Quite a few did not note that the origin is excluded

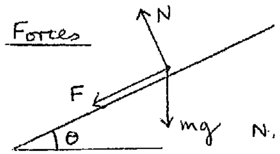
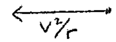
MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments
(c) (i) 		
Volume $= \lim_{\delta x \to 0} 2\pi(b-x)y\,\delta x$		Some students stated * without justification
$V = \int_{-a}^{a} 2\pi f(x)(b-x)\,dx$ *	1	and were not awarded full marks
$V = 2\pi b \int_{-a}^{a} f(x)\,dx - 2\pi \int_{-a}^{a} x f(x)\,dx$	1	Students needed to explain why $\int_{-a}^{a} f(x)\,dx = A$ and
$V = 2\pi b A$ since $\int_{-a}^{a} f(x) = A$		
and $\int_{-a}^{a} x f(x) = 0$ since $x f(x)$ is odd (odd func. x even func = odd func.)	1	$\int_{-a}^{a} x f(x)\,dx = 0$
(ii) From (i) $V = 2\pi b A$		Many students wasted time by not using c(i) but by finding the volume by integration.
$A =$ rectangle $-$ circle		
$= 4\times 2 - \pi(1)^2$		
$= 8-\pi$	1	
$b = 10$		A common error was to think that the area of the rectangle was 4
$V = 2\pi(10)(8-\pi)$		
$V = 20\pi(8-\pi)$ units^3	1	

2013 TRIAL X2 MATHEMATICS: Question 12 p1 of 2

Suggested Solutions	Marks	Marker's Comments
a) Vert Asy $x=-4$, $x=-2$ Hor. Asy $y=0$ Zero at $x=0$, also y intercept.	3×½	for each asymptote with either an equation or a line definitely finishing towards it.
	1	for shape - ½ off for each real error in main graph.
	½	for having a labelled point on each branch. (or associated scales)
b) i) $\frac{x^2}{(x+4)(x+2)} = A + \frac{B}{x+2} + \frac{C}{x+4}$ $\therefore A(x+2)(x+4) + B(x+4) + C(x+2) \equiv x^2$	1	Easy marks.
Equate coeffs of x^2 : $\underline{A = 1}$	1	
Put $x=-2$: $2B = 4$ $\therefore$ $\underline{B = 2}$	1	
Put $x=-4$: $-2C = 16$ $\therefore$ $\underline{C = -8}$	1	
ii) 	½	diagram
$\delta V \doteqdot 2\pi x y\,\delta x$ $\delta V = (\pi(x+\delta x)^2 - \pi x^2)y$	½	for $\doteqdot$ (type 2) or neglect 2^{nd} order terms
$= 2\pi x y\,\delta x$ (neglecting 2^{nd} order terms)	½	for limit of sum
$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x y\,\delta x$ $= 2\pi \int_0^1 \frac{x^2\,dx}{(x+2)(x+4)}$	½	for integral (except if baldly stated)

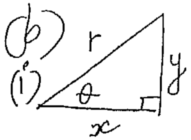
2013 TRIAL X2 MATHEMATICS: Question 12 p2 of 2

Suggested Solutions	Marks	Marker's Comments
b) ii) (cont) $\therefore V_\Omega = 2\pi \int_0^1 1 + \frac{2}{x+2} - \frac{8}{x+4}\,dx$ (using part i)		
$= 2\pi\left[x + 2\ln(x+2) - 8\ln(x+4)\right]_0^1$	1	
$= 2\pi\{1 + 2\ln 3 - 8\ln 5 - 2\ln 2 + 8\ln 4\}$	1	
$\underline{V_\Omega = 2\pi(1 + 2\ln 3 + 14\ln 2 - 8\ln 5)\ u^3}$		
c) Forces  Acc^n  N.B. $v = 110$ k/h $= \frac{275}{9}$ m/s	½	Most people got these 2½ marks.
Resolve vertically (V) $mg + F\sin\theta = N\cos\theta$	1	
Resolve horizontally (H) $F\cos\theta + N\sin\theta = \frac{mv^2}{r}$	1	
(Assuming $F = 0.22N$ means this is already the optimal angle θ.) Substituting numbers (V) → $N(\cos\theta - 0.22\sin\theta) = 20000$ (H) → $N(0.22\cos\theta + \sin\theta) = 12448.56$ Dividing: $\frac{\cos\theta - 0.22\sin\theta}{0.22\cos\theta + \sin\theta} = 1.6066\ldots$ $\frac{1 - 0.22\tan\theta}{0.22 + \tan\theta} = 1.6066\ldots$		Many mistakes in the numeric work. ½ for getting down to a simple equation in $\tan\theta$.
$\tan\theta = 0.3539$	½	Final mark for correct solution (29' or 30' accepted)
$\underline{\theta = 19^\circ 29'}$ (nearest minute)	1	

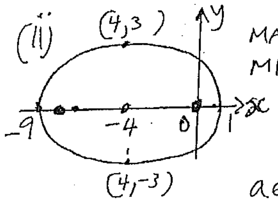
MATHEMATICS Extension 2: Question. 13

Suggested Solutions	Marks	Marker's Comments
(a) (i) Let $x = -u$ $dx = -du$ $x = 0 \quad u = 0$ $x = -a \quad u = a$ $\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du)$ $= \int_{0}^{a} f(-u)\,du$ $= \int_{0}^{a} f(-x)\,dx$ Changing the variable in a definite integral does not change its value	1	Well done by students Some students thought that the function must be even (or must be odd)
(ii) $\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ $= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$ $= \int_{0}^{a} [f(x) + f(-x)]\,dx$	1	Well done by students
(iii) $\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+\sin x)^2} = \int_{0}^{\pi/4} \frac{1}{(1+\sin x)^2} + \frac{1}{[1+\sin(-x)]^2}\,dx$ $= \int_{0}^{\pi/4} \frac{1}{(1+\sin x)^2} + \frac{1}{(1-\sin x)^2}\,dx$ $= \int_{0}^{\pi/4} \frac{(1-\sin x)^2 + (1+\sin x)^2}{(1-\sin^2 x)^2}\,dx$	1	Nearly all students used a(ii) correctly to begin

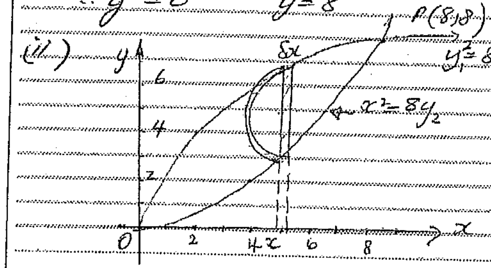
MATHEMATICS Extension 2: Question.. 13

Suggested Solutions	Marks	Marker's Comments
$\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+\sin x)^2} = \int_{0}^{\pi/4} \frac{2(1+\sin^2 x)}{\cos^4 x}\,dx \quad (A)$	1	Most students got to (A).
$= 2\int_{0}^{\pi/4} \sec^2 x(\sec^2 x + \tan^2 x)\,dx$ $= 2\int_{0}^{\pi/4} \sec^2 x(1 + 2\tan^2 x)\,dx$	1	Many failed to realise $\int \sec^2 x \tan^2 x\,dx = \frac{1}{3}\tan^3 x$
$= 2\left[\tan x + \frac{2}{3}\tan^3 x\right]_0^{\pi/4}$ $= 10/3$	1	Correct answer correctly done (by many of a variety of methods) for full marks
(b) (i)  $r = \sqrt{x^2+y^2}$ $\cos\theta = \frac{x}{r}$ $r = \frac{9}{5+4\cos\theta}$ $r = \frac{9}{5+4(\frac{x}{r})}$ $1 = \frac{9}{5r+4x}$ $5r = 9 - 4x$ $25r^2 = 81 - 72x + 16x^2$ $25(x^2+y^2) = 81 - 72x + 16x^2$	1	Most students failed to eliminate both θ and r so we unable to make progress
$9x^2 + 72x + 25y^2 = 81$ $9(x^2+8x+16) + 25y^2 = 81 + 9\times 16$ $9(x+4)^2 + 25y^2 = 225$	1	Arithmetic mistakes were common here
$\frac{(x+4)^2}{25} + \frac{y^2}{9} = 1$ This is an ellipse, centre $(-4, 0)$	1	Complete simplification required for full marks.

MATHEMATICS Extension 2: Question.. 13

Suggested Solutions	Marks	Marker's Comments
(ii)  MAJOR AXIS $= 2\times5 = 10$ units, MINOR AXIS $= 2\times 3 = 6$ units	1	Many students confused semiaxis (a or b) with axis (2a or 2b)
$e = \sqrt{1-\frac{b^2}{a^2}} = \frac{4}{5}$	1	
$ae = 4\times\frac{4}{5} = 4$	1	
FOCI: $(-4\pm4, 0) \Rightarrow (-8,0)$ and $(0,0)$	1	
(iii) AREA $= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$; $\pi ab = \int_0^{2\pi} \frac{1}{2}\left(\frac{9}{5+4\cos\theta}\right)^2 d\theta$; Area $= \pi\times3\times5 = 15\pi$	1	Many wasted time finding the area of the ellipse by integration instead of quoting $A=\pi ab$
$\therefore \int_0^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2} = 15\pi\times\frac{2}{81} = \frac{10\pi}{27}$	1	Full marks for correct answer correctly obtained.

MATHEMATICS Extension 2 : Question... 14

Suggested Solutions	Marks	Marker's Comments
14 a) (i) Find the points of intersection of $x^2=8y$ and $y^2=8x$		
$x^4 = (8y)^2$; $x^4 = 64\times8x$; $x^4 = 512x$; $x^4 - 512x = 0$; $x(x^3-512) = 0$; $x(x-8)(x^2+8x+64) = 0$; $x=0$ or $x=8$ or no solution; $\therefore y=0$, $y=8$	1	
(ii) 		
Area of Cross Section $= \frac{1}{2}\left(\frac{\pi D^2}{4}\right) = \frac{\pi}{8}(y_1-y_2)^2 = \frac{\pi}{8}\left(2\sqrt{2}x - \frac{x^2}{8}\right)^2$		
Volume of Slice $= A.\delta x$		
Volume $= \lim_{\delta x\to0} \sum_0^8 A(x)\cdot\delta x$		
$\therefore V = \frac{\pi}{8}\int_0^8 \left(2\sqrt{2}x - \frac{x^2}{8}\right)^2$		
$= \frac{\pi}{8}\int_0^8 8x - \frac{\sqrt{2}}{2}x^{5/2} + \frac{x^4}{64}\,dx$		
$= \frac{\pi}{8}\left[4x^2 - \frac{\sqrt{2}}{7}x^{7/2} + \frac{x^5}{320}\right]_0^8$		
$= \frac{\pi}{8}\left[4(8)^2 - \frac{\sqrt{2}}{7}(8)^{7/2} + \frac{8^5}{320} - 0\right]$		
$= 32\pi\left[1 - 8/7 + 2/5\right]$		
$\therefore V = \frac{288\pi}{35}$ units^3		

MATHEMATICS Extension 2 : Question 14

Suggested Solutions	Marks	Marker's Comments

14 b) Given $x = 4t$ and $y = 30t - 5t^2$
the particle hits the wall when $x = 20\,m$ and $y = 25\,m$

(i) $4t = 20$
$t = 5$

$y = 30 \times 5 - 5(5)^2$
$= 150 - 125$
$= 25$

$\therefore$ The particle does hit the wall 25 m above the ground.

And, $y' = 30 - 10t$
$= 30 - 10 \times 5$
$= -20$

$\therefore$ The particle has a downwards velocity of 20 m/s

3

(ii) After hitting the wall, the particle slides down with a resistance force of $0{\cdot}04mv^2$

$\downarrow + \quad \downarrow 10m \quad \uparrow R = 0{\cdot}04mv^2$

Sum of the forces $= m\ddot{x}$ (Newtons 2nd Law)

ie $m\ddot{x} = 10m - 0{\cdot}04mv^2$
$\ddot{x} = 10 - 0{\cdot}04v^2$
$\therefore \ddot{x} = -0{\cdot}04(v^2 - 250) \qquad v > \sqrt{250}$

(α) For velocity on return to the ground

$v \cdot \frac{dv}{dx} = -0{\cdot}04(v^2 - 250)$

$$\int_{20}^{v} \frac{v\,dv}{v^2 - 250} = \int_{0}^{25} -0{\cdot}04\,dx$$

$$\left[\tfrac{1}{2}\ln(v^2 - 250)\right]_{20}^{v} = -0{\cdot}04\left[x\right]_0^{25}$$

$$\tfrac{1}{2}\left[\ln(v^2 - 250) - \ln(20^2 - 250)\right] = -0{\cdot}04[25 - 0]$$

$$\tfrac{1}{2}\ln\left(\frac{v^2 - 250}{150}\right) = -1$$

4

8 35 pm   +1hr

955

MATHEMATICS Extension 2 : Question 14

Suggested Solutions	Marks	Marker's Comments

14 (b)(ii) (α) continued..

$$\ln\left(\frac{v^2 - 250}{150}\right) = -2$$

$$\frac{v^2 - 250}{150} = e^{-2} \quad \text{(take exponentials of both sides)}$$

$v^2 = 150e^{-2} + 250$
$= 270{\cdot}30029\ldots$

$v = 16{\cdot}4408\ldots$

$\therefore$ The velocity on return to the ground is approximately 16·44 m/s

(β) Find the total time for the particle to return to the ground.

$\ddot{x} = -0{\cdot}04(v^2 - 250)$ from (i)

ie $\frac{dv}{dt} = -0{\cdot}04(v^2 - 250)$

$$\int_{20}^{16\cdot44} \frac{dv}{v^2 - 250} = \int_0^T -0{\cdot}04\,dt.$$

NB $\frac{1}{v^2 - 250} = \frac{A}{v - \sqrt{250}} + \frac{B}{v + \sqrt{250}}$

$1 = A(v + \sqrt{250}) + B(v - \sqrt{250})$
$1 = v(A + B) + \sqrt{250}(A - B)$

$A - B = \frac{1}{\sqrt{250}} \qquad A + B = 0$
$\qquad\qquad\qquad A = -B$

$2A = \frac{1}{\sqrt{250}} \Rightarrow A = \frac{1}{2\sqrt{250}}$ & $B = -\frac{1}{2\sqrt{250}}$

Now,
$$\frac{1}{2\sqrt{250}}\int_{20}^{16\cdot44}\left(\frac{1}{v - \sqrt{250}} - \frac{1}{v + \sqrt{250}}\right)dt = -0{\cdot}04\left[t\right]_0^T$$

$$\frac{1}{2\sqrt{250}}\left[\ln\left(\frac{v - \sqrt{250}}{v + \sqrt{250}}\right)\right]_{20}^{16\cdot44} = -0{\cdot}04\left[T - 0\right]$$

$$T = \frac{-25}{2\sqrt{250}}\ln\left(\frac{16{\cdot}44 - \sqrt{250}}{16{\cdot}44 + \sqrt{250}} \times \frac{20 + \sqrt{250}}{20 - \sqrt{250}}\right)$$

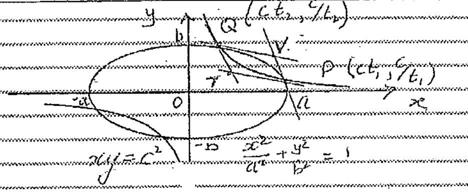
$$= -0{\cdot}790569 \times \ln\left(\frac{0{\cdot}628611\ldots}{32{\cdot}25138\ldots} \times \frac{35{\cdot}81138\ldots}{4{\cdot}18861\ldots}\right)$$

$= 1{\cdot}41662\ldots$

Total Time $= 1{\cdot}42 + 5$
$= 6{\cdot}42$ seconds (to 2 d.p.)

4 $+\frac{1}{2}$ hr

MATHEMATICS Extension 2: Question 15 ①

Suggested Solutions	Marks	Marker's Comments
(i) 	1	½ for correct position of P and Q; ½ for V and T
(ii) The point $(ct, \frac{c}{t})$ lies on $xy=c^2$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{(ct)^2}{a^2}+\frac{(c/t)^2}{b^2}=1$ $b^2c^2t^2+a^2c^2=a^2b^2t^2$ $b^2c^2t^2+a^2c^2-a^2b^2t^2=0$	2	1 sub $(ct, c/t)$ into $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 1 simplifying
(iii) Equation of tangent is $x+t^2y=2ct$ at P: $x+t_1^2y=2ct_1$ (i) at Q: $x+t_2^2y=2ct_2$ (ii) $y(t_1^2-t_2^2)=2c(t_1-t_2)$ $y(t_1-t_2)(t_1+t_2)=2c(t_1-t_2)$ $y=\frac{2c}{t_1+t_2}$ sub into (i) $x=2ct_1-t_1^2\left[\frac{2c}{t_1+t_2}\right]$ $=2c\left[\frac{t_1^2+t_1t_2-t_1^2}{t_1+t_2}\right]$ $x=\frac{2ct_1t_2}{t_1+t_2}$ $T=\left[\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right]$	2 $t_1\neq t_2$	no loss of mark if $t_1\neq t_2$ not written 1 x coordinate 1 y coordinate
(iv) Equation of tangent is $b^2x_1x+a^2y_1y=a^2b^2$ at P: $bct_1x+\frac{a^2c}{t_1}y=a^2b^2$ $\times t_2$ at Q: $bct_2x+\frac{a^2c}{t_2}y=a^2b^2$ $\times t_1$ ③ $bct_1t_2x+\frac{a^2ct_2}{t_1}y=a^2b^2t_2$ ④ $bct_1t_2x+\frac{a^2ct_1}{t_2}y=a^2b^2t_1$ ③−④ $a^2cy\left[\frac{t_2}{t_1}-\frac{t_1}{t_2}\right]=a^2b^2\left[t_2-t_1\right]$ $a^2c\left[\frac{t_2^2-t_1^2}{t_1t_2}\right]=a^2b^2\left[t_2-t_1\right]$		

MATHEMATICS Extension 2: Question 15 ②

Suggested Solutions	Marks	Marker's Comments
$y=\frac{b^2t_1t_2(t_2-t_1)}{c(t_1-t_2)(t_1+t_2)}$ $t_1\neq t_2$ $=\frac{b^2t_1t_2}{c(t_1+t_2)}$ (i)$\times t_1$ ⑤ $b^2ct_1^2x+a^2cy=a^2b^2t_1$ (ii)$\times t_2$ ⑥ $b^2ct_2^2x+a^2cy=a^2b^2t_2$ ⑤−⑥ $b^2cx\left[t_1^2-t_2^2\right]=a^2b^2\left[t_1-t_2\right]$ $x=\frac{a^2(t_1-t_2)}{c(t_1-t_2)(t_1+t_2)}=\frac{a^2}{c(t_1+t_2)}$ $V=\left[\frac{a^2}{c(t_1+t_2)}, \frac{b^2t_1t_2}{c(t_1+t_2)}\right]$	2	1 x coordinate 1 y coordinate
(v) Gradient of OT $m_{OT}=\frac{2c}{t_1+t_2}\Big/\frac{2ct_1t_2}{t_1+t_2}$ $=\frac{1}{t_1t_2}$ Gradient of OV $m_{OV}=\frac{b^2t_1t_2}{c(t_1+t_2)}\Big/\frac{a^2}{c(t_1+t_2)}$ $=\frac{b^2}{a^2}\left[t_1t_2\right]$		1 gradients OT, OV.
Roots of $b^2c^2t^4-a^2b^2t^2+a^2c^2=0$ t_1, t_2 and $-t_1, -t_2$ by symmetry product of roots $t_1^2t_2^2=\frac{a^2c^2}{b^2c^2}$ $\therefore t_1t_2=a/b$ as $t_1t_2>0$		1 $t_1t_2=a/b$.
$\therefore m_{OV}=\frac{b^2}{a^2}\times a/b=b/a$. $m_{OT}=\frac{1}{t_1t_2}=b/a$. $\therefore$ V, O, T collinear (2 equal gradients and common point) $\therefore$ VT passes through origin Alternatively Equation of TV $y-\frac{2c}{t_1+t_2}=\frac{b^2t_1t_2-2c^2}{a^2-2c^2t_1t_2}\left[x-\frac{2ct_1t_2}{t_1+t_2}\right]$ LHS = RHS when $x=0$ $y=0$ and $t_1t_2=a/b$ $\therefore$ TV passes through origin		1 Conclusion with working Alternatively 1 Gradient TV 1 Equation of TV and sub (0,0) 1 showing correctly LHS = RHS (using $t_1t_2=a/b$)

③

MATHEMATICS Extension 2 : Question 15

Suggested Solutions	Marks	Marker's Comments

(VI) (α) Focus $= V(c\sqrt{2}, c\sqrt{2})$ ②

$x_V = \dfrac{a^2}{c(t_1+t_2)} = c\sqrt{2}$

$y_V = \dfrac{b^2 t_1 t_2}{c(t_1+t_2)} = c\sqrt{2}$

① relating a and b

$\therefore \dfrac{x_V}{y_V} = \dfrac{a^2}{b^2 t_1 t_2} = 1$

$a^2 = b^2 t_1 t_2 \qquad t_1 t_2 = \frac{a}{b}$

$a^2 = b^2 \frac{a}{b} \qquad a^2 = ab$

$\therefore \frac{a}{b} = 1 \qquad a \neq 0 \; b \neq 0$

$a = b$.

① showing $a = b$ (with proof)

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad \therefore \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$

$\therefore x^2 + y^2 = a^2$

circle centre $(0,0)$ radius a units

(β) Focus lies on tangent to ellipse ③

Other methods possible.

$b^2 x x_1 + a^2 y_1 y = a^2 b^2$ (at x_1, y_1)

$a^2 x x_1 + a^2 y_1 y = a^4 \qquad a = b$

$x x_1 + y y_1 = a^2$

Tangent passes through focus $(c\sqrt{2}, c\sqrt{2})$

$c\sqrt{2} x_1 + c\sqrt{2} y_1 = a^2$

① expression for $x_1 + y_1$

$x_1 + y_1 = \dfrac{a^2}{c\sqrt{2}}$

But $x_1^2 + y_1^2 = a^2$ (circle).

$x_1^2 + 2x_1y_1 + y_1^2 = a^2 + 2x_1y_1$

$(x_1+y_1)^2 = a^2 + 2c^2$ as x_1, y_1 is on $xy = c^2$

$\dfrac{a^4}{2c^2} = a^2 + 2c^2$

① Quadratic Equation in a^2

$a^4 = 2c^2a^2 + 4c^4$

$a^4 - 2c^2a^2 - 4c^4 = 0$

$a^2 = \dfrac{2c^2 \pm \sqrt{4c^4 + 16c^4}}{2}$

$a^2 > 0 \qquad a^2 = \dfrac{2c^2 + c^2\sqrt{20}}{2} = c^2 + c^2\sqrt{5}$

① Solution

$a > 0$, $c > 0$ $\qquad a = c\sqrt{1+\sqrt{5}}$

MATHEMATICS Extension 2 : Question 16.

Suggested Solutions	Marks	Marker's Comments

9:15

16 a) $I_n = \displaystyle\int_0^{2\pi} (1+\cos\theta)^n \, d\theta, \quad n \geq 0$

(i) Show $I_{n+1} = \dfrac{2n+1}{n+1} I_n$

$I_{n+1} = \displaystyle\int_0^{2\pi} (1+\cos\theta)^{n+1} \, d\theta$

$= \displaystyle\int_0^{2\pi} (1+\cos\theta)(1+\cos\theta)^n \, d\theta$

$= \displaystyle\int_0^{2\pi} (1+\cos\theta)^n \, d\theta + \int_0^{2\pi} \cos\theta (1+\cos\theta)^n \, d\theta$

Integrating by parts:

$u = (1+\cos\theta)^n \qquad v' = \cos\theta$

$u' = -n(1+\cos\theta)^{n-1} \cdot \sin\theta \qquad v = \sin\theta$

$I_{n+1} = I_n + \left\{ \left[(1+\cos\theta)^n \cdot \sin\theta \right]_0^{2\pi} + n\displaystyle\int_0^{2\pi} \sin^2\theta (1+\cos\theta)^{n-1} \, d\theta \right\}$

$= I_n + \left[(1+\cos 2\pi)^n \cdot \sin 2\pi - (1+\cos 0)^n \sin 0 \right]$

$+ n\displaystyle\int_0^{2\pi} (1-\cos^2\theta)(1+\cos\theta)^{n-1} \, d\theta$

$= I_n + 0 - n\displaystyle\int_0^{2\pi} (\cos^2\theta + 2\cos\theta - 2\cos\theta + 2 - 2 - 1)(1+\cos\theta)^{n-1} \, d\theta$

$= I_n - n\displaystyle\int_0^{2\pi} \left[(1+\cos\theta)^2 - 2(1+\cos\theta) \right] (1+\cos\theta)^{n-1} \, d\theta$

$= I_n - n\displaystyle\int_0^{2\pi} (1+\cos\theta)^{n+1} - 2(1+\cos\theta)^n \, d\theta$

$I_{n+1} = I_n - n I_{n+1} + 2n I_n$

$(n+1) I_{n+1} = (2n+1) I_n$

$I_{n+1} = \dfrac{2n+1}{n+1} I_n \quad \#$

(ii) Find I_3: $\quad I_0 = \displaystyle\int_0^{2\pi} (1+\cos\theta)^0 \, d\theta = \int_0^{2\pi} d\theta = 2\pi$

$I_1 = I_0 = 2\pi$

$I_2 = \dfrac{2+1}{1+1} \cdot 2\pi = 3\pi$

$I_3 = \dfrac{2(2)+1}{2+1} \cdot 3\pi = 5\pi$

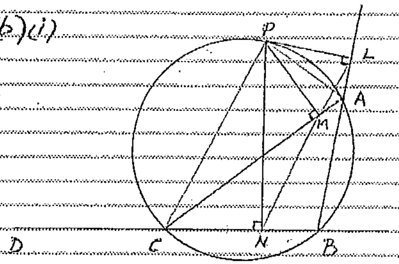
$\therefore I_3 = 5\pi \quad \#$

JC

MATHEMATICS Extension 2 : Question 16

Suggested Solutions | **Marks** | **Marker's Comments**

16(b)(i)



(1)

(ii) $P\hat{L}A + P\hat{M}A = 90° + 90°$ (L & M are the feet of the perpendiculars from P to AB & AC respectively)
$= 180°$
∴ PMAL is a cyclic quadrilateral (opposite angles are supplementary)
∴ P, M, A and L are concyclic points. (2)

(iii) $P\hat{M}C = P\hat{N}C = 90°$ (M & N are the feet of the perpendiculars from P to AC & CB resp'ly)
∴ PCNM is a cyclic quadrilateral (angles subtended by interval PC on the same side are equal)
∴ P, C, N & M are concyclic points (2)

(iv) Show: L, M and N are collinear

Constructions: Join ML, MN, PA & PC

Proof: $P\hat{C}B = P\hat{A}L$ (exterior angle of cyclic quad. PABC equals the interior opposite angle) (1)

$P\hat{A}L = P\hat{M}L$ (angles at the circumference in the same segment of cyclic quad. PMAL are equal) (1)

∴ $P\hat{C}B = P\hat{M}L$ (1)

Also, $P\hat{C}D = P\hat{M}N$ (exterior angle of cyclic quad. PCMN equals the interior opposite angle) (1)

$P\hat{M}L + P\hat{M}N = P\hat{C}B + P\hat{C}D$
$= 180°$ (straight angle BCD equals 180°)

Now $P\hat{M}L + P\hat{M}N = L\hat{M}N = 180°$
∴ $L\hat{M}N$ is a straight angle

∴ L, M & N are collinear (1)

+ 45 min
10:00

JC