Centre Number

Student Number

QATs
Quality Assessment Tasks

2013
HSC TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-14

Total marks – 70

Section I Pages 3-6
10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 7-12
60 marks
- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I (10 marks) **Marks**

(1) The point P divides the interval $A(\frac{17}{3},2)$ to $B(-3,4)$ **externally** in the ratio $2:3$. Which one of the following is the coordinates of point P? **1**

(A) $(-23,2)$.

(B) $(-9,-12)$.

(C) $(9,0)$.

(D) $(23,-2)$.

(2) Given that $xy = x+1$, the definite integral $\int_3^5 x\,dy$ equates to: **1**

(A) e^2.

(B) $\ln 2$.

(C) $-\ln\left(\frac{5}{3}\right)$.

(D) $e^{\frac{5}{3}}$.

(3) The number of ways that the letters of the word **TURRAMURRA** can be arranged is: **1**

(A) $151,200$.

(B) $37,800$.

(C) 10.

(D) $75,600$.

(4) A curve is defined by the parametric equations $x = \sin 2t$ and $y = \cos 2t$. Which of the following, in terms of t, equates to $\frac{dy}{dx}$? **Marks** **1**

(A) $-\tan 2t$.

(B) $2\tan 2t$.

(C) $2\sin 4t$.

(D) $\cos 4t$.

(5) Which of the following is the inverse function of $y = \frac{x-4}{x-2}$, $x \neq 2$? **1**

(A) $y = \frac{x-2}{x-4}$.

(B) $y = f^{-1}(y)$.

(C) $y = \frac{2(x-2)}{x-1}$.

(D) $y = \frac{x+4}{x+2}$.

(6) Which of the following is the coefficient of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^8$? **1**

(A) ${}^8C_3$.

(B) 1346.

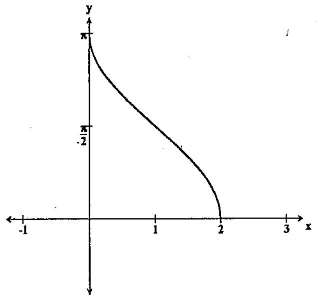
(C) 448.

(D) 1792

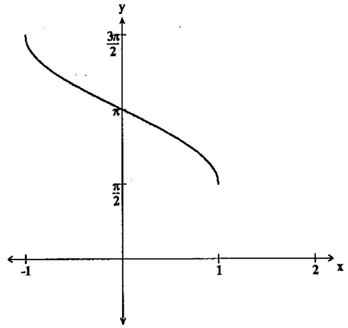
Marks

(7) Which of the following represents the graph of $y = \cos^{-1}(x+1)$. **1**

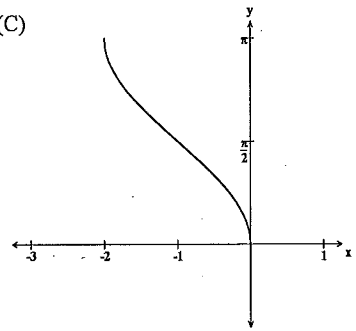
(A)



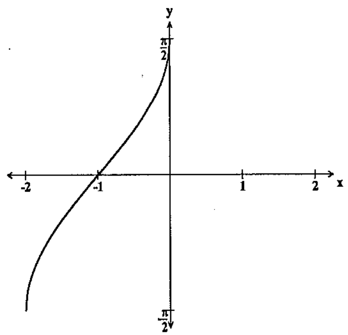
(B)



(C)



(D)



(8) The motion of a particle moving along the $x-axis$ executes simple harmonic motion. The maximum velocity of the particle is $4\ m/s$ and the period of motion is π seconds. Which of the following could be the displacement equation for this particle? **1**

(A) $x = 4\cos \pi t$.

(B) $x = -\sin 2t$.

(C) $x = 2\cos 2t$.

(D) $x = 2 + \cos 2t$.

Marks

(9) A particle moves with a velocity $v\ m/s$ where $v = \sqrt{x^2+1}$. Given that $x > 0$, which of the following is equal to the acceleration of the particle when $v = 4\,m/s$. **1**

(A) $\sqrt{17}\,m/s^2$.

(B) $-3\,m/s^2$.

(C) $\sqrt{15}\,m/s^2$.

(D) $2\sqrt{17}\,m/s^2$.

(10) Which of the following equates to the expression $\dfrac{1-e^{3x}}{1-e^{2x}}$. **1**

(A) $1+\dfrac{e^{2x}}{1+e^{x}}$.

(B) $1-e^{x}$.

(C) $1+e^{x}+e^{2x}$.

(D) None of the above.

Section II

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Write the domain of $y = \ln(x-2)$. **1**

(b) Find the exact value of $\sin 75^\circ$. **2**

(c) Given that the acute angle between the lines $y = mx$ and $2x - 3y = 0$ is 45°, find possible value(s) of x. **3**

(d) Using the substitution $u = 1 + x^2$, or otherwise, evaluate **3**

$$\int_0^{\sqrt{8}} \left(\frac{x}{\sqrt{1+x^2}} \right) dx.$$

(e) Solve the following inequality for x: **3**

$$\frac{1}{x} + \frac{x}{(x-2)} < 0$$

(f) (i) On the same number plane, graph the following functions: **2**

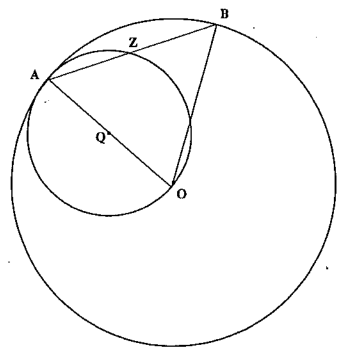
$y = 4 - x^2$ and $y = |3x|$

(ii) Hence or otherwise solve $4 - x^2 \le |3x|$ **1**

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of $\sin\left(2\cos^{-1}\frac{\sqrt{3}}{4}\right)$. **2**

(b) AB is a chord of a circle centre O. AO is a diameter of a circle centre Q. Z is the point where the circle centre Q meets AB.



NOT TO SCALE

(i) Explain why $AO = OB$. **1**

(ii) Hence or otherwise, prove that $AZ = ZB$. **2**

(c) The quadratic equation $x^2 - 4x + 9 = 0$ has roots $\tan A$ and $\tan B$. Hence, find the value(s) of $\angle(A+B)$, noting that $0 \le A+B \le 360^\circ$ *(leave your answer to the nearest degree)*. **3**

(d) (i) By use of long division, find the remainder, in terms of a and b when $P(x) = x^4 + 3x^3 + 6x^2 + ax + b$ is divided by $x^2 + 2x + 1$. **2**

(ii) If this remainder is $3x + 2$, find the values of a and b. **1**

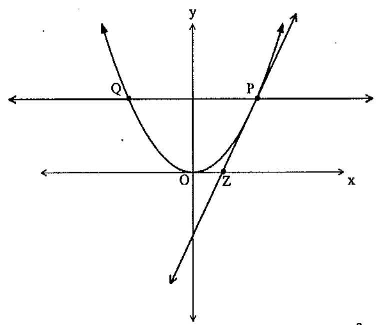
(e) (i) Prove that $\sin A \cos A \cos 2A = \frac{1}{4}\sin 4A$. **2**

(ii) Hence or otherwise solve $\sin A \cos A \cos 2A = 0$, for $0 \le x \le \frac{\pi}{2}$. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by mathematical induction that $5^n \geq 1+4n$ for all integers $n \geq 1$. **3**

(b)



$P(2ap, ap^2)$ and Q are variable points on the parabola $x^2 = 4ay$. The line PQ is parallel to the $x-axis$. The tangent at P meets the $x-axis$ at Z.

(i) Write down the equation of the tangent at P and hence show that $Z = (ap, 0)$. **2**

(ii) Find the locus of midpoints of QZ. **2**

(c) (i) Graph the function $y = 2\tan^{-1}(x)$. **1**

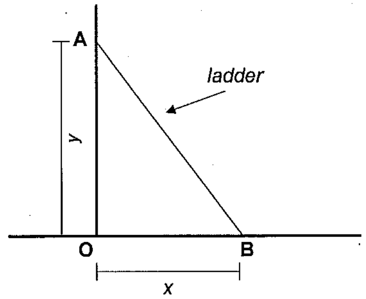
(ii) Graphically show why $2\tan^{-1}(x) - \frac{x}{4} = 0$ has one root, for $x > 0$. **1**

(iii) Taking $x_1 = 10$ as a first approximation to this root, use one application of Newton's method to find a better approximation, correct to 2 decimal places. **2**

Question 13 continues over the page

Marks

13 (d)



NOT TO SCALE

A ladder AB, 5 metres long, is leaning against a vertical wall OA (y metres), with its foot B, on horizontal ground OB (x metres). The foot of the ladder begins to slide along the ground away from the wall at a constant speed of 1 metre per second.

(i) Write down an equation relating x *and* y. **1**

(ii) Hence or otherwise, find the speed at which the top of the ladder A is moving down the wall at the time when the top of the ladder is 4 metres above the ground. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle is travelling in a straight line. Its displacement $(x\,cm)$ from O at a given time $(t\,\text{sec})$ after the start of motion is given by: $x = 2 + \sin^2 t$.

(i) Prove that the particle is undergoing simple harmonic motion. **2**

(ii) Find the centre of motion. **1**

(iii) Find the total distance travelled by the particle in the first $\frac{3\pi}{2}$ seconds. **2**

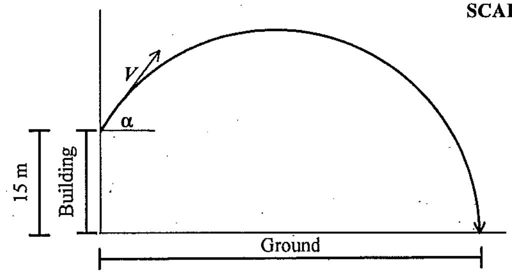
(b) For 10 consecutive sets of traffic light, the probabilities that each set would be green, red or yellow are 60%, 30%, 10% respectively. Andre drives through these sets of lights on his way home.

(i) Find the probability that all sets of lights are yellow (*leave your answer in index form*). **1**

(ii) Find the probability that exactly 2 sets of lights are yellow (*leave your answer to 3 decimal places*). **1**

(iii) Find the probability that at most 2 sets of lights are yellow (*leave your answer to 3 decimal places*). **2**

(c) Over 80 years ago, during training exercises, the Army fired an experimental missile from the top of a building 15 m high with initial velocity (v) where $v = 130 m/s$, at an angle (α) to the horizontal. Noting that $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ and taking $g = 10 m/s^2$.



Question 14 continues over the page

Question 14 (continued) **Marks**

14 (d)

(i) Write down the six equations of motion. **2**

(ii) The rocket hit its intended target when its velocity reached $60\sqrt{5}\ m/s$. Find the horizontal distance that the missile travelled to hit its target. **2**

(iii) The rocket was designed to hit its target once the angle to horizontal of its flight path in a downward direction lies between 20^o and 30^o. Find the range of times after firing that this could happen. **2**

END OF PAPER

QATs
Quality Assessment Tasks

2013 HSC TRIAL EXAMINATION

MATHEMATICS EXTENSION 1 – MARKING GUIDELINES

Question 11 (15 marks)

11(a) (1 mark)

Let $(x-2)>0$

$x>2$

Domain $=\{x:x>2\}$

11(b) (2 marks)

$$\sin 75^o = \sin\left(45^o+30^o\right)$$
$$=\sin 45^o \cos 30^o + \cos 45^o \sin 30^o$$
$$=\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}.\frac{1}{2}$$
$$=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

11(c) (3 marks)

$$\tan 45^o = \left|\frac{m-\frac{2}{3}}{1+\frac{2m}{3}}\right|$$
$$\left|\frac{3m-2}{3}\div\frac{3+2m}{3}\right|=1$$
$$\left|\frac{3m-2}{3+2m}\right|=1$$
$$\frac{3m-2}{3+2m}=1 \quad or \quad \frac{3m-2}{3+2m}=-1$$
$$m=5 \quad or \quad m=-\frac{1}{5}$$

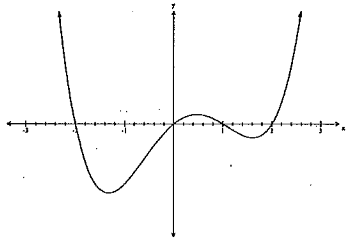
11(d) (3 marks)

$$u=1+x^2$$
$$\frac{1}{2}du = x\,dx$$
$$x=\sqrt{8}\rightarrow u=9$$
$$x=0 \quad \rightarrow u=1$$

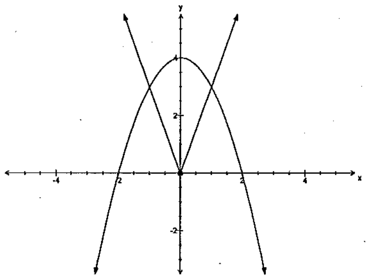
$$I=\int_0^{\sqrt{8}}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$$
$$I=\frac{1}{2}\int_1^9\left(\frac{du}{u^{\frac{1}{2}}}\right)$$
$$=\left[u^{\frac{1}{2}}\right]_1^9$$
$$=3-1$$
$$=2$$

11(e) (3 marks)

$$\frac{1}{x}+\frac{x}{(x-2)}<0$$
$$x(x-2)^2+x^3(x-2)<0$$
$$x(x-2)\left[(x-2)+x^2\right]<0$$
$$x(x-2)\left[x^2+x-2\right]<0$$
$$x(x-2)(x+2)(x-1)<0$$
$$-2<x<0 \qquad 1<x<2 \quad (\textit{from diagram})$$



11(f) (i) (2 marks)



11(f) (ii) (1 mark)

Solve

$4-x^2=|3x|$

$4-x^2=3x \quad or \quad 4-x^2=-3x$

$x^2+3x-4=0 \quad or \quad x^2-3x-4=0$

$(x+4)(x-1)=0 \;\; or \;\; (x-4)(x+1)=0$

$x=1,-4 \qquad x=4,-1$

check solutions

correct solutions $: x=\pm 1$

hence from diagram $x<-1$ *or* $x>1$

Question 12 (15 marks)

12(a) (2 marks)

$Let\ \alpha=\cos^{-1}\frac{\sqrt{3}}{4}$

$\therefore \sin\left(2\cos^{-1}\frac{\sqrt{3}}{4}\right)=\sin(2\alpha)$

Also $\cos\alpha=\frac{\sqrt{3}}{4}$, *hence* $\sin\alpha=\frac{\sqrt{13}}{4}$ *(pythagoras)*

$$\sin\left(2\cos^{-1}\frac{\sqrt{3}}{4}\right)=\sin(2\alpha)$$
$$=2\sin\alpha\cos\alpha$$
$$=2.\frac{\sqrt{13}}{4}.\frac{\sqrt{3}}{4}$$
$$=\frac{\sqrt{69}}{8}$$

12(b) (i) (1 mark)

$AO=OB$ (radii of circle centre O)

12(b) (ii) (2 marks)

If *from (i)* ΔOAB *is isosceles*

Also $\angle AZO=90^o$ *(angles in a semi-circle are right angles at the circumference)*

$\therefore AZ=ZB$ *(a line from the apex of an isosceles triangle which meets the base at right angles, bisects the base)*

12(c) (3 marks)

$\tan A+\tan B=-\frac{b}{a}$

$\tan A+\tan B=4$

$\tan A\tan B=\frac{c}{a}$

$\tan A\tan B=9$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$=\frac{4}{1-9}$$
$$=-\frac{1}{2}$$
$$\angle(A+B)=153^o26',\ 333^o26'$$
$$=153^o,\ 333^o$$

12(d) (i) (2 marks)

$P(x)=x^4+3x^3+6x^2+ax+b$

By long division

$P(x)=(x^2+2x+1)(x^2+x+3)+[(a-7)x+(b-3)]$

$\therefore R(x)=(a-7)x+(b-3)$

12(d) (ii) (1 mark)

$3x+2=(a-7)x+(b-3)$

$\therefore a=10$ and $b=5$

12(e) (i) (2 marks)

$\sin A \cos A \cos 2A = \frac{1}{4}\sin 4A$

$$
\begin{aligned}
LHS &= \frac{1}{2}[2\sin A\cos A \times \cos 2A] \\
&= \frac{1}{2}[\sin 2A \times \cos 2A] \\
&= \frac{1}{2}\times\frac{1}{2}[2\sin 2A \times \cos 2A] \\
&= \frac{1}{4}\sin 4A \\
&= RHS
\end{aligned}
$$

12(e) (ii) (2 marks)

$$
\begin{aligned}
\frac{1}{4}\sin 4A &= 0 \\
\sin 4A &= 0 \\
4A &= 0, \pi, 2\pi, 3\pi, 4\pi..... \\
A &= 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, ...
\end{aligned}
$$

Hence $A = 0, \frac{\pi}{4}, \frac{\pi}{2} \quad ; 0 \le A \le \frac{\pi}{2}$

Question 13 (15 marks)

13(a) (3 marks)

Step 1: Prove the expression is true for n=1

$5 \ge 5$ *(true)*

Step 2 : Assume the expression is true for n=k (where k is even)

$5^k \ge 1 + 4k$ (where k is a positive integer)

Step 3 : Prove the expression is true for n=k+1

$5^{k+1} \ge 1 + 4(k+1)$

$5.5^k \ge 4k + 5$

to prove $LHS \ge RHS$

prove $LHS - RHS \ge 0$

$LHS - RHS = 5.5^k - 4k - 5$

$\ge 5(1+4k) - 4k - 5$ (from assumption)

$= 16k \ge 0$ $(k \ge 1)$

Hence if the expression is true when n=k, it is true when n=k+1
If the expression is true for n=1, $\therefore$ it is true when n=2
If true for n=2, $\therefore$ it is true when n=3
Therefore the expression is true for all n, $n \ge 1$.

13(b) (i) (2 marks)

Tangent: $y = px - ap^2$

For Z: substitute $y=0$

$$
\begin{aligned}
px - ap^2 &= 0 \\
x &= ap \\
Z &= (ap, 0)
\end{aligned}
$$

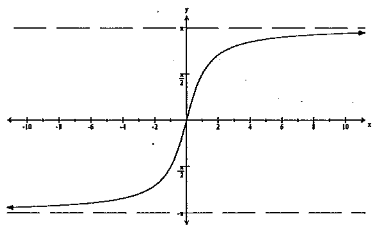
13(b) (ii) (2 marks)

$Q = (-2ap, ap^2)$ *symmetry of parabola*

$\therefore \text{midpoint}_{ZQ} = \left(\frac{-ap}{2}, \frac{ap^2}{2}\right)$

$$
\begin{aligned}
\text{hence } x &= \frac{-ap}{2} \\
p &= \frac{-2x}{a} \\
y &= \frac{ap^2}{2} \\
&= \frac{a}{2}\left(\frac{-2x}{a}\right) \\
y &= \frac{2x^2}{a} \\
2x^2 &= ay
\end{aligned}
$$

13(c) (i) (1 mark)

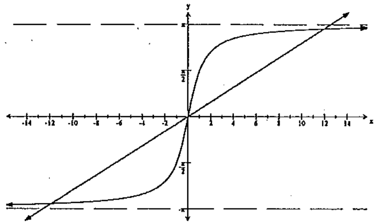


13(c) (ii) (1 mark)

$2\tan^{-1}(x) - \frac{x}{4} = 0$

$2\tan^{-1}(x) = \frac{x}{4}$

to solve, graph $y = 2\tan^{-1}(x)$ and $y = \frac{x}{4}$



13(c) (iii) (2 marks)

$P(x) = 2\tan^{-1}(x) - \frac{x}{4}$

$P'(x) = \frac{2}{1+x^2} - \frac{1}{4}$

$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$

$= 10 - \frac{P(10)}{P'(10)}$

$= 10 + \frac{0.4423}{0.230}$

$= 11.92$ (2 *decimal places*)

13(d) (i) (1 mark)

$x^2 + y^2 = 25$

13(d) (ii) (3 marks)

$x^2 + y^2 = 25$

$y = \sqrt{25 - x^2} \quad (y > 0)$

$\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$

Now $\frac{dx}{dt} = 1$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{-x}{\sqrt{25 - x^2}} \times 1$

Now when $y = 4$, $x = 3$

$\frac{dy}{dt} = \frac{-3}{\sqrt{25 - 3^2}} \times 1$

$= -\frac{3}{4}$ *(i.e. $\frac{3}{4}$ metres per second down the wall)*

Note: A quicker method using the implicit function rule can be used by Ext 2 students.

Question 14 (15 marks)

14(a) (i) (2 marks)

$x = 2 + \sin^2 t$

$v = 2\sin t \cos t$

$\ddot{x} = \cos t(2\cos t) + 2\sin t(-\sin t)$

$= 2\left[\cos^2 t - \sin^2 t\right]$

$= 2\left[1 - \sin^2 t - \sin^2 t\right]$

$= 2\left[1 - 2\sin^2 t\right]$

$= 2\left[1 - 2(x-2)\right]$

$= 2\left[1 - 2x + 4\right]$

$= 10 - 4x$

$= -4\left(x - \frac{5}{2}\right)$

Since in the form of $\ddot{x} = -n^2 x$, therefore SHM.

14(a) (ii) (1 mark)

$\ddot{x}=0$

$10-4x=0$

$x=\frac{5}{2}$

14(a) (iii) (2 marks)

$v=0$

$v=2\sin t\cos t$

$2\sin t\cos t=0$

$\therefore \sin t=0 \qquad \cos t=0$

$t=0,\pi,2\pi,3\pi... \qquad t=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}$

$t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}.....$

$t=0 \qquad x=2$

$t=\frac{\pi}{2} \qquad x=3$

$t=\pi \qquad x=2$

$t=\frac{3\pi}{2} \qquad x=2$

$\therefore$ total distance $=3\,cm$

14(b) (i) (1 mark)

$P(E)=\left(\frac{1}{10}\right)^{10}$

14(b) (ii) (1 mark)

$P(E)={}^{10}C_2\left(\frac{9}{10}\right)^8\left(\frac{1}{10}\right)^2$

$=0.1937$

$=0.194$

14(b) (iii) (2 marks)

$P(E)={}^{10}C_0\left(\frac{9}{10}\right)^{10}\left(\frac{1}{10}\right)^0+{}^{10}C_1\left(\frac{9}{10}\right)^9\left(\frac{1}{10}\right)^1+{}^{10}C_2\left(\frac{9}{10}\right)^8\left(\frac{1}{10}\right)^2$

$=0.3487+0.3874+0.1937$

$=0.9298$

$=0.930$

14(c) (i) (2 marks)

$\tan\alpha=\frac{5}{12}$, $\therefore \sin\alpha=\frac{5}{13}$ and $\cos\alpha=\frac{12}{13}$

$\ddot{x}=0 \qquad \ddot{y}=-10$

$\dot{x}=v\cos\alpha \qquad \dot{y}=-10t+v\sin\alpha$

$\dot{x}=130\times\frac{12}{13} \qquad \dot{y}=-10t+130\times\frac{5}{13}$

$\dot{x}=120 \qquad \dot{y}=-10t+50$

$x=120t \qquad y=-5t^2+50t+15$

14(c) (ii) (2 marks)

$$v^2=\left(\dot{x}\right)^2+\left(\dot{y}\right)^2$$

$$\left(60\sqrt{5}\right)^2=(120)^2+(-10t+50)^2$$

$$18000=14400+100t^2-1000t+2500$$

$100t^2-1000t-1100=0$

$t^2-10t-11=0$

$(t-11)(t+1)=0$

$t=11,-1$

$t=11 \ (t>0)$

$x=120(11)$

$x=1320m$

14(c) (iii) (2 marks)

$\tan\left(20^o\right)\le\left|\frac{\dot{y}}{\dot{x}}\right|\le\tan\left(30^o\right)$

$0.364\le\left|\frac{-10t+50}{120}\right|\le 0.577$

$43.68\le\left|-10t+50\right|\le 69.28$

As the flight path is in a downward direction $\dot{y}<0$.

Hence

$50-10t\le-48.68 \qquad 50-10t\ge-69.28$

$9.868\le t \qquad 11.928\ge t$

$9.868\le t\le 11.928$

$9.87\le t\le 11.93$