SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2015

HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Leave your answers in the simplest exact form, unless otherwise stated.
- Start each NEW question in a separate answer booklet.

Total Marks – 100

Section I Pages 1–5

10 Marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 6–13

90 marks

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Examiner: *P. Parker*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following represents $\dfrac{6}{3+\sqrt{3}i}$ in modulus-argument form?

(A) $\sqrt{3}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$

(B) $\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$

(C) $\sqrt{3}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

(D) $\sqrt{3}\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$

2 Which of the following is a correct expression for $\int x3^{x^2}\,dx$?

(A) $\dfrac{3^{x^2+1}}{x^2+1}+C$

(B) $\dfrac{3^{x^2}}{\ln 9}+C$

(C) $\dfrac{3^{x^2}}{\ln 3}+C$

(D) $3^{x^2}\ln 3+C$

3 Let $f(x)$ be a continuous, positive and decreasing function for $x>0$. Also, let $a_n=f(n)$.

Let $P=\int_1^6 f(x)\,dx$, $Q=\sum_{k=1}^{5} a_k$ and $R=\sum_{k=2}^{6} a_k$.

Which one of the following statements is true?

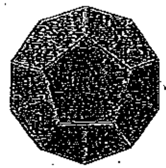
(A) $P<Q<R$

(B) $Q<P<R$

(C) $R<P<Q$

(D) $R<Q<P$

4 A 12-sided die is to be made by placing the integers 1 through 12 on the faces of a dodecahedron. How many different such dice are possible?



Here we consider two dice identical if one is a rotation of the other.

(A) $12!$

(B) $\dfrac{12!}{5}$

(C) $\dfrac{12!}{12}$

(D) $\dfrac{12!}{60}$

5 A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mk(v+v^3)$ Newtons when its speed is v m/s and k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity v m/s. Which of the following is an expression for x in terms of v?
Let g the acceleration due to gravity.

(A) $\dfrac{1}{k}\displaystyle\int \frac{1}{1+v^2}\,dv$

(B) $-\dfrac{1}{k}\displaystyle\int \frac{1}{1+v^2}\,dv$

(C) $\dfrac{1}{k}\displaystyle\int \frac{1}{v(1+v^2)}\,dv$

(D) $-\dfrac{1}{k}\displaystyle\int \frac{1}{v(1+v^2)}\,dv$

6 Let $g(x)$ be a function with first derivative given by $g'(x)=\int_0^x e^{-t^2}\,dt$.

Which of the following must be true on the interval $0<x<2$?

(A) $g(x)$ is increasing and the graph of $g(x)$ is concave up.

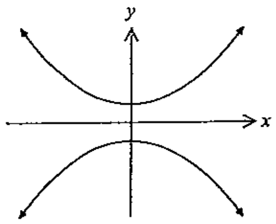
(B) $g(x)$ is increasing and the graph of $g(x)$ is concave down.

(C) $g(x)$ is decreasing and the graph of $g(x)$ is concave up.

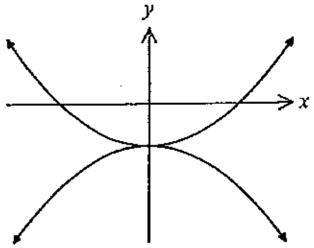
(D) $g(x)$ is decreasing and the graph of $g(x)$ is concave down.

7 Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?

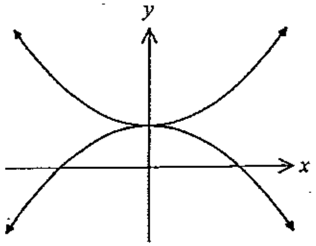
(A)



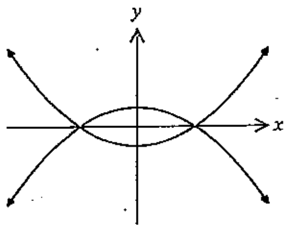
(B)



(C)



(D)



8 If $4x + \sqrt{xy} = y + 4$, what is the value of $\frac{dy}{dx}$ at (2, 8)?

(A) $\frac{20}{3}$

(B) $\frac{3}{20}$

(C) $-\frac{20}{3}$

(D) $-\frac{3}{20}$

9 For $z = a + ib$, $|z| = \sqrt{a^2 + b^2}$.

Let $\lambda = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$.

Which of the following is a correct expression for $|w|$, where $w = a + b\lambda$?

(A) $\sqrt{(a-b)^2 - ab}$

(B) $\sqrt{(a-b)^2 - 2ab}$

(C) $\sqrt{(a-b)^2 + ab}$

(D) $\sqrt{(a-b)^2 + 2ab}$

10 Kram was asked to evaluate $\binom{15}{0} + 3\binom{15}{1} + 5\binom{15}{2} + \ldots + (2n+1)\binom{15}{n} + \ldots + 31\binom{15}{15}$.

When told that he should use the fact that $\binom{15}{n} = \binom{15}{15-n}$, Kram was able to write down the value. What did he write down?

(A) 2^{15}

(B) 2^{16}

(C) 2^{19}

(D) 2^{31}

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hour and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a NEW Writing Booklet

(a) If $z = 2 - i$ express each of the following in the form $a + ib$, where a and b are real.

(i) $4z - 3$ **1**

(ii) $3z^2 - 2z + 1$ **2**

(b) Evaluate $\int_0^{\pi} x \cos \frac{1}{2} x \, dx$ **3**

(c) The complex number z moves such that $|z + 2| = -\text{Re } z$. **3**
Show that the locus of z is a parabola and find its focus and the equation of its directrix.

(d) Without the use of calculus, sketch the graph of $y = x - 1 - \dfrac{1}{(x-1)^2}$, **3**

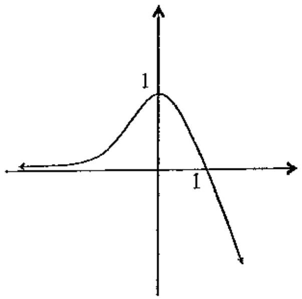
showing all intercepts and asymptotes.

(e) The region bounded by $y = x - x^2$ and $y = 0$ is rotated about the line $x = 2$. **3**

Using the method of cylindrical shells, find the volume of the solid formed.

Question 12 (15 Marks) Start a NEW Writing Booklet

(a) The graph of $y = f(x)$ is sketched below.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = e^{f(x)}$ **2**

(iii) $y = f(|x| + 1)$ **2**

(b) A curve is defined implicitly by $\tan^{-1} x^2 + \tan^{-1} y^2 = \dfrac{\pi}{4}$.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{x(1+y^4)}{y(1+x^4)}$. **2**

(ii) Using symmetry, or otherwise, sketch the curve. **3**

(c) The base of a solid S is the region enclosed by the graph of $y = \ln x$, the line $x = e$, and the x-axis. **4**
The cross sections of S perpendicular to the x-axis are squares.
What is the volume of S?

Question 13 (15 Marks) Start a NEW Writing Booklet

(a) A car, starting from rest, moves along a straight horizontal road.
The car's engine produces a constant horizontal force of magnitude 4000 newtons.
At time t seconds, the speed of the car is v m/s and a resistance force of magnitude $40v$ newtons acts upon the car.

The mass of the car is 1600 kg.

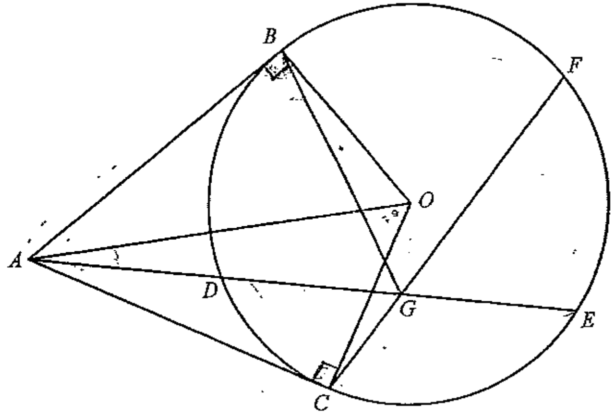
(i) Show that $\frac{dv}{dt} = \frac{100-v}{40}$ **2**

(ii) Find the velocity of the car at time t. **3**

(b) (i) Let $T = \tan\theta$ and $z = 1 + iT$.
Show that $z^3 = 1 - 3T^2 + i(3T - T^3)$ **1**

(ii) Hence find an expression for $\tan 3\theta$ only in terms of powers of $\tan\theta$. **2**

(c) In the diagram, AB and AC are tangents from A to the circle centre O, meeting the circle at B and C.
AE is a secant of the circle, intersecting it at D and E with G is the midpoint of DE.
CG produced meets the circle at F. You may assume that $ABOC$ is a cyclic quadrilateral.



Copy the diagram to your answer sheet.

(i) Show that $AOGC$ is a cyclic quadrilateral **3**

(ii) Construct BC and BF and let $\angle ABC = \theta$.
Prove that BF is parallel to AE. **4**

Question 14 (15 Marks) Start a NEW Writing Booklet

(a) The curve C has parametric equations

$$x = \frac{1}{\sqrt{(1+t^2)}} \text{ and } y = \ln\left(t + \sqrt{1+t^2}\right) \text{ for all real } t.$$

(i) Show that $\frac{dy}{dx} = -\frac{(1+t^2)}{t}$ **2**

(ii) Show that $\ln\left(-t + \sqrt{1+t^2}\right) = -\ln\left(t + \sqrt{1+t^2}\right)$ **1**

(iii) Deduce that C is symmetric about the x-axis. **1**

(iv) Show that the domain of C is $0 < x \leq 1$. **1**

(v) Sketch the graph of C. **2**

(b) A box contains six chocolates, two of which are identical.
From this box three chocolates are drawn without replacement.

(i) How many different selections could be made **2**

(ii) What is the probability that a selection will include the two identical chocolates? **1**

(c) For what values of k does the equation $3x^4 - 16x^3 + 18x^2 = k$ have four real solutions? **3**

(d) Find the polynomial equation of smallest degree that has rational coefficients and also has $-1+\sqrt{5}$ and $-6i$ as two of its roots. **2**

Question 15 (15 Marks) Start a NEW Writing Booklet

(a) By considering the expansion of $(1+i)^{2n}$ show that **3**

$$\sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k = 2^n \sin\left(\frac{n\pi}{2}\right)$$

(b) In an environment without resources to support a population greater than 1000, the population P at time t is governed by

$$\frac{dP}{dt} = P(1000-P)$$

(i) Show that $\ln\left(\frac{P}{1000-P}\right) = 1000t + C$, for some constant C. **3**

(ii) Hence show that $P = \frac{1000K}{K+e^{-1000t}}$, for some constant K. **3**

(iii) Given that initially there is a population of 200, determine at what time t, the population would reach 900. **2**

(c) Consider the real numbers $x_1, x_2, \ldots, x_n$, where $0 \le x_i \le 1$ for $i = 1, 2, \ldots, n$.

(i) Given that $(1-x_1)(1-x_2) \ge 0$, show that $2(1+x_1x_2) \ge (1+x_1)(1+x_2)$. **1**

(ii) Prove by mathematical induction that **3**

$$2^{n-1}(1+x_1 \times x_2 \times \cdots \times x_n) \ge (1+x_1)(1+x_2)\times \ldots \times (1+x_n)$$

for all positive integers n.

Question 16 (15 Marks) Start a NEW Writing Booklet

(a) A particle Q of mass 0.2 kg is released from rest at a point 7.2 m above the surface of the liquid in a container.
The particle Q falls through the air and into the liquid.
There is no air resistance and there is no instantaneous change of speed as Q enters the liquid.
When Q is at a distance of 0.8 m below the surface of the liquid, Q's speed is 6 m/s. The only force on Q due to the liquid is a constant resistance to motion of magnitude R newtons.
Take g, the acceleration due to gravity, to be 10 ms^{-2}.

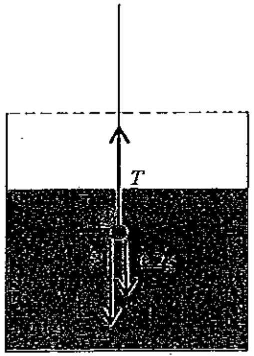
(i) Show that prior to entering the liquid that $\frac{dv}{dx} = \frac{10}{v}$. **1**

(ii) Hence find the speed as Q enters the liquid. **2**

(iii) Find the value of R. **3**

The depth of the liquid in the container is 3.6 m.
Q is taken from the container and attached to one end of a light inextensible string.
Q is placed at the bottom of the container and then pulled vertically upwards with constant acceleration.
The resistance to motion of R newtons continues to act.

The diagram below shows the forces acting on Q as it is being pulled out of the container.



The particle reaches the surface 4 seconds after leaving the bottom of the container.

(iv) By resolving the forces and finding an expression for $\frac{dv}{dt}$, find the tension in the string. **3**

Question 16 continues on page 13

Question 16 (continued)

(b) (i) Find the coordinates of the turning points of the curve $y = 27x^3 - 27x^2 + 4$. **2**

(ii) By sketching the curve, deduce that $x^2(1-x) \leq \frac{4}{27}$ for all $x \geq 0$. **2**

(iii) Three real numbers a, b and c lie between 0 and 1, prove that at least one of the numbers $bc(1-a)$, $ca(1-b)$ and $ab(1-c)$ is less than or equal to $\frac{4}{27}$. **2**

End of paper

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Sample Solutions

Question	Teacher
Q11	**JD**
Q12	**PB**
Q13	**BD**
Q14	**JD**
Q15	**AMG**
Q16	**AF**

MC Answers

Q1	B
Q2	B
Q3	C
Q4	D
Q5	B
Q6	A
Q7	B
Q8	A
Q9	C
Q10	C

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
B	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
C	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
D	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Section I 10 marks

1 Which of the following represents $\dfrac{6}{3+\sqrt{3}i}$ in modulus-argument form?

(A) $\sqrt{3}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$

(B) $\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$

(C) $\sqrt{3}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

(D) $\sqrt{3}\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$

$$\frac{6}{3+\sqrt{3}i}=\frac{6}{2\sqrt{3}\,\text{cis}\frac{\pi}{6}}=\frac{3}{\sqrt{3}}\text{cis}\left(-\tfrac{\pi}{6}\right)=\sqrt{3}\,\text{cis}\left(-\tfrac{\pi}{6}\right)$$

2 Which of the following is a correct expression for $\int x3^{x^2}\,dx$?

(A) $\dfrac{3^{x^2+1}}{x^2+1}+C$

(B) $\dfrac{3^{x^2}}{\ln 9}+C$

(C) $\dfrac{3^{x^2}}{\ln 3}+C$

(D) $3^{x^2}\ln 3+C$

$$\int x3^{x^2}\,dx=\frac{1}{2}\int 2x3^{x^2}\,dx=\frac{3^{x^2}}{2\ln 3}+C=\frac{3^{x^2}}{\ln 9}+C$$

Alternatively using the substitution $u=x^2$

$$\int x3^{x^2}\,dx=\frac{1}{2}\int 2x3^{x^2}\,dx=\frac{1}{2}\int 3^u\,du=\frac{3^u}{2\ln 3}+C=\frac{3^{x^2}}{\ln 9}+C$$

3 Let $f(x)$ be a continuous, positive and decreasing function for $x>0$. Also, let $a_n=f(n)$.

Let $P=\int_1^6 f(x)\,dx$, $Q=\sum_{k=1}^{5}a_k$ and $R=\sum_{k=2}^{6}a_k$.

Which one of the following statements is true?

(A) $P<Q<R$

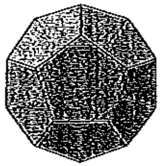
(B) $Q<P<R$

(C) $R<P<Q$

(D) $R<Q<P$

P is the exact value, Q is the upper sum since the graph is decreasing and R is the lower sum.

4 A 12-sided die is to be made by placing the integers 1 through 12 on the faces of a dodecahedron. How many different such dice are possible?



Here we consider two dice identical if one is a rotation of the other.

(A) $12!$

(B) $\frac{12!}{5}$

(C) $\frac{12!}{12}$

(D) $\frac{12!}{60}$

Since each face must receive a different number, start by counting 12! ways to assign the numbers.
However, there is no order to the faces on a die; it may be rolled around into many different orientations.
If the die is placed on a table, then any of the 12 faces (say, the one with the number 1 assigned to it) can be rotated to the top position.
Further, even after the location of this top face is chosen, there are still 5 ways in which it might be rotated about a line through the centers of the top and bottom faces (regular pentagons). That is, adjacent to the top face there are 5 faces from which to specify one as the front face.
Consequently, there are $12 \times 5 = 60$ ways to orient any numbering of the faces.
So the number of oriented numberings must be divided by 60.

Alternatively

Place the die on a surface. There are eleven possible numbers for the top face.
Below are two rings of 5 faces.
There are ${}^{10}C_5$ ways of selecting numbers for the top ring which can be arranged in 4! ways.
Then the remaining 5 faces can be numbered in 5! ways.

$\therefore 11 \times {}^{10}C_5 \times 4! \times 5! = \frac{11!}{5} = \frac{12!}{60}$

5 A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mk(v + v^3)$ Newtons when its speed is v m/s and k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity v m/s. Which of the following is an expression for x in terms of v?
Let g the acceleration due to gravity.

(A) $\frac{1}{k}\int \frac{1}{1+v^2}\,dv$

(B) $-\frac{1}{k}\int \frac{1}{1+v^2}\,dv$

(C) $\frac{1}{k}\int \frac{1}{v(1+v^2)}\,dv$

(D) $-\frac{1}{k}\int \frac{1}{v(1+v^2)}\,dv$

By inspection:
Being resistance it must be B or D
To get x in terms of v then the standard approach is $v\frac{dv}{dx}$ and so a v would get cancelled.
$\therefore$ B

Directly:

$mv\frac{dv}{dx} = -mk\left(v+v^3\right) \Rightarrow \frac{dv}{dx} = -k\left(\frac{v+v^3}{v}\right)$

$\therefore \frac{dx}{dv} = -\frac{1}{k}\left(\frac{1}{1+v^2}\right)$

6 Let $g(x)$ be a function with first derivative given by $g'(x) = \int_0^x e^{-t^2}\,dt$.

Which of the following must be true on the interval $0 < x < 2$?

(A) $g(x)$ is increasing and the graph of $g(x)$ is concave up.

(B) $g(x)$ is increasing and the graph of $g(x)$ is concave down.

(C) $g(x)$ is decreasing and the graph of $g(x)$ is concave up.

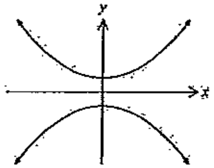
(D) $g(x)$ is decreasing and the graph of $g(x)$ is concave down.

As $e^{-t^2} > 0$, then $g'(x) = \int_0^x e^{-t^2}\,dt > 0$ i.e. $g(x)$ is increasing.

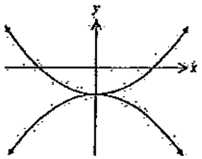
$g''(x) = \frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt\right) = e^{-x^2} > 0$ i.e. $g(x)$ is concave up

7 Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?

(A)



(B)

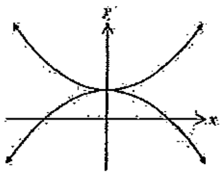


$x^4 = (y+1)^2$

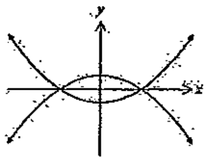
$\therefore y+1 = \pm x^2$

$\therefore y = \pm x^2 - 1$

(C)



(D)



8 If $4x + \sqrt{xy} = y + 4$, what is the value of $\frac{dy}{dx}$ at (2, 8)?

(A) $\frac{20}{3}$

(B) $\frac{3}{20}$

(C) $-\frac{20}{3}$

(D) $-\frac{3}{20}$

$4+\frac{1}{2}(xy)^{-\frac{1}{2}}\times(xy'+y)=y'$

$\therefore 4+\frac{1}{2}(16)^{-\frac{1}{2}}\times(2y'+8)=y'$

$\therefore 4+\frac{1}{8}\times(2y'+8)=y'$

$\therefore 4+1=y'-\frac{1}{4}y' \Rightarrow \frac{3}{4}y'=5$

$\therefore y'=\frac{20}{3}$

9 For $z = a + ib$, $|z| = \sqrt{a^2+b^2}$.

Let $\lambda = \frac{1}{2}\left(-1+i\sqrt{3}\right)$.

Which of the following is a correct expression for $|w|$, where $w = a + b\lambda$?

(A) $\sqrt{(a-b)^2 - ab}$

(B) $\sqrt{(a-b)^2 - 2ab}$

(C) $\sqrt{(a-b)^2 + ab}$

(D) $\sqrt{(a-b)^2 + 2ab}$

$a+b\lambda = a+\frac{1}{2}\left(-1+i\sqrt{3}\right)b$

$= \left(a-\frac{1}{2}b\right)+i\left(\frac{\sqrt{3}}{2}b\right)$

$|w| = \sqrt{\left(a-\frac{1}{2}b\right)^2+\left(\frac{\sqrt{3}}{2}b\right)^2}$

$= \sqrt{a^2-ab+\frac{1}{4}b^2+\frac{3}{4}b^2}$

$= \sqrt{a^2-ab+b^2}$

$= \sqrt{\left(a^2-2ab+b^2\right)+ab}$

$= \sqrt{\left(a-b\right)^2+ab}$

10 Kram was asked to evaluate $\binom{15}{0}+3\binom{15}{1}+5\binom{15}{2}+...+(2n+1)\binom{15}{n}+...+31\binom{15}{15}$.

When told that he should use the fact that $\binom{15}{n}=\binom{15}{15-n}$, Kram was able to write down the value. What did he write down?

(A) 2^{15}

(B) 2^{16}

(C) 2^{19}

(D) 2^{31}

Adding the *reverse* sum

$$\begin{array}{l} \binom{15}{0}+3\binom{15}{1}+5\binom{15}{2}+...+(2n+1)\binom{15}{n}+...+31\binom{15}{15} \\ 31\binom{15}{15}+29\binom{15}{14}+27\binom{15}{2}+...+(31-2n)\binom{15}{15-n}+...+\binom{15}{0} \\ \hline \end{array}$$

Now use the fact that $\binom{15}{n}=\binom{15}{15-n}$ i.e. $\binom{15}{0}=\binom{15}{15};\binom{15}{1}=\binom{15}{14};...$

$$\begin{aligned} \therefore 2\times\text{Sum} &= 32\times\left[\binom{15}{0}+\binom{15}{1}+\binom{15}{2}+...+\binom{15}{n}+...+\binom{15}{15}\right] \\ &= 32\times 2^{15} \\ &= 2^{20} \\ \therefore \text{Sum} &= 2^{19} \end{aligned}$$

SBHS MATHEMATICS EXT 2 TRIAL 2015 SOLUTIONS/COMMENTS

Q11. $z = 2 - i$

a) i) $4z - 3 = 4(2-i) - 3$
$= 5 - 4i$

1 mark. No problems

ii) $3z^2 - 2z + 1 = 3(2-i)^2 - 2(2-i) + 1$
$= 3(4 - 4i + i^2) - 4 + 2i + 1$
$= 3(3 - 4i) + 2i - 3$
$= 6 - 10i$

2 marks. A small number of students made simple errors. 1 mark awarded for error carried through.

b) $I = \int_0^{\pi} x \cos\frac{x}{2}\, dx$

LIATE. Let $u = x$ $\quad dv = \cos\frac{x}{2}\, dx$
$du = dx$ $\quad v = 2\sin\frac{x}{2}$

$I = \left[2x \sin\frac{x}{2} - 2\int \sin\frac{x}{2}\, dx\right]_0^{\pi}$

$= \left[2x \sin\frac{x}{2} + 4\cos\frac{x}{2}\right]_0^{\pi}$

$= 2\pi - 4$ 3 marks

Wrong use of limits −1
Some progress +1
None use of limits 2

c) $|z+2| = -\operatorname{Re} z$

Let $z = x + iy$

$|x + 2 + iy| = -x$

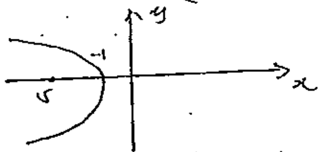
$\sqrt{(x+2)^2 + y^2} = -x$

$x^2 = x^2 + 4x + 4 + y^2$

$y^2 = -4x - 4$
$= 4(-1)(x+1)$

Parabola, vertex $(-1, 0)$

c) Continued (3 marks)

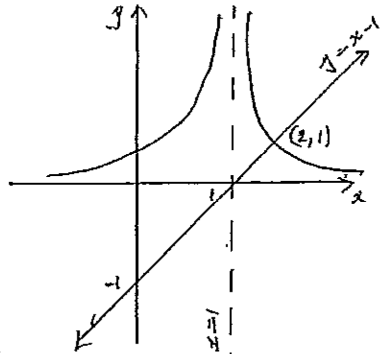


Focal length is −1
This gives focus as $(-2, 0)$ and directrix the y axis $x = 0$.

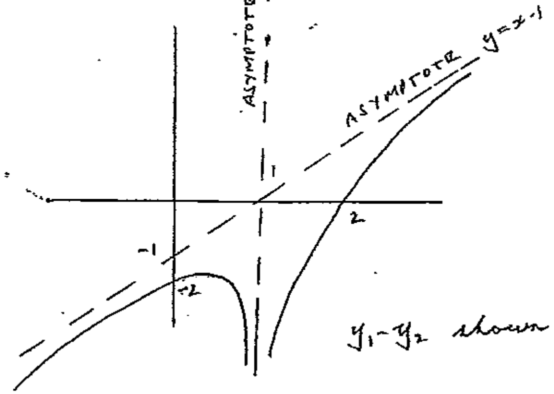
At least 20 students wrote $(x+2)^2 = x^2 + 2x + 4$ and finished up with
$y^2 = -2x - 4$
$y^2 = 4(-\tfrac{1}{2})(x+2)$
Being able to locate the focus at $(\tfrac{-5}{2}, 0)$ and directrix at $x = -3/2$ were able to score 2 marks.

Marks awarded were
Identifying parabola 1
Focus 1
Directrix 1

d) 3 MARKS



Graph shows $y_1 = x - 1$ and even function $y_2 = \frac{1}{(x-1)^2}$

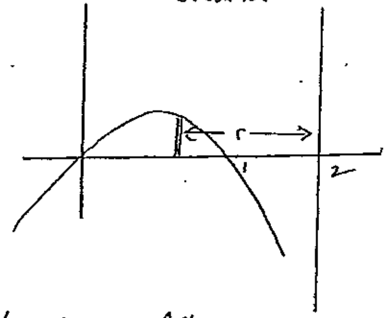


$y_1 - y_2$ shown by subtraction

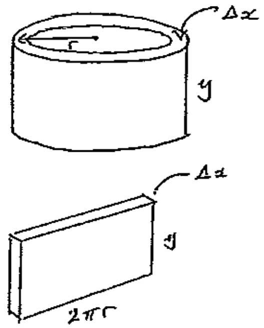
Intercepts 1
Asymptotes 1
Graph 1

Generally well done. Quite a few students went into too much detail when only a sketch was required.

d) 3 marks



$y = x - x^2$
$= x(1-x)$
$r = 2 - x$



$\Delta V \approx 2\pi r y \Delta x$

$V = \int_0^1 2\pi(2-x)(x-x^2)\,dx$

$= 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx$

$= 2\pi \left[x^2 - x^3 + \frac{1}{4}x^4\right]_0^1$

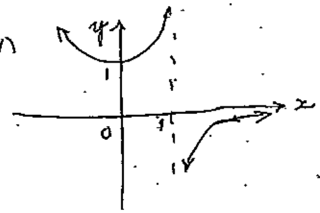
$= 2\pi\left[\frac{1}{4}\right]$

$= \frac{\pi}{2}$ cubic units.

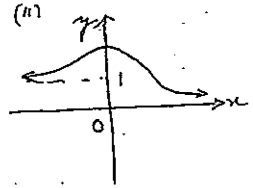
NOTES Many students used incorrect value of r (1½ marks)
2 marks for missing π
2 marks for simple mistake carried through
Correct using the wrong limits 2 marks
Small errors -½ each.
Question specifically asked for shell method.
No marks for volume by slicing.

QUESTION 12 (x2) ①

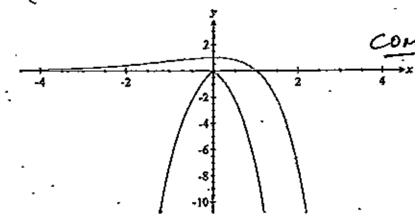
(a). (i)



(ii)



(iii)



COMMENT
(i) & (ii) were well done
(iii) proved more difficult.

(b) (i) $\tan^{-1}x^2 + \tan^{-1}y^2 = \frac{\pi}{4}$ (A)

Diff wrt x.

$\frac{2x}{1+x^4} + \frac{2y}{1+y^4}\cdot\frac{dy}{dx} = 0.$

$\frac{dy}{dx} = -\frac{x}{y}\cdot\frac{(1+y^4)}{(1+x^4)}$

COMMENT most were able to do this part.

(ii). Now $\tan(\tan^{-1}x^2 + \tan^{-1}y^2) = \tan\frac{\pi}{4}$

$\frac{x^2+y^2}{1-x^2y^2} = 1.$ (A1)

$\therefore y^2 = \frac{1-x^2}{1+x^2}$ or $x^2 = \frac{1-y^2}{1+y^2}$ (B)

$\therefore$ DOMAIN $|x| \le 1$

RANGE $|y| \le 1$.

Also at $x=0$, $y'=0$ $\therefore (0,1)$ & $(0,-1)$ are stationary

& at $y=0$ y' is undefined

$\therefore$ at $(1,0)$ & $(-1,0)$ are vertical tangents:

Also since the equation is SYMMETRICAL in $y = \pm x$.

(A) becomes $2\tan^{-1}x^2 = \frac{\pi}{4}$

$$\tan^{-1}x^2 = \frac{\pi}{8}.$$

$$x^2 = \tan\frac{\pi}{8}.$$

$$x = \pm\sqrt{\tan \pi/8}$$

$$\boxed{x \doteqdot \pm 0.64.}$$

OR. (A') becomes

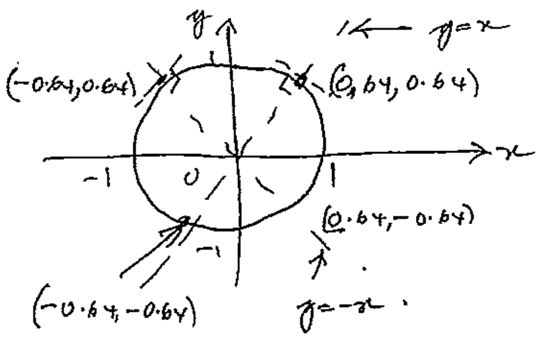
$$2x^2 = 1 - x^4$$

$$x^4 + 2x^2 - 1 = 0$$

$$x^2 = \frac{-2 \pm \sqrt{8}}{2}$$

$$= \sqrt{2} - 1. \Rightarrow x = \pm\sqrt{\sqrt{2}-1}.$$

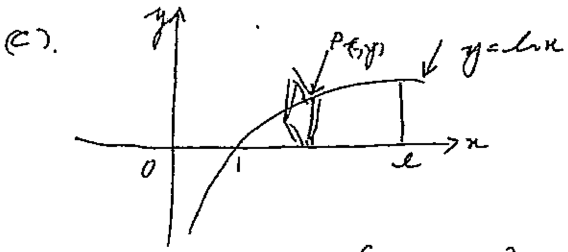
$$\boxed{x \doteqdot \pm 0.64}$$



COMMENT. Most were able to sketch a similar shape. Very few used the symmetry to find the intersection with $y = \pm x$.

④

(c).



$$\delta V = y^2 \delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=1}^{e} y^2 \delta x$$

$$= \int_1^e y^2 \, dx$$

$$= \int_1^e (\ln x)^2 \, dx \quad \text{(A)}$$

$$= \left[x (\ln x)^2 \right]_1^e - \int_1^e x \cdot 2 \ln x \times \frac{1}{x} \, dx$$

$$= e - 2 \int_1^e \ln x \, dx$$

$$= e - 2 \left[\left[x \ln x \right]_1^e - \int_1^e \frac{1}{x} \cdot x \, dx \right]$$

$$= e - 2 \left[e - (e-1) \right]$$

$$= e - 2 [1]$$

$$= (e - 2) \, u^3.$$

COMMENT most obtained full marks in this part. The integral in (A) was usually solved correctly.

X2 THSC 2015

Question 13

Average mark: 9.5/15

(a) $F = (4000 - 40v)$ N

(i) $\ddot{x} = \dfrac{4000 - 40v}{1600}$

$$\frac{dv}{dt} = \frac{100 - v}{40} \ \text{m s}^{-2}$$

②

Done well

0	0.5	1	1.5	2	Mean
2	3	3	0	107	1.9

(ii) $\displaystyle\int \frac{dv}{100 - v} = \frac{1}{40} \int dt$

$\therefore -\ln(100 - v) = \frac{1}{40} t + c$

When $t = 0$: $-\ln 100 = c$

$\therefore \frac{1}{40} t = \ln \dfrac{100}{100 - v}$

$\therefore e^{\frac{1}{40} t} = \dfrac{100}{100 - v}$

$\therefore \dfrac{100 - v}{100} = e^{-\frac{1}{40} t}$

$\therefore 100 - v = 100 \, e^{-\frac{1}{40} t}$

$\therefore v = 100 \left(1 - e^{-\frac{1}{40} t}\right)$

③

Done well

0	0.5	1	1.5	2	2.5	3	Mean
4	0	5	1	28	14	63	2.49

(b) (i) $z = 1 + iT$

$\therefore z^3 = (1 + iT)^3$

$= 1 + 3iT + 3(iT)^2 + (iT)^3$

$= 1 + 3iT - 3T^2 - iT^3$

$= (1 - 3T^2) + i(3T - T^3)$

①

Done well

0	0.5	1	Mean
1	1	113	0.99

(ii) $z^3 = \left(1 + i \dfrac{\sin\theta}{\cos\theta}\right)^3$

$= \dfrac{1}{\cos^3\theta} (\cos\theta + i\sin\theta)^3$

$= \dfrac{1}{\cos^3\theta} (\cos 3\theta + i \sin 3\theta)$

$$\tan 3\theta = \frac{\left(\dfrac{\sin 3\theta}{\cos^3\theta}\right)}{\left(\dfrac{\cos 3\theta}{\cos^3\theta}\right)}$$

$$= \frac{3T - T^3}{1 - 3T^2}$$

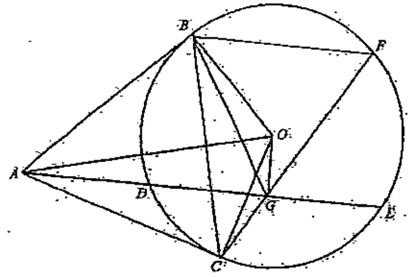
$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

②

A number of students did not follow the "Hence" instruction.

Some students invented a new DeMoivre's Theorem, suggesting that $(1 + i\tan\theta)^3 = 1 + i\tan 3\theta$

0	0.5	1	1.5	2	Mean
35	7	31	8	34	1.00



(c)

(i) As G is midpoint of DE,

$OG \perp DE$
(line joining midpoint of chord to centre of circle is perpendicular to chord).

$\angle ACO = 90^\circ$ (radius $\perp$ tangent at point of contact)

As $\angle AGO = \angle ACO$, $AOGC$ is a cyclic quadrilateral.
(angles on same segment equal) (3)

Realising that $OG \perp DE$ usually lead to a good attempt.

0	0.5	1	1.5	2	2.5	3	Mean
38	4	25	0	8	3	37	1.40

(ii) $\angle ABC = \angle AOC = \theta$
(angles in same segment, circle $ABOC$).

$\angle AOC = \angle AGC = \theta$
(angles in same segment, circle $AOGC$)

$\angle ABC = \angle BFC = \theta$
(alternate segment theorem).

$\therefore \angle BFC = \angle AGC = \theta$

$\therefore BF \parallel AE$
(corresponding $\angle$s equal) (4)

There are number of ways of proving the result. Those who used the cyclic quadrilateral had the greatest success.

0	0.5	1	1.5	2	2.5	3	3.5	4	Mn
39	1	18	10	11	4	11	4	27	1.79

Q 14

$x = (1+t^2)^{-1/2}$

$\frac{dx}{dt} = -\frac{1}{2}(1+t^2)^{-3/2} \cdot 2t$

$= \frac{-t}{(1+t^2)^{3/2}}$

$y = \ln\left(t + (1+t^2)^{1/2}\right)$

$\frac{dy}{dt} = \frac{1 + \frac{1}{2}(1+t^2)^{-1/2} \cdot 2t}{t + (1+t^2)^{1/2}}$

$= \frac{1 + t(1+t^2)^{-1/2}}{t + (1+t^2)^{1/2}}$

MULTIPLY TOP/BOTTOM by $(1+t^2)^{1/2}$

gives

$\frac{dy}{dt} = \frac{(1+t^2)^{1/2} + t}{(1+t^2)^{1/2}\left[t + (1+t^2)^{1/2}\right]}$

$= \frac{1}{\sqrt{1+t^2}}$

Then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{1}{(1+t^2)^{1/2}} \times \frac{(1+t^2)^{3/2}}{-t}$

$= -\frac{1+t^2}{t}$

2 marks

1 mark for $\frac{dy}{dt}$ and $\frac{dx}{dt}$ correct.

This question was extremely poorly set out by the majority of students. In many cases the standard of presentation was far below reasonable expectations. Some work was barely legible. This needs to be worked on. Many students presented the given answer without the adequate lead up. Consequently they did not score the marks.

ii RHS $= -\ln\left(t + \sqrt{1+t^2}\right)$

$= \ln\left(t + \sqrt{1+t^2}\right)^{-1}$

$= \ln\left[\frac{1}{t+\sqrt{1+t^2}} \times \frac{t - \sqrt{1+t^2}}{t - \sqrt{1+t^2}}\right]$

$= \ln\left(\frac{t - \sqrt{1+t^2}}{-1}\right)$

$= \ln\left(-t + \sqrt{1+t^2}\right)$

$=$ LHS

OR RHS $-$ LHS

$= \ln\left(-t + \sqrt{1+t^2}\right) + \ln\left(t + \sqrt{1+t^2}\right)$

$= \ln\left(\left(t + \sqrt{1+t^2}\right)\left(-t + \sqrt{1+t^2}\right)\right)$

$= \ln 1$

$= 0$

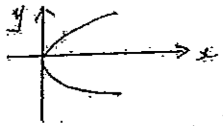
1 mark

Generally well done

Q 14 iii

This question was not answered very well at all by the majority. Many students have the comment "Not Shown" on their papers.

Consider the simple parabola



$y^2 = 4ax$ with parametric eqns $x = at^2$, $y = 2at$

Symmetric about the x axis

$y = f(t)$

$f(-t) = -2at$

$-f(t) = -2at$

ie $f(-t) = -f(t)$

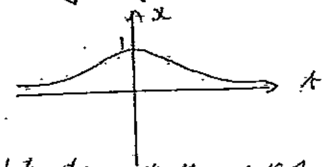
This is the statement that needs to be shown in this question

Hence symmetric about x axis

Many students used the fact that since $x = \dfrac{1}{\sqrt{1+t^2}}$

Then x has the same value for each $\pm t$.

This only refers to the x, t axes



Not the x, y axes

Hence they scored zero marks when they presented no further explanation.

SOLUTION (1 mark)

$y = \ln(t + \sqrt{1+t^2})$

$y = f(t)$

$f(-t) = \ln(-t + \sqrt{1+t^2})$

$-f(t) = -\ln(t + \sqrt{1+t^2})$

from answer ii

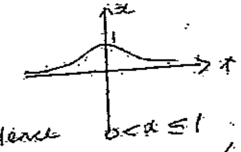
Q 14 iv

$x = \dfrac{1}{\sqrt{1+t^2}}$

$x \neq 0$, $x > 0$ since $\sqrt{1+t^2} > 0$

As $t \to \pm\infty$, $x \to 0$,

When $t = 0$, $x = 1$



Hence $0 < x \leq 1$

1 mark or no marks.

Many students went into far too much detail (unnecessary) for the 1 mark.

v) $\dfrac{dx}{dy} = \dfrac{-t}{1+t^2}$, $\dfrac{dx}{dy} = 0$ when $t = 0$

When $t = 0$, $x = 1$, $y = 0$

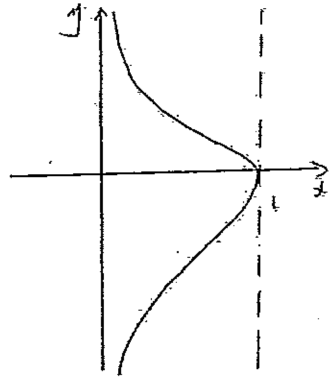
$\dfrac{d^2x}{dy^2} = \dfrac{(1+t^2)(-1) + (-t)(2t)}{(1+t^2)^2}$

$= \dfrac{-1-3t^2}{(1+t^2)^2}$

Then $\dfrac{d^2x}{dy^2} = -1$ when $t = 0$

Hence MAX.

NO student did this operation



If x is very small positively $y \doteqdot \ln\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\right)$

(Since $y = \ln\left[\dfrac{\sqrt{1-x^2}}{x} + \dfrac{1}{x}\right]$ on eliminating t between x & y)

ie y is very large

ε is a very small number

This information provides the sketch in quadrant 1

Symmetry provides the sketch in quadrant 4

NOTE

Despite being told symmetry exists. Many students drew the graph only in quadrant 1.

Q.14 b CHOCOLATES.

Generally very poorly answered.
Consider 6 different chocolates

A, B, C, D, E, F

Number of ways of choosing 3 from 6
is $^6C_3 = 20$

If 2 of the chocolates are the same
say A & B
Then choosing A and omitting B is the same choice as choosing B and omitting A.
That is, now only 2 chocolates need to be chosen
from the remaining 4 C, D, E, F. $= {}^4C_2 = 6$
That is there are 6 pairs of combinations from the 20 that are identical

Then there are $20 - 6 = 14$ different selections.

ie $^6C_3 - {}^4C_2 = 14$ (2 marks)

If both A and B are chosen then there is now only 1 selection to be made from 4

$= {}^4C_1$
$= 4$

Hence $P(\text{choosing 2 identical}) = \frac{4}{14}$

$= \frac{2}{7}$ (1 mark)

ie $\dfrac{^4C_1}{^6C_3 - {}^4C_2}$

NOTE
Students who had part i incorrect but still had the (4 choices identical) still scored 1 mark SEE OVER

CHOCOLATES.

Consider the 6 chocolates as different A, B, C, D, E, F
choosing 3 from 6 gives $^6C_3 = 20$ combinations
To be ABSOLUTELY CLEAR it is not too difficult to list the 20

1) ABC
2) ABD
3) ABE
4) ABF
5) ACD
6) ACE
7) ACF
8) ADE
9) ADF
10) AEF
11) BCD
12) BCE
13) BCF
14) BDE
15) BDF
16) BEF
17) CDE
18) CDF
19) CEF
20) DEF

If 2 chocolates are the same say A and B

Then the 12 combinations arrowed appear in pairs

ie 6 different combinations

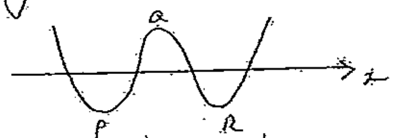
That is a total of 14 different selections

ie $P(\text{2 identical}) = \frac{4}{14}$

Selections 1) 2) 3) 4) contain both A & B

14c

Consider $y = 3x^4 - 16x^3 + 18x^2 - k$ having 4 distinct real roots then the graph must be



with turning points P, Q, R

These occur when $y' = 0$

$12x^3 - 48x^2 + 36x = 0$

$12x(x^2 - 4x + 3) = 0$

$12x(x-1)(x-3) = 0$

ie when $x = 0, 1, 3$

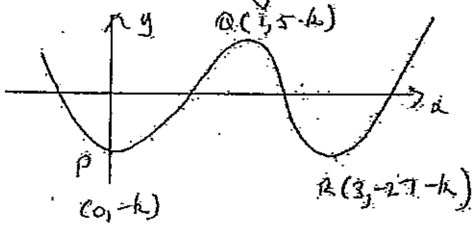
Substituting these values for y

$x = 0$, $y = -k$

$x = 1$, $y = 5 - k$

$x = 3$, $y = -27 - k$

Hence the above graph now becomes

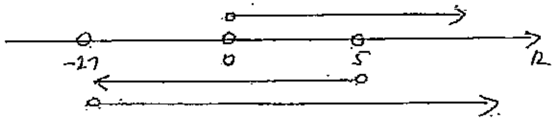


so From diagram $-k < 0$ ie $k > 0$

$5 - k > 0$ ie $k < 5$

$-27 - k < 0$ ie $k > -27$

The only solution common to the above is $0 < k < 5$



Q 14c continued

3 marks

Part marks awarded for either but not all

1 for turning points or y values (not both)

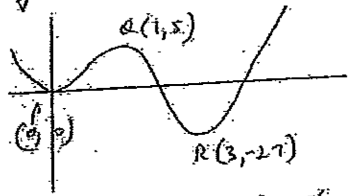
1 for some progress

No penalty for $0 \leq x \leq 5$

Solution "almost" complete but without explanation

2 marks.

Many students looked at the graph $y = 3x^4 - 16x^3 + 18x^2$



and came up with the correct answer $0 < k < 5$ but DID NOT provide EXPLANATION

These students scored 2 marks out of the 3.

- All necessary working should be shown in every question if full marks are to be awarded.

QUESTION 14d (3 marks)

If $-6i$ is a root of the polynomial so is $6i$

$(x-6i)(x+6i) = x^2+36$

The quadratic that has a root $-1+\sqrt{5}$

ie has $\frac{-2+2\sqrt{5}}{2}$ as a root

ie $\frac{-2+\sqrt{20}}{2}$ is a root

$\frac{-2\pm\sqrt{20}}{2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Hence $a=1$, $b=2$, $b^2-4ac=20$

$4-4c=20$

$-4c=16$

$c=-4$

ie x^2+2x-4

Hence polynomial of smallest degree with rational coefficients is $(x^2+36)(x^2+2x-4)$

$x^4+2x^3+32x^2+72x-144=0$

OR

Use $x=-1+\sqrt{5}$

$x+1=\sqrt{5}$

$(x+1)^2=5$

$x^2+2x+1=5$

$x^2+2x-4=0$

{2 marks for simple mistake carried through

{"Some progress" $1\frac{1}{2}$

$-\frac{1}{2}$ for each simple mistake eg + instead of −

Some students used the roots as $6i, -6i, -1+\sqrt{5}, -1-\sqrt{5}$ as the roots & then used the sum-products results for the roots of polynomials to find the coefficients. This is much more difficult and prone to error.

Question 15

(a) $(1+i)^{2n} = \left(\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^{2n}$

LHS

$= {}^{2n}C_0 + {}^{2n}C_1 i + {}^{2n}C_2 i^2 + {}^{2n}C_3 i^3 + {}^{2n}C_4 i^4 + {}^{2n}C_5 i^5 + \ldots + {}^{2n}C_{2n-1} i^{2n-1} + {}^{2n}C_{2n} i^{2n}$

$= {}^{2n}C_0 + {}^{2n}C_1 i - {}^{2n}C_2 - {}^{2n}C_3 i + {}^{2n}C_4 + {}^{2n}C_5 i - \ldots + {}^{2n}C_{2n-1} i^{2n-1}$

Now $\text{Im[LHS]} = {}^{2n}C_1 - {}^{2n}C_3 + {}^{2n}C_5 - {}^{2n}C_7 + \ldots - {}^{2n}C_{2n-1}$

$= \sum_{k=0}^{n-1} {}^{2n}C_{2k+1}(-1)^k$

RHS $= \left(\sqrt{2}\right)^{2n}\left(\cos\left(\frac{2n\pi}{4}\right) + i\sin\left(\frac{2n\pi}{4}\right)\right)$ (de Moivre's Theorem)

$= 2^n\left(\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)\right)$

Thus $\text{Im[RHS]} = 2^n \sin\left(\frac{n\pi}{2}\right)$

$= \text{Im[LHS]}$

Hence $\sum_{k=0}^{n-1} {}^{2n}C_{2k+1}(-1)^k = 2^n \sin\left(\frac{n\pi}{2}\right)$ as required.

Comments: Well answered generally. Those who lost marks failed to see the connection between the imaginary parts.

(b) $\frac{dP}{dt} = P(1000-P)$

(i) $\frac{dt}{dP} = \frac{1}{P(1000-P)}$

Integrating w.r.t. P:

$t = \int \frac{1}{P}.\frac{1}{1000-P}\,dP + C$

Partial Fractions:

$\frac{1}{P(1000-P)} \equiv \frac{A}{P} + \frac{B}{1000-P}$

$1 \equiv A(1000-B) + BP$

Hence $A = B = \frac{1}{1000}$

$\therefore t = \frac{1}{1000}\left(\int \frac{dP}{P} + \int \frac{dP}{1000-P}\right) + C$

$= \frac{1}{1000}\left(\ln P - \ln(1000-P)\right) + C$

$$\therefore \ln\left(\frac{P}{1000-P}\right)=1000t+C \qquad \text{as required.}$$

Alternatively:

$$1000t+C=\ln\left(\frac{P}{1000-P}\right)$$

Differentiating w.r.t. P:

$$1000\frac{dt}{dP}=\frac{1}{\left(\frac{P}{1000-P}\right)}\frac{d}{dP}\left(\frac{P}{1000-P}\right)$$

$$=\frac{1000-P}{P}\left[\frac{(1000-P).1-P.(-1)}{(1000-P)^2}\right]$$

$$=\frac{1}{P}\left[\frac{1000}{1000-P}\right]$$

$$\therefore \frac{dt}{dP}=\frac{1}{P(1000-P)}$$

Thus $\frac{dP}{dt}=P(1000-P)$, and $1000t+C=\ln\left(\frac{P}{1000-P}\right)$

is a solution.

Comments: Again very well answered, with most candidates using partial fractions, some by observation rather than formally.

(ii) From above, taking exponentials:

$$\frac{P}{1000-P}=e^{1000t+C}$$

$$=Ke^{1000t}$$

Thus $P=1000Ke^{1000t}-PKe^{1000t}$

$$P\left(1+Ke^{1000t}\right)=1000Ke^{1000t}$$

$$P=\frac{1000Ke^{1000t}}{1+Ke^{1000t}}$$

$$\therefore P=\frac{1000K}{K+e^{-1000t}} \quad \text{on division by } e^{1000t}.$$

Comments: Again very well answered, with most candidates getting the full 3 marks.

(iii) When $t=0,\ P=200$

So $200=\frac{1000K}{K+1}$

Hence $K=\frac{1}{4}$.

When the population is 900

$$900=\frac{1000\times 0.25}{0.25+e^{-1000t}}$$

$$\therefore e^{-1000t}=\frac{250}{900}-\frac{1}{4}$$

Taking natural logarithms:

$$-1000t=\ln\left(\tfrac{1}{36}\right)$$

$$t=\frac{\ln(36)}{1000}$$

$t\approx 0.0036$ (Assumedly the units are years)

Comments: Almost every candidate obtained this rather alarming result.

(c) $0\le x_i\le 1,\ \ i=1, 2, \dots, n$

(i) Given $(1-x_1)(1-x_2)\ge 0$, RTP $2(1+x_1x_2)\ge(1+x_1)(1+x_2)$

$$(1-x_1)(1-x_2)\ge 0$$
$$1-x_2-x_1+x_1x_2\ge 0$$
$$1-(x_1+x_2)+x_1x_2\ge 0$$
$$1+x_1x_2\ge x_1+x_2 \qquad \text{-----(1)}$$

Consider $2(1+x_1x_2)-(1+x_1)(1+x_2)$

$$=2(1+x_1x_2)-\left(1+(x_1+x_2)+x_1x_2\right)$$
$$=2(1+x_1x_2)-(1+x_1x_2)-(x_1+x_2)$$
$$=(1+x_1x_2)-(x_1+x_2)$$
$$\ge 0 \qquad \text{From (1)}$$

Thus $2(1+x_1x_2)\ge(1+x_1)(1+x_2)$

Comments: This was generally well done, although some assumed the result, and proceeded to beg the question.

(ii) $P(n):\ 2^{n-1}(1+x_1x_2\dots x_n)\ge(1+x_1)(1+x_2)\dots(1+x_n)$

$P(1):\ 2^0(1+x_1)\ge 1+x_1$

$LHS=1+x_1;\quad RHS=1+x_1$

$\therefore P(1)$ is true (equality)

$P(k)$: Assume the proposition is true for some positive integer k

Thus $2^{k-1}(1+x_1x_2\dots x_k)\ge(1+x_1)(1+x_2)\dots(1+x_k)$

$P(k+1)$: RTP that $P(k)$ implies $P(k+1)$

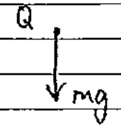
that is $2^k(1+x_1x_2\dots x_{k+1})\ge(1+x_1)(1+x_2)\dots(1+x_{k+1})$

$$RHS = (1+x_1)(1+x_2)\ldots(1+x_k)(1+x_{k+1})$$
$$\le 2^{k-1}(1+x_1x_2\ldots x_k)(1+x_{k+1}) \quad \text{by the assumption}$$
$$\le 2^{k}(1+x_1x_2\ldots x_kx_{k+1}) \quad \text{by part (i)}$$
$$= LHS$$
$$\therefore LHS \ge RHS$$

Hence by the principle of mathematical induction, the proposition is true for all $n \ge 1$.

Comments: Almost no candidates took the short route to proof shown above, but most who attempted it found a way.

16)a) i)



$ma = mg$

$\cancel{a = 10}$

$\cancel{\frac{v\,dv}{dx} = 10}$

$\frac{dv}{dx} = \frac{10}{v}$

ii) $\frac{dx}{dv} = \frac{v}{10}$

$x = \frac{v^2}{20} + C$

when $t=0$, $x=0$, $v=0$

$\therefore c = 0$

$x = \frac{v^2}{20}$

when $x = 7.2$

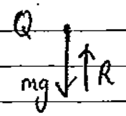
$7.2 = \frac{v^2}{20}$

$v^2 = 144$

$v = +12$

$v = 12\ ms^{-1}$

iii)



$ma = mg - R$ Note: R is a constant

$0.2a = 0.2 \times 10 - R$

$a = 10 - 5R$

$\frac{d(\frac{1}{2}v^2)}{dx} = 10 - 5R$

$\frac{1}{2}v^2 = (10 - 5R)x + C$

when $x = 0$, $v = 12$

$\frac{1}{2}(12)^2 = C$

$c = 72$

$\frac{1}{2}v^2 = (10-5R)x + 72$

when $x = 0.8$, $v = 6$

$\frac{1}{2}(6)^2 = (10-5R)(0.8) + 72$

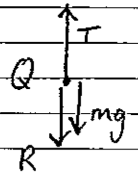
$(10-5R)(0.8) = -54$

$10 - 5R = -67.5$

$-5R = -77.5$

$R = 15.5\ N$

iv)



$ma = T - R - mg$

$0.2a = T - 15.5 - 0.2 \times 10$

$0.2a = T - 17.5$

$a = 5T - 87.5$

Note: since a is a constant T is a constant.

$\frac{dv}{dt} = 5T - 87.5$ ———— *

$v = (5T - 87.5)t + c$

when $t = 0$, $v = 0$

$\therefore c = 0$

$\frac{dx}{dt} = (5T - 87.5)t$

$x = (5T - 87.5)\frac{t^2}{2} + c$

when $t = 0$ $x = 0$

$\therefore c = 0$

$x = (5T - 87.5)\frac{t^2}{2}$

when $t = 4$, $x = 3.6$

$3.6 = (5T - 87.5)\frac{(4)^2}{2}$

$5T - 87.5 = 0.45$

$5T = 87.95$

$T = 17.59\ N$

COMMENT:

- Students should approach these questions by resolving forces. Many started with acceleration.
- Students should not use $\begin{cases} v = u + at \\ v^2 = u^2 + 2aS \\ S = ut + \frac{1}{2}at^2 \end{cases}$
- Definite integrals can be used. However, mistakes were made in (iv).

$\frac{dv}{dt} = 5T - 87.5$ ———— *

$dv = (5T - 87.5)dt$

$$\boxed{\int_0^v dv = \int_0^4 (5T - 87.5)dt}$$

this was a common mistake.

It should have been

$\int_0^v dv = \int_0^t (5T - 87.5)dt$

$v = (5T - 87.5)t$

$\frac{dx}{dt} = (5T - 87.5)t$

$\int_0^{3.6} dx = \int_0^4 (5T - 87.5)t\,dt$

$3.6 = \left[(5T - 87.5)\frac{t^2}{2}\right]_0^4$

$3.6 = (5T - 87.5)\frac{(4)^2}{2}$

$5T - 87.5 = 0.45$

$5T = 87.95$

$T = 17.59\ N$

b) i) $y = 27x^3 - 27x^2 + 4$

$y' = 81x^2 - 54x$

For stationary points let $y' = 0$

$27x(3x - 2) = 0$

$x = 0, \frac{2}{3}$

when $x = 0$, $y = 4$

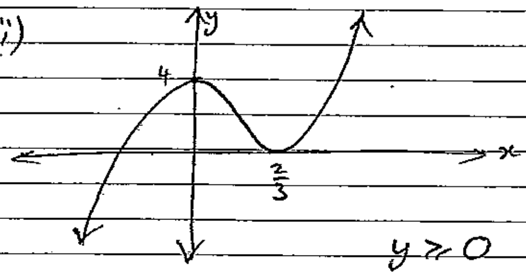
when $x = \frac{2}{3}$

$y = 27\left(\frac{2}{3}\right)^3 - 27\left(\frac{2}{3}\right)^2 + 4$

$= 0$

$\therefore$ Turning points at $(0, 4)$ & $\left(\frac{2}{3}, 0\right)$

ii)



$y \geq 0$

$27x^3 - 27x^2 + 4 \geq 0$

$4 \geq 27x^2 - 27x^3$

$4 \geq 27x^2(1 - x)$

$\frac{4}{27} \geq x^2(1 - x)$

$\therefore x^2(1 - x) \leq \frac{4}{27}$

iii) Consider $0 < a \leq b \leq c < 1$ —— (1)

$0 > -a \geq -b \geq -c > -1$

$-1 < -c \leq -b \leq -a < 0$

$0 < 1 - c \leq 1 - b \leq 1 - a < 1$ —— (2)

From (1) $a \leq b$

$ab \leq b^2 \qquad (b > 0)$

$ab(1-c) \leq b^2(1-c) \qquad (1 - c > 0)$

From (2) $b^2(1-c) \leq b^2(1-b)$

$\therefore ab(1-c) \leq b^2(1-c) \leq b^2(1-b)$

From (ii) $b^2(1-b) \leq \frac{4}{27}$

$\therefore ab(1-c) \leq \frac{4}{27}$

And so at least one of $ab(1-c)$, $bc(1-a)$, $ca(1-b)$ is less than or equal to $\frac{4}{27}$.

iii) since $a > 0$

$$a^2(1-a) \le \frac{4}{27} \quad \text{from (ii)}$$

Also $0 < a^2(1-a)$

$$0 < a^2(1-a) \le \frac{4}{27} \quad \text{—— (1)}$$

Similarly, $0 < b^2(1-b) \le \frac{4}{27}$ —— (2)

$$0 < c^2(1-c) \le \frac{4}{27} \quad \text{—— (3)}$$

(1) × (2) × (3)

$$0 < a^2(1-a).b^2(1-b).c^2(1-c) \le \left(\frac{4}{27}\right)^3$$

$$0 < bc(1-a).ca(1-b).ab(1-c) \le \left(\frac{4}{27}\right)^3 \quad \text{—— (4)}$$

Proof by contradiction:

Assume that $bc(1-a)$, $ca(1-b)$ and $ab(1-c)$ are all greater than $\frac{4}{27}$

ie $bc(1-a) > \frac{4}{27}$ —— (5)

$$ca(1-b) > \frac{4}{27} \quad \text{—— (6)}$$

$$ab(1-c) > \frac{4}{27} \quad \text{—— (7)}$$

(5) × (6) × (7)

$$bc(1-a).ca(1-b).ab(1-c) > \left(\frac{4}{27}\right)^3$$

This contradicts (4)

Assumption is false.

∴ At least one of $bc(1-a)$, $ca(1-b)$ and $ab(1-c)$ is less than or equal to $\frac{4}{27}$.

COMMENT:

Part (i) & (ii) were done well by students. A small number of students assumed the result in (ii) which is not a valid form of proof.

Not many students made any progress with (iii)