Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12
Mathematics Extension 1

Trial HSC Examination

2015

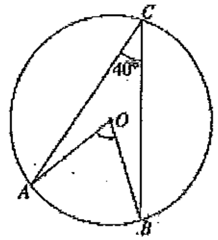
Time allowed: 120 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1	Multiple Choice Questions 1-10 10 Marks	
Section II	Questions 11-14 60 Marks	

1. The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle ACB$ is $40°$.



NOT TO SCALE

What is the size of $\angle AOB$?

(A) $20°$

(B) $40°$

(C) $70°$

(D) $80°$

2. The angle θ satisfies $\sin\theta = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$.

What is the value of $\sin 2\theta$?

(A) $\frac{10}{13}$

(B) $-\frac{10}{13}$

(C) $\frac{120}{169}$

(D) $-\frac{120}{169}$

3. Which of the following is the correct expression for $\int \frac{dx}{\sqrt{49-x^2}}$?

(A) $-\cos^{-1}\frac{x}{7}+c$

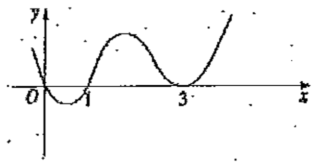
(B) $-\cos^{-1}7x+c$

(C) $-\sin^{-1}\frac{x}{7}+c$

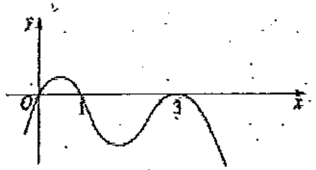
(D) $-\sin^{-1}7x+c$

4. Which diagram best represents the graph $y = x(1-x)^3(3-x)^2$?

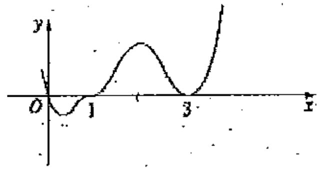
(A)



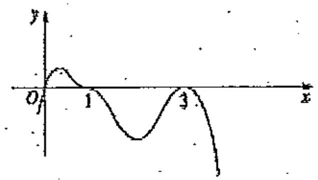
(B)



(C)



(D)



5. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

6. Which of the following is an expression for $\int \sin^2 6x dx$?

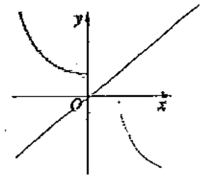
(A) $\frac{x}{2} - \frac{1}{24}\sin 6x + c$

(B) $\frac{x}{2} + \frac{1}{24}\sin 6x + c$

(C) $\frac{x}{2} - \frac{1}{24}\sin 12x + c$

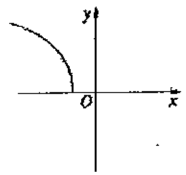
(D $\frac{x}{2} + \frac{1}{24}\sin 12x + c$

7. The diagram shows the graph $y = f(x)$.

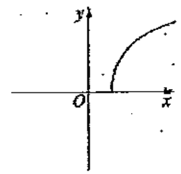


Which diagram shows the graph $y = f^{-1}(x)$?

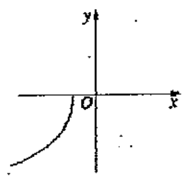
(A)



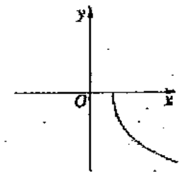
(B)



(C)

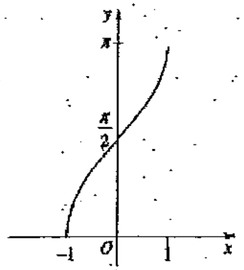


(D)



8.

The diagram shows the graph of a function.



Which function does the graph represent?

(A) $y=\cos^{-1}x$

(B) $y=\frac{\pi}{2}+\sin^{-1}x$

(C) $y=-\cos^{-1}x$

(D) $y=-\frac{\pi}{2}-\sin^{-1}x$

9.

Which inequality has the same solution as $|x+2|+|x-3|=5$?

(A) $\frac{5}{3-x}\geq 1$

(B) $\frac{1}{x-3}-\frac{1}{x+2}\leq 0$

(C) $x^2-x-6\leq 0$

(D) $|2x-1|\geq 5$

10.

After t minutes the temperature (T) of a bottle of water placed in a refrigerator is given by $T=A+Be^{kt}$ where A is the temperature inside the refrigerator and B and k are constants.

A bottle of water with a temperature of $20^\circ C$ is placed in a refrigerator with temperature of $2^\circ C$. If 20 minutes later the temperature of the bottle of water has decreased to $10^\circ C$ then the value of k is?

(A) $k=-\frac{1}{20}\log_e\frac{9}{4}$

(B) $k=-\frac{1}{10}\log_e\frac{4}{9}$

(C) $k=\frac{1}{20}\log_e\frac{9}{4}$

(D) $k=\frac{1}{10}\log_e\frac{4}{9}$

Question 11 (15 Marks) Start a fresh sheet of paper. **Marks**

(a) Solve the inequality $\dfrac{4-2x}{x+5} \le 2$ **2**

(b) Find $\lim\limits_{x \to 0} \dfrac{\sin\left(\frac{\pi}{4}x\right)}{2x}$ **2**

(c) Find, to the nearest degree, the acute angle between the lines:

$x - 2y + 1 = 0$ and $y = 5x - 4$ **2**

(d) Differentiate $\ln(\sin^{-1} 2x)$ **2**

(e) Find the Cartesian equation of the parabola given $x = t - 2$ and $y = 3t^2 - 1$. **1**

(f) i. Prove that $\sin\theta \sec\theta = \tan\theta$ **1**

ii. Hence solve $\sin\theta \sec\theta = \sqrt{3}$. $(0 \le \theta \le 2\pi)$ **1**

(g) Find the remainder when the polynomial $P(x) = x^3 - 4x$ is divided by $x + 3$. **1**

(h) Given that $\log_3 7 = m$, find an expression for $\log_3 21$ **1**

(i) Differentiate the following, expressing your answer as a single fraction: **2**

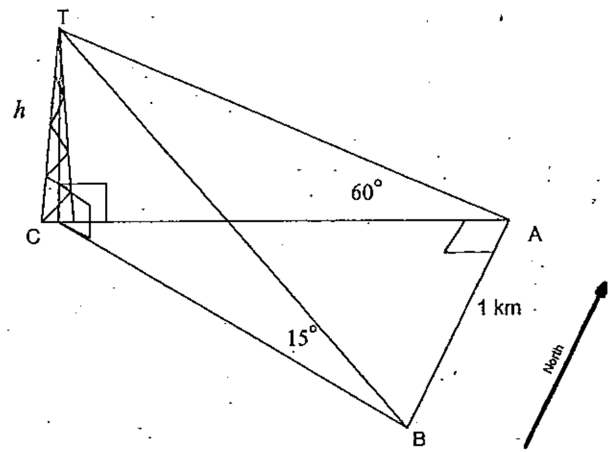
$$\frac{2x^2 - 4x}{x\sqrt{x}}$$

Question 12 (15 Marks) Start a fresh sheet of paper. **Marks**

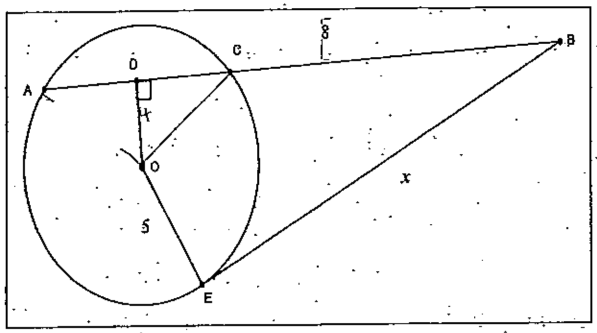
(a) Use the process of mathematical induction to show that:

$$1 + 3 + 9 + \ldots\ldots\ldots\ldots + 3^{n-1} = \frac{1}{2}\left(3^n - 1\right) \text{ for all positive integers n}$$

(b) The angle of elevation of the top of a tower (T) from a point A due East of the tower is 60°. From a point B due South of A, the angle of elevation of T is 15°. A and B are at the same elevation as the base of the tower. If the distance AB = 1 km, find the height (h) of the tower to the nearest metre. **3**



(c) 2



Given the length OD is 4cm, OE is 5cm, BC is 8cm, and BE is a tangent to the circle, centre O. OD is perpendicular to the secant AC. Find the value of x, correct to 2 decimal places.

(d) If $\alpha = \sin^{-1}\left(\frac{2}{3}\right)$ and $\beta = \sin^{-1}\left(\frac{3}{5}\right)$, find the value of $\sin(\alpha+\beta)$ 3

(e) Using the substitution $u = x-2$, evaluate $\int_3^4 \frac{x^2}{(x-2)^2}\,dx$ 3

End of Question 12

Question 13 (15 Marks) Start a fresh sheet of paper. **Marks**

(a) Simplify $1+\cos^2 x+\cos^4 x+\cos^6 x+\ldots\ldots\ldots\ldots \quad (0<x<\frac{\pi}{2})$ 2

(b) (i) Show that $\frac{d}{dx}\ln\left(\frac{x-1}{x+1}\right)=\frac{2}{(x-1)(x+1)}$ 2

(ii) Hence evaluate $\int_2^3 \frac{4dx}{(x-1)(x+1)}$ (answer in exact form) 2

(c) Let P$(2ap, 2ap^2)$ and Q$(2aq, 2aq^2)$ be points on the parabola $y=\frac{x^2}{2a}$.

i Find the equation of the chord PQ. 2

ii. If PQ is a focal chord, find the relationship between p and q. 2

iii. Show that the locus of the midpoint of PQ is a parabola. 2

(d) Find the area under the curve $y=\frac{1}{\sqrt{4-x^2}}$ from $x=1$ to $x=2$. 2

(e) Write $\sqrt{3}\cos x - \sin x$ in the form $2\cos(x+\alpha)$
where $0<\alpha<\frac{\pi}{2}$ 1

End of Question 13

Question 14 (15 Marks) Start a fresh sheet of paper. **Marks**

(a) The polynomial $x^3 - 4x^2 + 5x - 1 = 0$ has 3 roots. If the roots are α, β and γ.

i. Find the value of $\alpha + \beta + \gamma$ **1**

ii. Find the value of $\alpha\beta\gamma$ **1**

iii. Find the equation of the polynomial with roots 2α, 2β and 2γ. **2**

(b) The two equal sides of an isosceles triangle are of length 6cm. If the angle between them is increasing at the rate of 0·05 radians per second, find the rate at which the area of the triangle is increasing when the angle between the equal sides is $\frac{\pi}{6}$ radians. **2**

(c) A particle moves so that its distance x centimetres from a fixed point O at time t seconds is $x = 8\sin 3t$.

i. Using the equation $\ddot{x} = -n^2 x$, show that the particle is displaying simple harmonic motion **1**

ii. What is the period of the motion? **1**

iii. Find the velocity of the particle when it first reaches 4 centimetres to the right of the origin. **2**

(d) The function $f(x) = \sec x$ for $0 \le x < \frac{\pi}{2}$, and is not defined for other values of x.

(i) State the domain of the inverse function $f^{-1}(x)$. **1**

(ii) Show that $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$. **2**

(iii) Hence find $\frac{d}{dx}f^{-1}(x)$. **2**

End of Examination

SYDNEY TECHNICAL HIGH SCHOOL

MULTIPLE CHOICE ANSWER SHEET

Name : ....

Teacher: ....

Course: Trial HSC Examination 2015

Completely fill the response oval representing the most correct answer.

Do not remove this sheet from the answer booklet.

1. A ○ B ○ C ○ D ● ✓
2. A ○ B ○ C ○ D ● ✓
3. A ● B ○ C ○ D ○ ✓
4. A ○ B ○ C ○ D ● ✓
5. A ○ B ○ C ● D ○ ✓
6. A ○ B ○ C ● D ○ ✓
7. A ○ B ○ C ○ D ● ✓
8. A ○ B ● C ○ D ○ ✓
9. A ○ B ○ C ● D ○ ✓
10. A ● B ○ C ○ D ○ ✓

4

Question 11:

a) $\frac{4-2x}{x+5} \leq 2$

$(4-2x)(x+5) \leq 2(x+5)^2$

$\therefore 4x+20-2x^2-10x \leq 2(x^2+10x+25)$

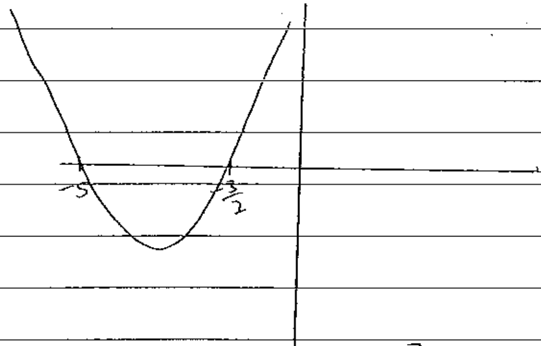
$-2x^2-6x+20 \leq 2x^2+20x+50$

$\therefore 4x^2+26x+30 \geq 0$

$2x^2+13x+15 \geq 0$

$(2x+3)(x+5) \geq 0$

$\therefore x = \frac{-3}{2}$ or -5



$\therefore$ Answer: $x \geq \frac{-3}{2}$ or $x < -5$

1/2

b) $\lim_{x \to 0} \frac{\sin(\frac{\pi}{4}x)}{2x} \times \frac{\frac{\pi}{8}}{\frac{\pi}{8}}$

$= \lim_{x \to 0} \frac{\pi}{8} \frac{\sin(\frac{\pi}{4}x)}{\frac{\cancel{x}\pi x}{8}}$

$= \lim_{x \to 0} \frac{\pi}{8} \frac{\sin\frac{\pi}{4}x}{\frac{\pi}{4}x} = \frac{\pi}{8}$

1/2

c) $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$x - 2y + 1 = 0$ $\quad$ $y = 5x - 4$

$\therefore 2y = x + 1$ $\quad$ $\therefore m_2 = 5$

$y = \frac{x}{2} + \frac{1}{2}$

$\therefore m_1 = \frac{1}{2}$

$\therefore \tan\theta = \dfrac{\frac{1}{2} - 5}{1 + (\frac{1}{2} \cdot 5)}$

$= \dfrac{-\frac{9}{\cancel{2}}}{\frac{7}{\cancel{2}}}$

$= \dfrac{-9}{7}$

$\therefore \theta = \tan^{-1}\left(-\frac{9}{7}\right)$

$= -52°7'30.06''$

$\therefore$ acute angle is ~~52° 8' to the nearest~~
52° to the nearest degree

2

d) $\ln(\sin^{-1} 2x)$

$\dfrac{d}{dx}\left(\ln(\sin^{-1} 2x)\right) = \dfrac{f'(x)}{f(x)}$

$= \dfrac{\frac{1}{\sqrt{1-(2x)^2}} \cdot 2}{\sin^{-1} 2x} = \dfrac{2}{\sin^{-1} 2x\sqrt{1-4x^2}}$

2

e) $x = t - 2$ ③ $\quad$ $y = 3t^2 - 1$ ②

$t = x + 2$ ①

sub ① into ②:

$\therefore y = 3(x+2)^2 - 1$

$= 3(x^2 + 4x + 4) - 1$

$= 3x^2 + 12x + 12 - 1$

$= 3x^2 + 12x + 11$

$\therefore$ Cartesian eqn is $y = 3x^2 + 12x + 11$

1

f) i) $\sin\theta\sec\theta = \tan\theta$

LHS $= \sin\theta \cdot \frac{1}{\cos\theta}$

$= \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

$=$ RHS.

1

ii) $\therefore$ For $\sin\theta\sec\theta = \sqrt{3}$

$\tan\theta = \sqrt{3}$ (1st, 3rd)

$\theta = \tan^{-1}\sqrt{3}$

For $0 \le \theta \le 2\pi$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

1

g) $P(x) = x^3 - 4x$

$P(-3) = -27 - 4(-3)$

$= -27 + 12$

$= -15$

$\therefore$ remainder is -15

1

h) $\log_3 7 = m$

$\log_3(21) = \log_3(3\times 7)$

$= \log_3 3 + \log_3 7$

$= 1 + m$

1

i) $\dfrac{2x^2-4x}{x\sqrt{x}} = \dfrac{2x^2-4x}{x^{\frac{3}{2}}}$ $\qquad \dfrac{d}{dx} = \dfrac{vu'-uv'}{v^2}$

$\therefore \dfrac{d}{dx}\left(\dfrac{2x^2-4x}{x\sqrt{x}}\right) = \dfrac{x^{\frac{3}{2}}\cdot(4x-4)-(2x^2-4x)\cdot\frac{3\sqrt{x}}{2}}{\left(x^{\frac{3}{2}}\right)^2}$

$\dfrac{d}{dx}\left(x^{\frac{3}{2}}\right) = \dfrac{3}{2}x^{\frac{1}{2}}$

$= \dfrac{x\sqrt{x}(4x-4) - \cancel{2}(x^2-2x)\frac{3\sqrt{x}}{\cancel{2}}}{x^3}$

$= \dfrac{4x^2\sqrt{x} - 4x\sqrt{x} - [3x^2\sqrt{x} - 6x\sqrt{x}]}{x^3}$

$= \dfrac{x^2\sqrt{x} + 2x\sqrt{x}}{x^3}$

$= \dfrac{x\sqrt{x}+2\sqrt{x}}{x^2}$

15

2

Question 12:

a) Step 1: Show true for $n=1$

LHS $= \cancel{3^{x-1}} = \frac{n}{2}(1+3^{n-1})$ $\qquad$ RHS $= \frac{1}{2}(3^n-1)$

$= \cancel{3^{1-1}} = \frac{1}{2}(1+3^{1-1})$ $\qquad = \frac{1}{2}(3^1-1)$

$= \cancel{3^0}$ $\quad = \frac{1}{2}(1+3^0)$ $\qquad = \frac{1}{2}(2)$

$= \cancel{1}$ $\quad = \frac{1}{2}(2)$

$= 1$ $\qquad = 1$

$\therefore$ True for $n=1$

Step 2: Assume for for $n=k$

ie. $1+3+9+\ldots+3^{k-1} = \frac{1}{2}(3^k-1)$

Prove true for $n=k+1$

ie. $1+3+9+\ldots+3^{(k+1)-1} = \frac{1}{2}(3^{k+1}-1)$

or $1+3+9+\ldots+3^k = \frac{1}{2}(3^{k+1}-1)$

LHS $= 1+3+9\ldots+3^k$

$S_{k+1} = S_k + T_{k+1}$

$= \frac{1}{2}(3^k-1) + 3^k$ (from assumption)

$= \frac{3^k}{2} - \frac{1}{2} + 3^k$

$= \frac{3^k}{2} - \frac{1}{2} + \frac{2\times 3^k}{2}$

$= \frac{3\times 3^k}{2} - \frac{1}{2}$

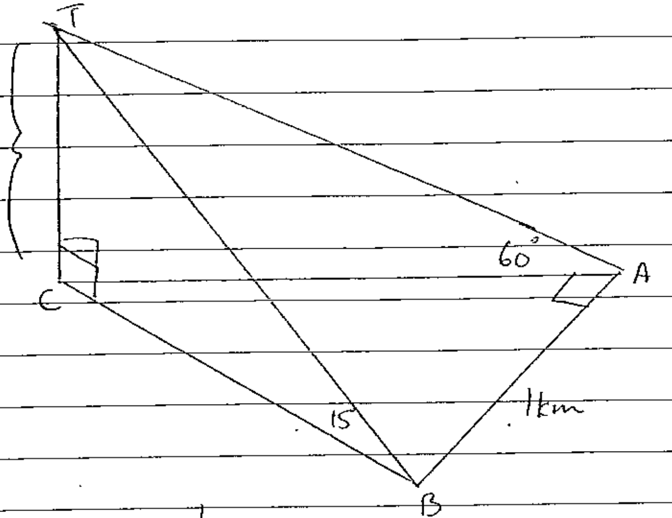
$= \frac{1}{2}(3^{k+1}-1)$ which is the required result

$\therefore$ true for $n=k+1$

Step 3: Since true for n=1 and n=k+1, it is also true for n=2 ⇒ ie, n=1+1, and so on for all positive integers n.

4

b)



14/15

$\tan 15^\circ = \dfrac{h}{BC}$ $\qquad$ $\tan 60^\circ = \dfrac{h}{AC}$

$\therefore BC = \dfrac{h}{\tan 15^\circ}$ $\qquad$ $\therefore AC = \dfrac{h}{\tan 60^\circ}$

By pythagoras' theorem:

$$BC^2 = 1^2 + AC^2$$

$$\therefore \frac{h^2}{(\tan 15^\circ)^2} = 1 + \frac{h^2}{(\sqrt{3})^2}$$

$$\therefore \frac{h^2}{(\tan 15^\circ)^2} - \frac{h^2}{3} = 1$$

$$\therefore \frac{3h^2 - (\tan 15^\circ)^2 h^2}{\cancel{(\tan 15^\circ)^2 (3)}} = (\tan 15^\circ)^2 (3)$$

Cont. →

$$\therefore \frac{3h^2 - (\tan 15^\circ)^2 h^2}{\cancel{3(\tan 15^\circ)^2}} = 3(\tan 15^\circ)^2$$

$$h^2(3 - (\tan 15^\circ)^2) = [3(\tan 15^\circ)^2]$$

$$h^2 = \frac{[3(\tan 15^\circ)^2]}{3 - (\tan 15^\circ)^2}$$

$$= \frac{0.2153903092}{2.92820323}$$

$$= 0.07355715852$$

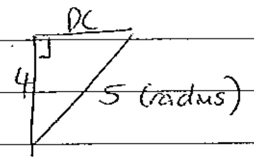
$$h = 0.2712142299$$

3

$\therefore h = 271$ metres.

c) Value of x:

Using $\triangle OCD$



$\therefore DC = \sqrt{5^2 - 4^2}$

$= \sqrt{25 - 16}$

$= \sqrt{9} = 3$ cm

$\therefore AD = DC = 3$ cm (line from centre O of circle that makes a right angle with secant bisects AC)

$\therefore AC = 6\text{cm}$

Finding x:

$AB \times BC = (BE)^2$ (length of lines from circumference to points is equal to the square of the length of tangent from the point of contact of the circle to the point)

no sq

$\therefore (8+6)^2 \times (8^2) = x^2$

$14^2 \times 8^2 = x^2$

$196 \times 64 = x^2$

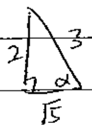
$x^2 = 12544$

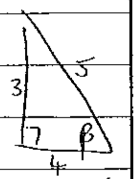
$\therefore x = \sqrt{112}\ \text{cm}$

1

d) $\alpha = \sin^{-1}\left(\frac{2}{3}\right)$ $\quad\quad$ $\beta = \sin^{-1}\left(\frac{3}{5}\right)$

$\therefore \sin\alpha = \frac{2}{3}$ $\quad\quad$ $\therefore \sin\beta = \frac{3}{5}$





$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \frac{2}{3}\cdot\frac{4}{5} + \frac{3}{5}\cdot\frac{\sqrt{5}}{3}$

$= \frac{8}{15} + \frac{3\sqrt{5}}{15} = \frac{8+3\sqrt{5}}{15}$

3

e) Using $u = x-2$

$\therefore \frac{du}{dx} = 1$ $\quad\quad$ when $x = 4,\ u = 2$; $\ x = 3,\ u = 1$

$du = dx$ $\quad\quad$ If $u = x-2$, $x = u+2$

$\therefore \int_3^4 \frac{x^2}{(x-2)^2}\,dx$

$= \int_1^2 \frac{(u+2)^2}{(u+\cancel{2}-\cancel{2})^2}\,du$

$= \int_1^2 \frac{(u+2)^2}{u^2}\,du$

$= \int_1^2 \frac{u^2+4u+4}{u^2}\,du$

$= \int_1^2 \frac{u^2}{u^2} + \frac{4}{u} + 4u^{-2}\,du$

$= \left[u + 4\ln(u) - \frac{4}{u}\right]_1^2$

$= \left[(\cancel{2} + 4\ln(2) - \cancel{2}) - (1 + 4\ln(1) - 4)\right]$

$= 4\ln(2) + 3 - 4\ln(1)$

$= 4\ln(2) + 3$

3

Question 13:

a) $1 + \cos^2 x + \cos^4 x + \cos^6 x + \ldots$

$\cancel{\frac{1}{\cos^2 x} + 1 + \cos^2 x + \cos^4 x + \ldots}$

$a = 1$, $r = \cos^2 x$

$\therefore$ as $0 < x < \frac{\pi}{2}$

$\cancel{-1 \le r \le 1}$ $-1 \le \cos^2 x \le 1$

$\therefore$ There is a limiting sum $\therefore -1 \le r \le 1$

Limiting sum $\Rightarrow \frac{a}{1-r}$

$= \frac{1}{1-\cos^2 x} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x$

2

Show

b) i) $\frac{d}{dx} \ln\left(\frac{x-1}{x+1}\right) = \frac{2}{(x-1)(x+1)}$

$LHS = \frac{d}{dx}\left[\ln\left(\frac{x-1}{x+1}\right)\right]$

$= \frac{d}{dx}\left[\ln(x-1) - \ln(x+1)\right]$

$= \frac{1}{x-1} - \left(\frac{1}{x+1}\right)$

$= \frac{x+1}{(x-1)(x+1)} - \left(\frac{(x-1)}{(x-1)(x+1)}\right)$

2

$= \frac{\cancel{x}+1-\cancel{x}+1}{(x-1)(x+1)}$

$= \frac{2}{(x-1)(x+1)} = RHS.$

ii) Hence: $\int_2^3 \frac{4\,dx}{(x-1)(x+1)}$

$= 2\int_2^3 \frac{2\,dx}{(x-1)(x+1)}$

$= 2\left[\ln\left(\frac{x-1}{x+1}\right)\right]_2^3$

$= 2\left[\left(\ln\left(\frac{2}{4}\right)\right) - \left(\ln\left(\frac{1}{3}\right)\right)\right]$

$= 2\left[\left(\ln\left(\frac{1}{2}\right) - \left(\ln\left(\frac{1}{3}\right)\right)\right)\right]$

$= 2\left[\ln\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)\right]$

$= 2\left(\ln\left(\frac{3}{2}\right)\right)$

$= 2\ln\left(\frac{3}{2}\right)$

2

c) $P(2ap, 2ap^2)$ $Q(2aq, 2aq^2)$

$y = \frac{x^2}{2a}$

i) Chord PQ

Gradient $M \Rightarrow \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2aq^2 - 2ap^2}{2aq - 2ap}$

$= \frac{\cancel{2a}(q^2 - p^2)}{\cancel{2a}(q-p)}$

$= \frac{\cancel{(q-p)}(q+p)}{\cancel{q-p}}$

$= q+p$

~~Using $y - y_1 = m(x - x_1)$~~

$\therefore y - 2ap^2 = (q+p)(x - 2ap)$

$y - \cancel{2ap^2} = qx - 2apq + px - \cancel{2ap^2}$ 2

$\therefore y = qx - 2apq + px$

ii) If ~~A~~ PQ is a focal chord:

then A passes through $(0, \cancel{a})$

Sub $x = 0$, $y = a$

$\therefore a = -2apq$

$pq = \frac{\cancel{a}}{-2\cancel{a}}$ $\therefore pq = -\frac{1}{2}$

iii) Midpoint of PQ:

$M_{PQ}^{\text{Midpoint}} \left(\frac{2ap + 2aq}{2}, \frac{2ap^2 + 2aq^2}{2} \right)$

$\left(a(p+q), a(p^2+q^2) \right)$

From $x = a(p+q)$

$(p+q) = \frac{x}{a}$

From $y = a(p^2+q^2)$

$= a\left[(p+q)^2 - 2pq\right]$

$= a\left[\left(\frac{x}{a}\right)^2 - \left(2 \times -\frac{1}{2}\right)\right]$ as $pq = -\frac{1}{2}$

$= a\left[\frac{x^2}{a^2} + 1\right]$

$\therefore \cancel{y} = \frac{x^2}{a} + a$ C. T. E.

$\therefore y = \frac{x^2}{a} + a$ 2

which is in the form of a parabola with gradient $\frac{1}{a}$

d) $y = \frac{1}{\sqrt{4 - x^2}}$ From $x=1$ to $x=2$

$= (4 - x^2)^{-\frac{1}{2}}$

$\Rightarrow$ PTO.

$y = (4-x^2)^{-\frac{1}{2}}$

Area: $\int_1^2 (4-x^2)^{-\frac{1}{2}}\,dx$

$= \left[\dfrac{(4-x^2)^{\frac{1}{2}} \div -2x}{\frac{1}{2}}\right]_1^2$

$= \left[-4x(4-x^2)^{\frac{1}{2}}\right]_1^2$

$= \left[(0) - (-4(3)^{\frac{1}{2}})\right]$

$= \cancel{-}(\cancel{+}4\sqrt{3})$

$= 4\sqrt{3}$ units^2 1

e) $\sqrt{3}\cos x - \sin x = 2\cos(x+\alpha)$

$\tan x = \dfrac{1}{\sqrt{3}}$

$\therefore x = \tan^{-1}\dfrac{1}{\sqrt{3}}$

$= \dfrac{\pi}{6}$ 1

$\therefore \sqrt{3}\cos x - \sin x = 2\cos\left(x + \dfrac{\pi}{6}\right)$

Question 14:

a) $x^3 - 4x^2 + 5x - 1 = 0$

Let roots be α, β, γ

i) $\alpha + \beta + \gamma = \Sigma$(roots one at a time)

$= \dfrac{-b}{a}$

$= \dfrac{-(-4)}{1}$ 1

$= 4$

ii) $\alpha\beta\gamma$ = product of roots

$= \dfrac{-d}{a}$

$= \dfrac{-(-1)}{1}$ 1

$= 1$

iii) $y = 2x$

$\therefore x = \dfrac{y}{2}$

sub $x = \dfrac{y}{2}$ into ~~eq~~ polynomial:

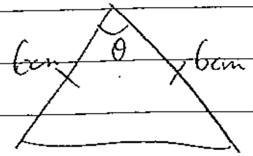
$\left(\dfrac{y}{2}\right)^3 - 4\left(\dfrac{y}{2}\right)^2 + 5\left(\dfrac{y}{2}\right) - 1 = 0$

$\therefore \dfrac{y^3}{8} - \dfrac{\cancel{4}y^2}{\cancel{4}} + \dfrac{5y}{2} - 1 = 0$ 2

$\therefore y^3 - 8y^2 + 20y - 8 = 0$ is the eqn with roots $2\alpha, 2\beta, 2\gamma$

4

b)



angle increasing at 0.05 radians/sec

when $\theta = \frac{\pi}{6}$ radians.

Area of triangle: $\frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 6 \times 6 \times \sin\theta$

$= 18 \sin\theta$

Rate at which area is increasing with respect to angle: $\frac{dA}{d\theta} = 18 \cos\theta$

$\cancel{\frac{dA}{d\theta} = \frac{dA}{dx} \cdot \frac{dx}{d\theta}}$

$\cancel{\frac{dA}{dx} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dx}}$

$\cancel{= 18\cos\theta \cdot 0.05 \text{ radians.}}$

$\cancel{\text{sub } \theta = \frac{\pi}{6}}$

$\cancel{\therefore \frac{dA}{dx} = \frac{18\sqrt{3}}{2} \cdot 0.05}$

$\cancel{= \frac{18\sqrt{3}}{2} \cdot \frac{1}{20}}$

$\cancel{= \frac{18\sqrt{3}}{40}}$

$=$

$\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt}$

$= 18\cos\theta \cdot 0.05$

sub $\theta = \frac{\pi}{6}$

$\therefore \frac{dA}{dt} = \frac{18\sqrt{3}}{2} \cdot 0.05$

$= \frac{18\sqrt{3}}{2} \cdot \frac{1}{20}$

$= \frac{18\sqrt{3}}{40}$

$= \frac{9\sqrt{3}}{20}$ cm^{2}s^{-1}

2

6

c) $x = 8\sin(3t)$

i) using $\ddot{x} = -n^2 x$

$x = 8\sin(3t)$

$\dot{x} = 24\cos(3t)$

$\ddot{x} = -72\sin(3t)$

~~which is in the form $\ddot{x} = -n^2 x$ where $n^2 = 72$~~

$= -9(8\sin(3t))$

$= -9x$

which is in the form $\ddot{x} = -n^2 x$

$\therefore$ SHM.

1

ii) The period T is $T = \frac{2\pi}{n}$

and $n^2 = 9$

$\therefore n = 3$

$\therefore T = \frac{2\pi}{3}$

1

iii) ~~when t =~~ $x = 8\sin(3t)$

when $x = 4$

$4 = 8\sin(3t)$

$\frac{1}{2} = \sin(3t)$

$\therefore 3t = \frac{\pi}{6}$

$\therefore t = \frac{\pi}{18}$

$\dot{x} = 24\cos(3t)$

sub $t = \frac{\pi}{18}$

$\dot{x} = 24\cos\left(\frac{3\pi}{18}\right)$

$= 24\cos\left(\frac{\pi}{6}\right)$

$= 24 \times \frac{\sqrt{3}}{2}$

$\therefore \dot{x} = 12\sqrt{3}$ cms^{-1}

2

10

d) $f(x) = \sec x$ for $0 \leq x \leq \frac{\pi}{2}$

i) Domain of $f^{-1}(x)$ is range of $f(x)$

Range of $f(x)$:

$f(x) = \sec x$

sub $x = 0$

$\therefore \sec 0 = \frac{1}{\cos 0}$

$= \frac{1}{1}$

$= 1$

14

sub $x = \frac{\pi}{2}$

$\therefore \sec \frac{\pi}{2} = \frac{1}{\cos(\frac{\pi}{2})}$

$= \text{undefined}$

0

$\therefore$ Domain of $f^{-1}(x)$ is $0 < x \leq 1$

ii) $f(x) = \sec x$

$= \frac{1}{\cos x}$

For inverse:

$y = \frac{1}{\cos x}$

$x = \frac{1}{\cos y}$

$\cos y = \frac{1}{x}$

$\therefore y = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$

2

iii) Hence, $\frac{d}{dx} f^{-1}(x)$

$= \frac{d}{dx}\left(\cos^{-1}\frac{1}{x}\right)$

$= \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot -\frac{1}{x^2}$

$= \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$

$\frac{d}{dx}(x^{-1}) = -x^{-2}$

$= -\frac{1}{x^2}$

$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$= \frac{1}{x\sqrt{x^2-1}}$