CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2013
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Morning Session
Friday, 9 August 2013

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a SEPARATE sheet
- In Questions 11–14, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and the Student Number at the top of this page

Total marks – 70

Section I Pages 2-5

10 marks
- Attempt Questions 1–10
- Allow 15 minutes for this section

Section II Pages 6-11

60 marks
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

6300-1

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\ x\neq 0,\ \textit{if}\ n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is an expression for $\int 2\sin^2 x\, dx$?

(A) $x+\frac{1}{2}\sin 2x+C$

(B) $x-\frac{1}{2}\sin 2x+C$

(C) $x+\sin 2x+C$

(D) $x-\sin 2x+C$

2 Let A be the point $(-1,4)$ and B the point $(9,-6)$. Find the coordinates of the point which divides AB internally in the ratio 3:2.

(A) (1,3)

(B) (3,1)

(C) (–2,5)

(D) (5,–2)

3 The polynomial $P(x)=8x^3+ax^2-4x+1$ has a factor of $2x+1$. What is the value of a?

(A) –8

(B) 0

(C) 3

(D) 8

4 The letters of the word TWITTER are arranged randomly. What is the probability that the three Ts are grouped together?

(A) $\frac{1}{42}$

(B) $\frac{1}{35}$

(C) $\frac{1}{7}$

(D) $\frac{6}{7}$

5 The polynomial equation $x^3-5x^2+x-4=0$ has roots α, β and γ. What is the value of $\alpha^2+\beta^2+\gamma^2$?

(A) 17

(B) 23

(C) 25

(D) 27

6 What is the coefficient of x^3 in the expansion of $(1-3x)^7$?

(A) –945

(B) –35

(C) 35

(D) 945

7 The expression $\sin x - \sqrt{3}\cos x$ can be written in the form $2\sin(x+\alpha)$.

Find the value of α.

(A) $\alpha = \frac{\pi}{3}$

(B) $\alpha = -\frac{\pi}{3}$

(C) $\alpha = \frac{\pi}{6}$

(D) $\alpha = -\frac{\pi}{6}$

8 Differentiate $\sin^{-1}\left(\frac{3x}{4}\right)$ with respect to x.

(A) $\sqrt{16-9x^2}$

(B) $\frac{\sqrt{16-9x^2}}{3}$

(C) $\frac{1}{\sqrt{16-9x^2}}$

(D) $\frac{3}{\sqrt{16-9x^2}}$

9 Three stones, A, B and C, are released from the top of a vertical cliff to reach the horizontal ground below.

Stone A is projected horizontally at $15\,\text{ms}^{-1}$, stone B is projected horizontally at $20\,\text{ms}^{-1}$ and stone C is dropped from rest.

Which statement below is correct?

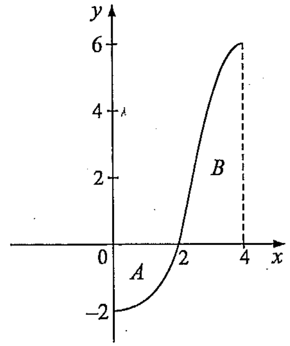
(A) Stone A reaches the ground first.

(B) Stone B reaches the ground first.

(C) Stone C reaches the ground first.

(D) The three stones all reach the ground at the same time.

10 The graph of the function $y = f(x)$ is drawn below.



Area A is equal to 3 square units and Area B is equal to 7 square units.

What is the value of $\int_{-2}^{6} f^{-1}(x)\,dx$?

(A) 4

(B) 10

(C) 20

(D) 22

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$, giving your answer in terms of π. **2**

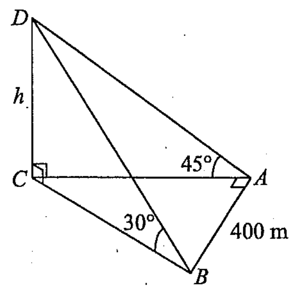
(b) Solve the inequality $\frac{t-2}{t+3} > -2$. **3**

(c) Find $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\,dx$ using the substitution $u=\sqrt{x}$. **3**

(d) (i) Find $\frac{d}{dx}(x\sin 2x)$. **2**

(ii) Hence, or otherwise, find $\int x\cos 2x\,dx$. **2**

(e) **3**



The angle of elevation of a tower, *CD*, from a point *A* due East of the tower is 45°. From a point *B* due South of *A*, the angle of elevation is 30°. The distance from *A* to *B* is 400 metres.

Find the height, *h*, of the tower.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A bowl of soup is cooling in a room that has a constant temperature of 20°C. At time *t*, measured in minutes, the temperature, *T*, of the soup is decreasing according to the differential equation

$$\frac{dT}{dt} = -k(T-20)$$

where *k* is a positive constant.

The initial temperature of the soup is 100°C and it cools to 70°C after 5 minutes.

(i) Verify that $T = 20 + Ae^{-kt}$ is a solution to the differential equation, where A is a constant. **1**

(ii) Find the values of A and k. **2**

(iii) Find the temperature of the soup after 15 minutes. Give your answer correct to the nearest degree. **1**

(b) Sketch the graph of $y = \sin^{-1}\frac{x}{3}$. **2**

(c) In how many ways can seven students sit in a circle if Jack and Jill refuse to sit next to each other? **2**

(d) (i) Show that the function **2**

$$f(x) = \log_e x - \sin x + 1$$

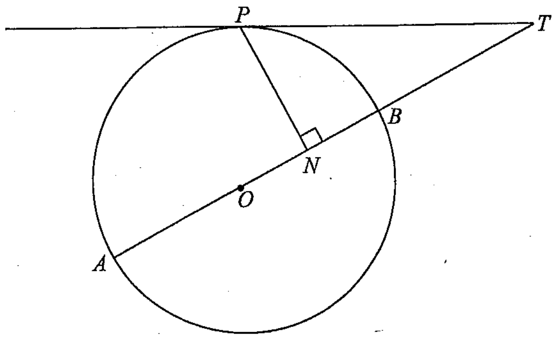
has a zero between 0.7 and 0.8.

(ii) Hence use *halving-the-interval* method to find the value of this zero, correct to one decimal place. **2**

Question 12 continues on page 8

Question 12 (continued)

(e) AB is a diameter of the circle, centre O, shown below and P is a point on the circumference. The tangent to the circle at P meets AB produced at T. Let N be the foot of the perpendicular from P to AB.



Prove that BP bisects $\angle NPT$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The velocity $v\,\text{ms}^{-1}$ of a particle moving in simple harmonic motion along the x-axis is given by

$$v^2 = -x^2 - 4x + 12.$$

(i) Find the acceleration of the particle in terms of x. **1**

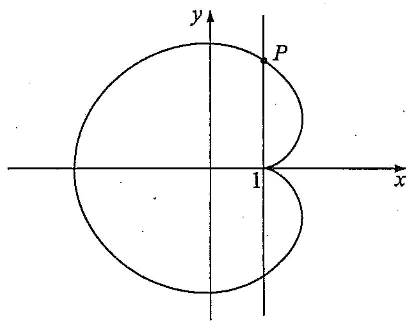
(ii) State the centre and period of the motion. **2**

(iii) What is the maximum speed of the particle? **1**

(b) A cardioid curve is defined by the following pair of parametric equations **3**

$$x = 2\cos\theta - \cos 2\theta$$
$$y = 2\sin\theta - \sin 2\theta$$

for $0 \le \theta < 2\pi$. The graph is drawn below.



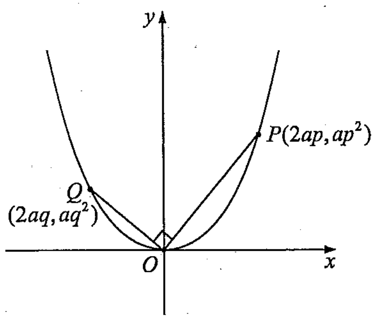
The cardioid curve and the vertical line $x = 1$ intersect in the 1st quadrant at a point P.

Find the coordinates of the point P.

Question 13 continues on page 10

Question 13 (continued)

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ such that OP is perpendicular to OQ.



(i) Prove that $pq = -4$. **2**

(ii) Let R be the point such that $OPRQ$ is a rectangle. **2**
R has coordinates $\left(2a(p+q), a(p^2+q^2)\right)$. (Do not prove this).
Show that the locus of R is a parabola.

Suppose the point $P(2ap, ap^2)$, $p > 0$, is moving along the original parabola $x^2 = 4ay$ such that its y coordinate, y_P, is increasing at a rate of 1 unit per second.

(iii) Using part (i) show that $\frac{dq}{dt} = \frac{-q}{p} \times \frac{dp}{dt}$. **2**

(iv) Hence, or otherwise, show that the y coordinate of Q, y_Q, is decreasing at a rate of $\frac{q^2}{p^2}$ units per second. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by induction that $n^3 + 5n$ is divisible by 6, for integers $n \geq 1$. **4**

(b) Let n be a positive even integer.

(i) Expand and simplify $(a+b)^n + (a-b)^n$. **2**

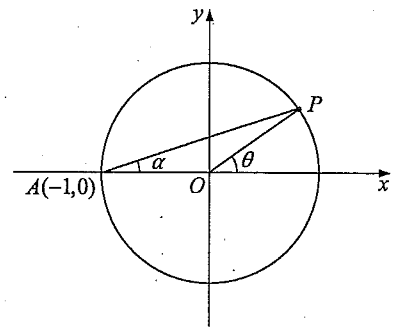
Suppose n fair dice are thrown, each with faces numbered 1 to 6.

(ii) Write down the probability of obtaining exactly two sixes. **1**

(iii) Show that the probability of obtaining an even number of sixes is given by **2**

$$\frac{1}{2}\left(1+\left(\frac{2}{3}\right)^n\right).$$

(c)



Let A be the point $(-1,0)$ on the unit circle and suppose that the line through A with gradient t meets the circle at P in the first quadrant. OP and AP make angles of θ and α respectively with the positive x-axis.

(i) What range of values can t take? **1**

(ii) Prove that P has x-coordinate $\frac{1-t^2}{1+t^2}$ and find the y-coordinate of P. **3**

(iii) Deduce that $\tan\theta = \frac{2t}{1-t^2}$, for $t \neq 1$. **1**

(iv) Find the value of t if $\alpha = 22\frac{1}{2}°$. **1**

End of Paper

Section I
10 marks

Questions 1-10 (1 mark each)

Q1 Solution	Answer	Mark
$\int 2\sin^2 x\,dx = \int (1-\cos 2x)\,dx$ $= x - \frac{1}{2}\sin 2x + C$	B	1

Q2 Solution	Answer	Mark
$(x,y) = \left(\frac{3(9)+2(-1)}{3+2}, \frac{3(-6)+2(4)}{3+2}\right)$ $=(5,-2)$	D	1

Q3 Solution	Answer	Mark
$P\left(\frac{-1}{2}\right)=0 \Rightarrow 8\left(\frac{-1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right)+1=0$ $\therefore a=-8$	A	1

Q4 Solution	Answer	Mark
Total number of arrangements of TWITTER $=\frac{7!}{3!}$ Number of arrangements with TTT grouped is the same as the number of arrangements of TWIER $=5!$ $\therefore$ probability Ts are grouped together $=\frac{5!}{\left(\frac{7!}{3!}\right)}=\frac{1}{7}$	C	1

Q5 Solution	Answer	Mark
$\alpha+\beta+\gamma = -\frac{-5}{1}=5$ $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{1}{1}=1$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $=5^2-2\times1$ $=23$	B	1

Q6 Solution	Answer	Mark
The coefficient of x^3 in $(1-3x)^7$ is $\binom{7}{3}1^4(-3)^3=-945$	A	1

Q7 Solution	Answer	Mark
$\sin x-\sqrt{3}\cos x \equiv 2\sin x\cos\alpha + 2\cos x\sin\alpha$ $\therefore 2\cos\alpha=1, 2\sin\alpha=-\sqrt{3} \Rightarrow \alpha=-\frac{\pi}{3}$	B	1

Q8 Solution	Answer	Mark
$\frac{d}{dx}\sin^{-1}\left(\frac{3x}{4}\right)=\frac{1}{\sqrt{1-\left(\frac{3x}{4}\right)^2}}\times\frac{3}{4}$ $=\frac{3}{\sqrt{16-9x^2}}$	D	1

Q9 Solution	Answer	Mark
The vertical motion is the same for all three stones. $\therefore$ All three stones reach the ground at the same time.	D	1

Q10 Solution	Answer	Mark
The graph of $y=f^{-1}(x)$ is the reflection of the graph of $y=f(x)$ about the line $y=x$. $\int_{-2}^{6} f^{-1}(x)dx =$ area A + area of 4×6 rectangle − area B $=3+24-7$ $=20$ square units	C	1

Section II

60 marks

Question 11 (15 marks)

(a) (2 marks)

Sample answer:

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$$
$$=\sin^{-1}\frac{1}{2}-\sin^{-1}0$$
$$=\frac{\pi}{6}$$

(b) (3 marks)

Sample answer:

$$\frac{t-2}{t+3}>-2$$
$$(t+3)(t-2)>-2(t+3)^2$$
$$3t^2+13t+12>0$$
$$(3t+4)(t+3)>0$$
$$\therefore t<-3, t>\frac{-4}{3}$$

(c) (3 marks)

Sample answer:

Let $u=\sqrt{x}$, $du=\dfrac{dx}{2\sqrt{x}}$

$$\int\frac{e^{3\sqrt{x}}}{\sqrt{x}}dx=2\int\frac{e^{3\sqrt{x}}}{2\sqrt{x}}dx$$
$$=2\int e^{3u}\,du$$
$$=\frac{2}{3}e^{3u}+C$$
$$=\frac{2}{3}e^{3\sqrt{x}}+C$$

(d)(i) (2 marks)

Sample answer:

$$\frac{d}{dx}(x\sin 2x)=2x\cos 2x+\sin 2x$$

(d)(ii) (2 marks)

Sample answer:

$$\int 2x\cos 2x\,dx+\int\sin 2x\,dx=x\sin 2x$$
$$\int x\cos 2x\,dx=\frac{1}{2}x\sin 2x-\frac{1}{2}\int\sin 2x\,dx$$
$$=\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+C$$

(e) (3 marks)

Sample answer:

$$\tan 30°=\frac{h}{BC}\Rightarrow BC=\sqrt{3}h$$
$$\tan 45°=\frac{h}{AC}\Rightarrow AC=h$$
$$\left(\sqrt{3}h\right)^2-h^2=400^2$$
$$2h^2=400^2$$
$$h=200\sqrt{2}$$

∴ The height of the tower is $200\sqrt{2}$ metres.

Question 12 (15 marks)

(a)(i) (1 mark)

Sample answer:

$$T=20+Ae^{-kt}$$
$$\frac{dT}{dt}=-kAe^{-kt}$$
$$=-k(T-20)\text{ since }Ae^{-kt}=T-20$$

(a)(ii) (2 marks)

Sample answer:

$$t=0, T=100\Rightarrow A=80$$
$$t=5, T=70\Rightarrow 70=20+80e^{-5k}$$
$$e^{-5k}=\frac{5}{8}$$
$$\therefore k=\frac{-1}{5}\ln\left(\frac{5}{8}\right)\approx 0.094\text{ (3 d.p.)}$$

(a)(iii) (1 mark)

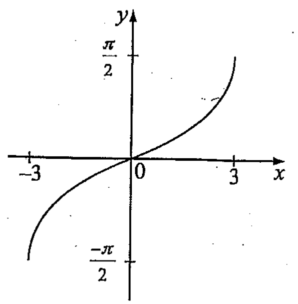
Sample answer:

$$T=20+80e^{\frac{1}{5}\ln\left(\frac{5}{8}\right)\times 15}$$
$$=39.531\ldots$$
$$\approx 40\text{ (nearest degree)}$$

∴ The temperature of the soup after 15 minutes is 40°C (to the nearest degree)

(b) (2 marks)

Sample answer:



(c) (2 marks)

Sample answer:

If Jack is seated first, there are 4 seats (not next to Jack) in which Jill can sit. There are then 5! ways to seat the remaining students.

∴ The total number of ways is $1\times4\times5!=480$.

(d)(i) (2 marks)

Sample answer:

$$f(0.7)=\ln 0.7-\sin 0.7+1=-0.00089\ldots<0$$
$$f(0.8)=\ln 0.8-\sin 0.8+1=0.05950\ldots>0$$

Since $f(x)$ is continuous for $x>0$ and changes sign from $x=0.7$ to $x=0.8$, $f(x)$ has a zero between 0.7 and 0.8.

(d)(ii) (2 marks)

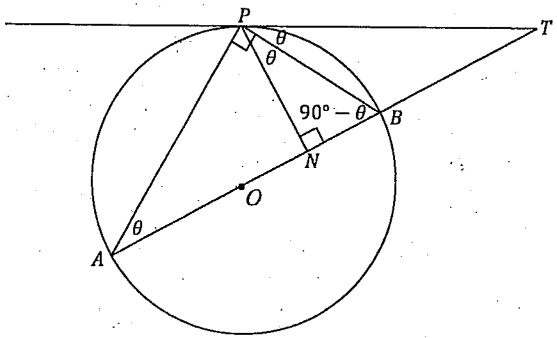
Sample answer:

$$f(0.75)=\ln 0.75-\sin 0.75+1=0.030679\ldots>0$$

∴ $f(x)$ has a zero between 0.7 and 0.75, hence the zero is 0.7 correct to one decimal place.

(e) (3 marks)

Sample answer:



Let $\angle TPB = \theta$

Hence, $\angle PAB = \theta$ (The angle between a tangent and a chord equals the angle in the alternate segment).

$\angle APB = 90°$ (angle subtended at the circumference by a diameter is 90°)

$\angle PBA = 90° - \theta$ (angle sum of ΔAPB is 180°)

$\angle BPN = \theta$ (angle sum of ΔNPB is 180°)

$\therefore \angle TPB = \angle BPN$, hence BP bisects $\angle NPT$.

Question 13 (15 marks)

(a)(i) (1 mark)

Sample answer:

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(-\frac{1}{2}x^2 - 2x + 6\right) = -x - 2$$

(a)(ii) (2 marks)

Sample answer:

Since $\ddot{x} = -1^2(x+2)$,

Centre of motion is $x = -2$.

Period of motion is $\frac{2\pi}{n} = \frac{2\pi}{1} = 2\pi$ seconds.

(a)(iii) (1 mark)

Sample answer:

Maximum speed occurs at the centre of motion, $x = -2$.

$$v^2 = -(-2)^2 - 4(-2) + 12 = 16$$

$$v = \pm 4$$

$\therefore$ The maximum speed of the particle is $4\,\text{ms}^{-1}$.

(b) (3 marks)

Sample answer:

Solve $x = 1$ and $x = 2\cos\theta - \cos 2\theta$ simultaneously for points of intersection:

$$2\cos\theta - \cos 2\theta = 1$$

$$2\cos\theta - 2\cos^2\theta + 1 = 1$$

$$2\cos\theta(1 - \cos\theta) = 0$$

$$\cos\theta = 0,\ \cos\theta = 1$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, 0 \text{ for } 0 \le \theta < 2\pi$$

$\theta = \frac{\pi}{2} \Rightarrow y = 2\sin\frac{\pi}{2} - \sin\pi = 2$ which is in the first quadrant,

Hence P has coordinates $(1, 2)$.

Note: $\theta = \frac{3\pi}{2} \Rightarrow y = 2\sin\frac{3\pi}{2} - \sin 3\pi = -2$ which is the point of intersection in the 4th quadrant and $\theta = 0 \Rightarrow y = 2\sin 0 - \sin 0 = 0$ which is the point of intersection on the x-axis.

(c)(i) (2 marks)

Sample answer:

$$m_{OP} = \frac{ap^2}{2ap} = \frac{p}{2}$$

$$m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$$

$$OP \perp OQ \Rightarrow \frac{p}{2} \times \frac{q}{2} = -1.$$

$$\therefore pq = -4.$$

(c)(ii) (2 marks)

Sample answer:

$$x = 2a(p+q) \Rightarrow p + q = \frac{x}{2a}$$

$$y = a(p^2 + q^2) \Rightarrow p^2 + q^2 = \frac{y}{a}$$

$pq = -4$ (from part i)

$$(p+q)^2 = p^2 + q^2 + 2pq$$

$$\Rightarrow \left(\frac{x}{2a}\right)^2 = \frac{y}{a} + 2(-4)$$

$$\Rightarrow \frac{x^2}{4a^2} = \frac{y}{a} - 8$$

$$\Rightarrow x^2 = 4a(y - 8a)$$

$\therefore$ The equation of the locus of R is a parabola.

(c)(iii) (2 marks)

Sample answer:

$$pq = -4 \Rightarrow q = \frac{-4}{p} \Rightarrow \frac{dq}{dp} = \frac{4}{p^2}$$

$$\frac{dq}{dt} = \frac{dq}{dp} \times \frac{dp}{dt} = \frac{4}{p^2} \times \frac{dp}{dt} = \frac{-q}{p} \times \frac{dp}{dt} \left(\text{since } pq = -4, \frac{4}{p} = -q\right).$$

(c)(iv) (2 marks)

Sample answer:

Given $\frac{dy_P}{dt}=1$, we are required to prove $\frac{dy_Q}{dt}=\frac{-q^2}{p^2}$

$$y_P=ap^2 \Rightarrow \frac{dy_P}{dy_p}=2ap$$

$$y_Q=aq^2 \Rightarrow \frac{dy_Q}{dy_q}=2aq$$

$$\frac{dy_P}{dt}=\frac{dy_P}{dp}\times\frac{dp}{dt}$$

$$1=2ap\times\frac{dp}{dt}$$

$$\frac{dp}{dt}=\frac{1}{2ap}$$

$$\frac{dy_Q}{dt}=\frac{dy_Q}{dq}\times\frac{dq}{dt}$$

$$=2aq\times\frac{-q}{p}\times\frac{dp}{dt} \text{ (using part (i))}$$

$$=2aq\times\frac{-q}{p}\times\frac{1}{2ap} \text{ (from above)}$$

$$=\frac{-q^2}{p^2}$$

$\therefore$ The y-coordinate of Q is decreasing at the rate $\frac{q^2}{p^2}$ units per second.

Question 14 (15 marks)
(a) (4 marks)

Sample answer:

Let $P(n)$ be the given proposition. $P(1)$ is true since $1^3+5(1)=6$ which is divisible by 6.

Assume $P(k)$ is true for some positive integer k.
i.e. $k^3+5k=6M$ for some integer M.

Prove $P(k+1)$ is true:

$$(k+1)^3+5(k+1)=k^3+3k^2+8k+6$$
$$=6M-5k+3k^2+8k+6 \text{ (using assumption)}$$
$$=6M+6+3k^2+3k$$
$$=6(M+1)+3k(k+1).$$

Since either k or $k+1$ is even, $k(k+1)$ is divisible by 2. $\therefore 3k(k+1)$ is divisible by 6.
Also, as M is an integer, $6(M+1)$ is divisible by 6.
Hence, the above expression is divisible by 6.

$\therefore$ By the Principle of Mathematical Induction, $P(n)$ is true for integers $n\geq 1$.

(b)(i) (2 marks)

Sample answer:

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\binom{n}{3}a^{n-3}b^3+\binom{n}{4}a^{n-4}b^4+\ldots+\binom{n}{n}b^n$$

$$(a-b)^n=\binom{n}{0}a^n-\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2-\binom{n}{3}a^{n-3}b^3+\binom{n}{4}a^{n-4}b^4+\ldots+\binom{n}{n}b^n$$

$$(a+b)^n+(a-b)^n=2\binom{n}{0}a^n+2\binom{n}{2}a^{n-2}b^2+2\binom{n}{4}a^{n-4}b^4+\ldots+2\binom{n}{n}b^n$$

(b)(ii) (1 mark)

Sample answer:

The probability of obtaining exactly two sixes is given by $\binom{n}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2}$.

(b)(iii) (2 marks)

Sample answer:

$$P(\text{even number of sixes})=\binom{n}{0}\left(\frac{5}{6}\right)^n+\binom{n}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2}+\binom{n}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{n-4}+\ldots+\binom{n}{n}\left(\frac{1}{6}\right)^n$$

$$=\frac{1}{2}\left(\left(\frac{5}{6}+\frac{1}{6}\right)^n+\left(\frac{5}{6}-\frac{1}{6}\right)^n\right) \text{ using part i}$$

$$=\frac{1}{2}\left(1+\left(\frac{2}{3}\right)^n\right)$$

(c)(i) (1 mark)

Sample answer:
$0\leq t\leq 1$

(c)(ii) (3 marks)

Sample answer:
The equation of the line through $A(-1,0)$ with gradient t is $y=t(x+1)$.

For the coordinates of P solve $y=t(x+1)$ and $x^2+y^2=1$ simultaneously:

$$x^2+\left(t(x+1)\right)^2=1$$
$$\left(1+t^2\right)x^2+2xt^2+\left(t^2-1\right)=0$$

Since one of the roots of this quadratic is $x=-1$ (x-coordinate of A) and the product of roots is $\frac{t^2-1}{1+t^2}$, the other root is $\frac{1-t^2}{1+t^2}$.

Hence the x coordinate of P is $\frac{1-t^2}{1+t^2}$.

Substituting into $y=t(x+1)$ gives $y=t\left(\frac{1-t^2}{1+t^2}+1\right)=\frac{2t}{1+t^2}$

Hence the y coordinate of P is $\frac{2t}{1+t^2}$.

(c)(iii) (1 mark)

Sample answer:

$$\tan\theta = m_{OP}$$

$$= \frac{2t}{1+t^2} \div \frac{1-t^2}{1+t^2}$$

$$= \frac{2t}{1-t^2},\ t \neq 1.$$

(c)(iv) (1 mark)

Sample answer:

$\theta = 2\alpha$ since the angle at the centre of a circle is twice the angle at the circumference, subtended by the same arc.

$\alpha = 22\frac{1}{2}° \Rightarrow \theta = 45°$

$\therefore \tan\theta = 1$

Hence, $\frac{2t}{1-t^2} = 1$

$t^2 + 2t - 1 = 0$

$t = \frac{-2 \pm \sqrt{8}}{2}$

$t = -1 \pm \sqrt{2}$

$t = -1 + \sqrt{2}$, since $0 \leq t \leq 1$