2005 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension I

General Instructions
- Reading time – 5 minutes
- Working time –2 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Answer each question in a SEPARATE Writing Booklet

Total marks – 84
- Attempt Questions 1 – 7
- All questions are of equal value

Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of standard trial papers. Moreover, some questions have been adapted from previous HSC examinations as well as from trial examinations from various schools and other sources, in an attempt to provide students with exposure to a broad range of possible questions.
However, there is no guarantee whatsoever that the 2005 HSC examination will have similar content, style or format. This paper is intended only as a trial for the HSC examination or as revision leading to the examination.

Question 1 (12 marks) Use a separate writing booklet **Marks**

(a) Evaluate: $\lim_{x \to 0} \frac{\sin 3x}{2x}$ **1**

(b) Find: $\frac{d}{dx}\left[\ln\sqrt{\frac{1+x}{1-x}}\right]$ **2**

(c) Evaluate: $\int_{-3}^{3} \frac{dx}{x^2+9}$ **2**

(d) State the domain and range of the function : $f(x) = 2\cos^{-1} 3x$ **2**

(e) The variable point (2cosθ, 3sinθ) lies on a curve. Find the cartesian equation of this curve. **2**

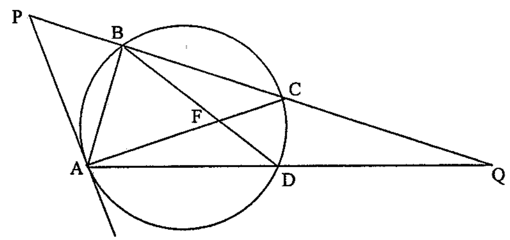
(f) Use the substitution $\sqrt{x} = u$ to evaluate: $\int_{1}^{4} \frac{dx}{x+\sqrt{x}}$ **3**

Question 2 (12 marks) Use a separate writing booklet

(a) Solve: $3^{x+1} = 5$. Give your answer correct to two decimal places. **2**

(b) Solve: $x^3 + 2x^2 - 5x - 6 = 0$ **2**

(c) NOT TO SCALE



In the above figure, AP is a tangent to the circle at A. PBCQ and ADQ are straight lines. Prove that $\angle PAB = \frac{1}{2}(\angle CFD + \angle CQD)$ **3**

(d) Evaluate: $2\int_{0}^{\frac{\pi}{4}} \cos^2 4x \, dx$ **3**

(e) Find the general solution to: $\cos 5\theta - \cos 2\theta = 0$ **2**

Question 3 (12 marks) Use a separate writing booklet **Marks**

(a) Seven people occupy 7 seats at a circular table. In how many ways can they be seated if :

(i) any person can be seated in any seat? **1**

(ii) two particular persons must be seated next to each other? **1**

(iii) two particular persons refuse to sit next to each other? **1**

(b) (i) Show that $f(x) = 3\sin 2x - x$ has a root between

1·33 and 1·34. **1**

(ii) Starting with $x = 1{\cdot}33$, use one application of Newton's method to find a better approximation for this root correct to 4 decimal places. **3**

(c) Consider the graph of $y = \dfrac{x^2}{4-x^2}$

(i) Write down the asymptotes of the function. **1**

(ii) Show that it is an even function. **1**

(iii) Find any stationary points and determine their nature. **2**

(iv) Sketch the graph. **1**

Question 4 (12 marks) Use a separate writing booklet **Marks**

(a) (i) Prove by mathematical induction

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 \quad \text{for } n \geq 1$$ **4**

(ii) Hence evaluate: $2^3 + 4^3 + 6^3 + \cdots + 20^3$ **1**

(b) The polynomial $P(x) = 2x^3 - 5x^2 - 3x + 1$ has zeros α, β and γ. Find the values of

(i) $3\alpha + 3\beta + 3\gamma - 4\alpha\beta\gamma$ **2**

(ii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ **1**

(iii) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(c) A particle moves in SHM on a horizontal line and its acceleration is $\dfrac{d^2x}{dt^2} = 36 - 9x$, where x is the displacement after t seconds.

(i) Find the centre of its motion. **1**

(ii) State its period. **1**

(iii) If the particle is initially at rest at $x = 6$, find the amplitude. **1**

Question 5 (12 marks) Use a separate writing booklet **Marks**

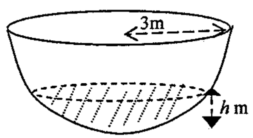
(a) A student takes a test with 10 questions and guesses on each question. If the probability of guessing the correct answer is 0·3 on each question, calculate the probability that the student answers

(i) exactly 6 questions correctly. Give correct to 3 decimal places **1**

(ii) at least one question correctly. Give correct to 3 decimal places **1**

(b) (i) Show that 1 is a root of $h^3 - 9h^2 + 8 = 0$ and find the other roots. **2**

NOT TO SCALE



(ii) A hemi-spherical bowl has a radius of 3m. Oil is poured into the container at a constant rate of $\pi/3$ m^3/min. When the depth is h metres, the volume of oil is

$$V = \frac{\pi}{3}\left(9h^2 - h^3\right) m^3.$$

(α) How deep is the oil after 8 minutes? **2**

(β) At what rate is h increasing at this time? **2**

(c) Three functions $S(x)$, $C(x)$ and $T(x)$ are defined as

$$S(x) = \frac{1}{2}\left(e^x - e^{-x}\right),\quad C(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \text{ and } T(x) = \frac{S(x)}{C(x)}$$

Show that

(i) $[C(x)]^2 - [S(x)]^2 = 1$ **1**

(ii) $\frac{d}{dx}[S(x)] = C(x)$ **1**

(iii) If $T(x) = U$, then $x = \frac{1}{2}\left(\frac{1+U}{1-U}\right)$ **1**

(iv) By using (iii) or otherwise, find the value of x for which $T(x) = 0{\cdot}9$. **1**

Question 6 (12 marks) Use a separate writing booklet **Marks**

(a) From a cliff 100 m high, a projectile is launched at 90 m/s. Take the origin to be at the point of projection with upward direction as positive and $g = 10$ m/s^2.

(i) Show that if the angle of projection is α, the equation of trajectory is given by

$$y = \frac{-5x^2}{8100}\left(1 + \tan^2\alpha\right) + x\tan\alpha$$ **3**

(ii) If the projectile lands 500m away from the base of the cliff,

show that $125\tan^2\alpha - 405\tan\alpha + 44 = 0$. **2**

(iii) At what angle must it be launched to hit the target 500 m away from the base of the cliff? **2**

(b) (i) Using the result for tan (A+B), prove that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$ **2**

(ii) Given A, B and C are angles of a triangle and

$\frac{\tan A}{5} = \frac{\tan B}{6} = \frac{\tan C}{7} = k$, show that $k = \sqrt{\frac{3}{35}}$ **2**

(iii) Hence calculate the smallest of the angles to the nearest minute. **1**

Question 7 (12 marks) Use a separate writing booklet **Marks**

(a) A cup of soup with a temperature 95^0C is placed in a room which has a temperature of 20^0C. In 10 minutes the cup of soup cools to 70^0C. Assuming the rate of heat loss is proportional to the excess of its temperature above room temperature, that is

$$\frac{dT}{dt} = -k(T - 20),$$

(i) show that $T = 20 + Ae^{-kt}$ is a solution of $\frac{dT}{dt} = -k(T-20)$. **1**

(ii) find the temperature of the soup after a further 5 min. to the nearest degree. **2**

(iii) how long will it take the soup to cool to 35^0C? Give your answer correct to the nearest minute. **1**

(iv) find the rate of cooling when the soup is 35^0C. Give your answer correct to 1 decimal place. **1**

(b) (i) State the binomial theorem for $(1+x)^n$ where n is a positive integer. **1**

(ii) If k is a positive integer, show that $\left(1+\frac{1}{n}\right)^k$ approaches 1 as $n \to \infty$ **1**

(iii) Show that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all positive integral $n \geq 3$. **3**

(iv) Use (iv) or otherwise deduce that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = k$ where $2 < k < 3$. **2**

STANDARD INTEGRALS

$\int x^n dx = \frac{1}{n+1}x^{n+1}, n \neq -1; x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} dx = \ln x, \ x > 0$

$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$

$\int \cos ax dx = \frac{1}{a}\sin ax, \ a \neq 0$

$\int \sin ax dx = -\frac{1}{a}\cos ax, \ a \neq 0$

$\int \sec^2 ax dx, = \frac{1}{a}\tan ax, \ a \neq 0$

$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \ a \neq 0$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(x+\sqrt{x^2-a^2}), \ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x+\sqrt{x^2+a^2})$

NOTE : $\ln x = \log_e x, \quad x > 0$

ANSWERS QUESTION 1

Question1 (a) (b)

Criteria	Marks
• (a) One mark for the answer.	1
• (b) One mark for differentiation and one for simplification.	2

Answers:

(a) $\lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\frac{3\times\sin 3x}{3\times 2x}=\lim_{x\to 0}\left[\frac{3\sin 3x}{2\times 3x}\right]=\frac{3}{2}\times 1=\frac{3}{2}$

(b) $y=\ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

$y=\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)$

$\frac{dy}{dx}=\frac{1}{2(1+x)}+\frac{1}{2(1-x)}=\frac{1}{2}\left(\frac{1-x+1+x}{1-x^2}\right)=\frac{1}{1-x^2}$

Answers:

Question1 (c) (d) (e)

Criteria	Marks
• (c) One mark for integration and one for simplification	2
• (d) One mark for the domain and one for the range	2
• (e) One mark for using the identity $\sin^2\theta+\cos^2\theta=1$ and one for finding the curve	2

(c)

$\int_{-3}^{3}\frac{dx}{x^2+9}=\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_{-3}^{3}$

$=\frac{1}{3}\left[\tan^{-1}1-\tan^{-1}(-1)\right]$

$=\frac{1}{3}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi}{6}$

(d) Domain: $-1\le 3x\le 1$ or

$-\frac{1}{3}\le x\le\frac{1}{3}$

Range: $0\le y\le 2\pi$

(e) $x=2\cos\theta$ $\therefore \cos\theta=x/2$

$y=3\sin\theta$ $\therefore \sin\theta=y/3$

since $\sin^2\theta+\cos^2\theta=1$

$\frac{x^2}{4}+\frac{y^2}{9}=1$

Question 1 (f)

Criteria	Marks
• One mark for finding du, one mark for integration and one for simplification	3

Answer:

$I=\int_1^4\frac{dx}{x+\sqrt{x}}$

Let $\sqrt{x}=u$ then $\frac{1}{2}x^{-\frac{1}{2}}dx=du$

$dx=2u\,du$

When $x=4$, $u=2$ and x=1, u=1

$I=\int_1^2\frac{2u\,du}{u^2+u}=\int_1^2\frac{2\,du}{1+u}$

$=2[\ln(1+u)]_1^2=2(\ln 3-\ln 2)=2\ln\frac{3}{2}$

ANSWERS QUESTION 2

Question 2 (a) (b)

Criteria	Marks
• (a) one for taking logs and one for simplification	2
• (b) one for finding a factor and one for finding the roots	2

(a)

$3^{x+1}=5$

$(x+1)\ln 3=\ln 5$

$\therefore x+1=\frac{\ln 5}{\ln 3}$

$x=\frac{\ln 5}{\ln 3}-1=0.464..=0\cdot 46$ (2 dec. place)

(b) $P(x)=x^3+2x^2-5x-6$

Constant term is $-6=-1\times 2\times 3$, ...

We try ± 1, ± 2, ± 3

$P(-1)=-1+2+5-6=0$ $\therefore x+1$ is a factor

Dividing x^3+2x^2-5x-6 by $x+1$ we get

x^2+x-6

Factors of x^2+x-6 are $(x+3)$ and $(x-2)$

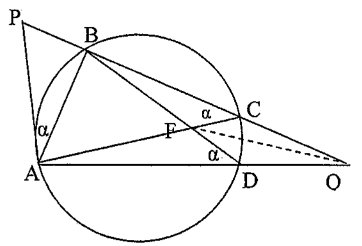
$\therefore$ $x^3+2x^2-5x-6=(x+1)(x+3)(x-2)$

$\therefore$ Roots of $x^3+2x^2-5x-6=0$ are -1,-3&2

Question2(c)

Criteria	Marks
• one for ∠ PAB = ∠ ACB (Angle between the tangent the chord is equal to the angle in the alternate segment), one for ∠ BCA +∠ ADF =∠CQF + ∠CFQ+∠ DQF +∠DFQ and one for conclusion	3

Answer:



Proof: Join FQ

∠ PAB = ∠ ACB (Angle between the tangent the chord is equal to the angle in the alternate segment)

= ∠ ADB = α (Angles in the same segment)

In Δ FQC, External ∠ BCA = ∠CQF + ∠CFQ (Sum of interior opposite angles)

In ΔFQD, External ∠ADF (α) = ∠ DQF +∠ DFQ

∠ BCA +∠ ADF =∠CQF + ∠CFQ+∠ DQF +∠DFQ

= (∠CQF+∠ DQF) +(∠CFQ+∠DFQ)

= ∠ CQD + ∠ CFD

But ∠ BCA = ∠ ADF = α

∴ 2 α = ∠ CQD + ∠ CFD

α = ½ (∠ CQD + ∠ CFD)

Question 2 (d) (e)

Criteria	Marks
• (d) one for $2\cos^2 4x = \cos 8x + 1$, one for integration and one for evaluation	3
• (e) one for each answer	2

Answers:

(d) $I = 2\int_0^{\frac{\pi}{4}} \cos^2 4x \, dx$

$\cos 8x = 2\cos^2 4x - 1$ *or* $2\cos^2 4x = \cos 8x + 1$

$\therefore I = \int_0^{\frac{\pi}{4}} (\cos 8x + 1)\, dx = \left[\frac{\sin 8x}{8} + x\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{\sin 2\pi}{8} + \frac{\pi}{4}\right] - \left(\frac{\sin 0}{8} - 0\right) = \frac{\pi}{4}$

(e) $\cos 5\theta = \cos 2\theta$

$\therefore 5\theta = 2n\pi \pm 2\theta$, where n is an integer

i.e. $3\theta = 2n\pi$ *or* $7\theta = 2n\pi$

$\therefore \theta = \frac{2n\pi}{3}$ *or* $\frac{2n\pi}{7}$

ANSWERS QUESTION 3

Question 3 (a) (i) (ii) (iii)

Criteria	Marks
• (i) one for the correct answer	1
• (ii) one for the correct answer	1
• (iii) one for the correct answer	1

(a) (i) Without restrictions, seven people can be seated around a table in (7-1)! = 6! = 720 ways

(ii) Two people can be grouped together and hence they can be seated in (6-1)! = 5! ways and the 2 grouped people can be seated in 2 ways.

∴ Altogether they can be seated in 5! × 2 = 240 ways.

(iii) The number of arrangements in which those two are not together = 720 – 240 = 480 ways

Question 3(b) (i) (ii) (c) (i) (ii) (iii) (iv)

Answers:

Criteria	Marks
• (b) (i) one for the correct conclusion	1
• (b) (ii) one for evaluating $f'(x_1)$, one for substituting into formula and one for simplification	3
• (c) (i) one for the correct answer	1
• (c) (ii) one for differentiation and one for nature of stationary point	2
• (c) (iii) one for the correct answer	1
• (c) (iv) one for the correct sketch	1

(b)

(i) $f(1{\cdot}33) = 3\sin(2{\cdot}66) - 1{\cdot}33 = 0.05957.. > 0$

and $f(1{\cdot}34) = 3\sin(2{\cdot}68) - 1{\cdot}34 = -0.00387.. < 0$

∴ There is a root between 1·33 and 1·34

(ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x) = 3\sin 2x - x \quad \therefore f'(x) = 6\cos 2x - 1$

$f'(1{\cdot}33) = 6\cos(2{\cdot}66) - 1 = -6{\cdot}31755$

$f(x_1) = 3\sin(2{\cdot}66) - 1{\cdot}33 = 0{\cdot}05957$

$x_2 = 1{\cdot}33 - \frac{0{\cdot}05957}{-6{\cdot}31755} = 1.33942 = 1{\cdot}3394\,(4dpl)$

(c) (i) $y = \frac{x^2}{4 - x^2} = \frac{1}{\frac{4}{x^2} - 1}$ *(dividing x^2)*

As $x \to \infty$, $y \to -1$ ∴ $y = -1$ is a horizontal asymptote.

If $x \to \pm 2$, $y \to \infty$ ∴ $x \pm 2$ are vertical asymptotes.

(ii) $\frac{dy}{dx} = \frac{(4 - x^2) \times 2x - x^2 \times -2x}{(4 - x^2)^2} = \frac{8x}{(4 - x^2)^2}$

∴ stationary point at $x = 0$

When $x = 0^-$, $\frac{dy}{dx} < 0$ and when $x = 0^+$, $\frac{dy}{dx} > 0$

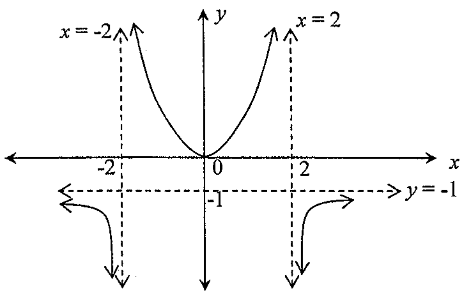
∴ minimum at $x = 0$ and min.point is (0,0)

(iii) $y = f(x) = \frac{x^2}{4 - x^2}$ and

$f(-x) = \frac{(-x)^2}{4 - (-x)^2} = \frac{x^2}{4 - x^2} = f(x)$

∴ *f(x)* is even and the graph is symmetrical about the *y*- axis

(iv)



ANSWERS QUESTION 4

Question 4 (a) (i) (ii) (b) (i) (ii) (iii)

Criteria	Marks
• (a) (i) one for substituting assumption, one for showing L.H.S= $[\frac{1}{4}k^2(k+1)^2]+(k+1)^3$, one for proving $n=k+1$, and one for conclusion.	4
• (a) (ii) one for the correct answer	1
• (b) (i) one for finding $\alpha+\beta+\gamma$ and $\alpha\beta\gamma$, and one for simplification	2
• (b) (ii) one for the correct answer	1
• (b) (iii) one for the correct answer	1

(i) For $n=1$, LHS $=1$ RHS $=\frac{1}{4}\times1^2\times2^2=1$ $\therefore$ It is true for $n=1$ Assume it to be true for $n=k$ i.e. $1^3+2^3+\cdots k^3=\frac{1}{4}k^2(k+1)^2$ To prove that it is true for $n=k+1$ i.e. $1^3+2^3+\cdots k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$ Now L.H.S= $[[1^3+2^3+\cdots+k^3]+(k+1)^3]$ $=[\frac{1}{4}k^2(k+1)^2]+(k+1)^3$ $=\frac{1}{4}(k+1)^2[k^2+4k+4]$	$=\frac{1}{4}(k+1)^2(k+2)^2$ = R.H.S conclusion: by assuming the result is true for $n=k$ we have proved it true for $n=k+1$. Since it is true for $n=1$ $\therefore$ it is true for $n=1+1=2$, $n=2+1=3$ and so it is true for all positive integers (ii) $2^3+4^3+6^3+\cdots+20^3$ $=(2^3\times1^3)+(2^3\times2^3)+(2^3\times3^3)+\cdots(2^3\times10^3)$ $=2^3(1^3+2^3+3^3+\cdots+10^3)$ $=8\times\frac{1}{4}\times10^2\times11^2=2\times100\times121=24200$

(b) (i) $\alpha+\beta+\gamma=\frac{5}{2}$ and $\alpha\beta\gamma=-\frac{1}{2}$ $3(\alpha+\beta+\gamma)-4\alpha\beta\gamma=3\times\frac{5}{2}-\left(4\times-\frac{1}{2}\right)$ $=\frac{15}{2}+2=9\frac{1}{2}$	(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\lambda}=\frac{-\frac{3}{2}}{-\frac{1}{2}}=3$ (iii) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ $=\left(\frac{5}{2}\right)^2-2\left(-\frac{3}{2}\right)=9\frac{1}{4}$

Question4(c) (i) (ii) (iii)

Criteria	Marks
• (i) one mark for the correct answer	1
• (ii) one mark for the correct answer	1
• (iii) one mark for the correct answer	1

Answers:

(i) At the centre of motion the acceleration is 0. $\therefore 9x=36$ or $x=4$ (ii) $\frac{d^2x}{dt^2}=-9(x-4)$ *i.e.* $n^2=9$ *or* $n=3$	$\therefore$ Period= $\frac{2\pi}{n}=\frac{2\pi}{3}$ seconds (iii) Centre of motion is $x=4$, velocity $=0$ at $x=6$ $\therefore$ amplitude $=6-4=2$

ANSWERS QUESTION 5

Question5 (a) (i) (ii) (b) (i) (ii) (α) (β)

Criteria	Marks
• (a) (i) one for the correct answer	1
• (a) (ii) one for the correct answer	1
• (b) (i) one for showing 1 is a root and one for finding the other roots	2
• (b) (ii) (α) one for h^3-9h^2+8 and one for simplification	2
• (b) (ii) (β) one for $\frac{dV}{dt}=\frac{\pi}{3}(18h-3h^2)\frac{dh}{dt}$ and one for simplification	2

(a) n=10 p= prob.correct answers = 0.3 q = prob. Incorrect answers= 1- 0·3 = 0·7 X= no.of correct answers	(i) P(X=6) = ${}^{10}c_6q^4p^6$ $=210\times0{\cdot}2401\times0.000729=0{\cdot}037$ (3 dp) (ii) P(X≥1) = 1- $q^{10}=1-(0{\cdot}7)^{10}=0{\cdot}972$(3 dp)

(b) (i) $f(h)=h^3-9h^2+8$

$f(1)=1-9+8=0$

$\therefore$ h − 1 is a factor

Dividing h^3-9h^2+8 by $h-1$ we get h^2-8h-8

Solving, $h^2-8h-8=0$

$$h=\frac{8\pm\sqrt{64+32}}{2}$$

$$=\frac{8\pm4\sqrt{6}}{2}=4\pm2\sqrt{6}$$

$\therefore$ The roots are 1, $4\pm2\sqrt{6}$ ---------- (1)

(b) (ii) (α) when t = 8, $V=\frac{8\pi}{3}m^3$

$$\therefore\ \frac{8\pi}{3}=\frac{\pi}{3}\left(9h^2-h^3\right)$$

or $h^3-9h^2+8=0$

$\therefore h=1$ or $4\pm2\sqrt{6}$ from (1)

$4-2\sqrt{6}<0$ and $4+2\sqrt{6}>3$ and hence we reject these two values for h.

$\therefore h=$ height of bowl after 8 min is 1 m

(β) Given $\frac{dV}{dt}=\frac{\pi}{3}m^3/\text{min}$

and $V=\frac{\pi}{3}\left(9h^2-h^3\right)$

$$\frac{dV}{dt}=\frac{\pi}{3}\left(18h-3h^2\right)\frac{dh}{dt}$$

When t = 8, h = 1 and $\frac{dV}{dt}=\frac{\pi}{3}$

$$\therefore\ \frac{\pi}{3}=\frac{\pi}{3}(18-3)\frac{dh}{dt}\quad\text{or}\quad\frac{dh}{dt}=\frac{1}{15}$$

i.e. h is increasing at $\frac{1}{15}$ m/min.

Question5(c) (i) (ii) (iii) (iv)

Criteria	Marks
• (c) (i) one for the correct answer	1
• (c) (ii) one for the correct answer	1
• (c) (iii) one for the correct answer	1
• (c) (iv) one for the correct answer	1

Answers:

(i) $[C(x)]^2-[S(x)]^2=$

$$=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$=\frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4}=\frac{4}{4}=1$$

(ii) $\frac{d}{dx}[S(x)]=\frac{1}{2}\left[e^x+e^{-x}\right]=C(x)$

(iii) $U=\frac{e^x-e^{-x}}{e^x+e^{-x}}\Rightarrow Ue^x+Ue^{-x}=e^x-e^{-x}$

$\Rightarrow e^{-x}(U+1)=e^x(1-U)$

$\Rightarrow e^{2x}=\frac{1+U}{1-U}\quad\Rightarrow 2x=\ln\left(\frac{1+U}{1-U}\right)$

$\Rightarrow x=\frac{1}{2}\ln\left(\frac{1+U}{1-U}\right)$

(iv) If $T(x)=0{\cdot}9$, then x

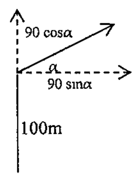
$=\frac{1}{2}\left(\frac{1\cdot9}{0\cdot1}\right)=\frac{1}{2}\ln19\cong1\cdot472$

ANSWERS QUESTION 6

Question 6 (a) (i) (ii) (iii) (b) (i) (ii)(iii)

Criteria	Marks
• (a) (i) one for $t=\frac{x}{90\cos\alpha}$, one for $y=-5t^2+90\sin\alpha\cdot t$, and one for conclusion	3
• (a) (ii) one for $-100=-\frac{5}{8100}\times250\,000\left(1+\tan^2\alpha\right)+500\tan\alpha$, and one for simplification	2
• (a) (iii) one for $\tan\alpha=\frac{405\pm\sqrt{405^2-4\times125\times44}}{250}$, and one for simplification	2
• (b) (i) $\tan[A+(B+C)]=\dfrac{\tan A+\frac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\cdot\frac{\tan B+\tan C}{1-\tan B\tan C}}$, and one for simplification	2
• (b) (ii) $5k+6k+7k=5k\times6k\times7k$ and one for simplification	2
• (b) (iii) one for the correct answer	1

(a) (i)



Horizontal motion:

$$\frac{d^2x}{dt^2}=0,$$

Integrating, $\frac{dx}{dt}=C_1$; when $t=0$; $\frac{dx}{dt}=90\cos\alpha$

$\therefore C_1=90\cos\alpha$

$\therefore\frac{dx}{dt}=90\cos\alpha$

Integrating, $x=(90\cos\alpha)\,t+C_2$

When $t=0; x=0\ \therefore C_2=0\ \therefore t=\frac{x}{90\cos\alpha}$ - - - (1)

Vertical motion:

$$\frac{d^2y}{dt^2}=-10$$

Integrating, $\frac{dy}{dt}=-10t+C_3$

When $t=0$, $\frac{dy}{dt}=90\sin\alpha$; $\therefore C_3=90\sin\alpha$

$\frac{dy}{dt}=-10t+90\sin\alpha$

Integrating, $y=-5t^2+90\sin\alpha\cdot t+C_4$

When $t=0, y=0\ \therefore C_4=0$

$\therefore y=-5t^2+90\sin\alpha\cdot t$

Substituting for t the value from (1), we get

$$y=-5\left(\frac{x}{90\cos\alpha}\right)^2+90\sin\alpha\cdot\frac{x}{90\cos\alpha}$$

$$=-\frac{5x^2\sec^2\alpha}{8100}+x\tan\alpha$$

$$=-\frac{5}{8100}x^2\left(1+\tan^2\alpha\right)+x\tan\alpha$$

(a) (ii) When $x = 500$, $y = -100$

$$-100 = -\frac{5}{8100}\times 250\,000\left(1+\tan^2\alpha\right)+500\tan\alpha$$

$$-1 = -\frac{125}{81}\left(1+\tan^2\alpha\right)+5\tan\alpha$$

$$-81 = -125-125\tan^2\alpha+405\tan\alpha$$

or $125\tan^2\alpha-405\tan\alpha+44=0$

(iii) $\therefore \tan\alpha = \dfrac{405\pm\sqrt{405^2-4\times125\times44}}{250}$

$\doteqdot 3{\cdot}127448\ldots$ or $0{\cdot}1125182..$

$\alpha = 72^0 16'$ or $6^0 25'$

(b) (i) $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$$\tan[A+(B+C)] = \frac{\tan A+\tan(B+C)}{1-\tan A\tan(B+C)}$$

$$= \frac{\tan A+\dfrac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\cdot\dfrac{\tan B+\tan C}{1-\tan B\tan C}}$$

$$= \frac{\dfrac{\tan A-\tan A\tan B\tan C+\tan B+\tan C}{1-\tan B\tan C}}{\dfrac{1-\tan B\tan C-\tan A\tan B-\tan A\tan C}{1-\tan B\tan C}}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

(b) (ii) In ΔABC, A+B+C = 180^0

$\therefore$ tan (A+B+C) = tan $180^0 = 0$

$\therefore \tan A+\tan B+\tan C-\tan A\tan B\tan C = 0$

or $\tan A+\tan B+\tan C = \tan A\tan B\tan C$

Given $\dfrac{\tan A}{5}=\dfrac{\tan B}{6}=\dfrac{\tan C}{7}=k$

$\therefore \tan A = 5k$, $\tan B = 6k$ and $\tan C = 7k$

$5k+6k+7k = 5k\times 6k\times 7k$

$18k = 210k^3$ or $k(35k^2-3) = 0$

$\therefore k=0$ $k = \pm\sqrt{\dfrac{3}{35}}$ The only valid value of k = $+\sqrt{\dfrac{3}{35}}$ since $\tan A>0$, $\tan B>0$ and $\tan C>0$ and A,B and C are acute.

(b) (iii) Smallest angle is A where $\tan A = 5k$

$\tan A = 5\times\sqrt{\dfrac{3}{35}}$ $\therefore A = 55^0 40^1$ (to the nearest minute)

ANSWERS QUESTION 7

Question 1 (a) (i) (ii) (iii) (iv)

Criteria	Marks
• (a) (i) one for the correct answer	1
• (a) (ii) one for finding k and one for simplification	2
• (a) (iii) one for the correct answer	1
• (a) (iv) one for the correct answer	1

$(a)(i)$ $T = 20+Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T-20)$$

(ii) *when* $t=0, T=95$

$\therefore 95 = 20+A \therefore A = 75$

$T = 20+75e^{-kt}$

$t=10, T=70$

$\therefore 70 = 20+75e^{-10k}$

$\therefore e^{-10k} = \dfrac{2}{3} \therefore k = \dfrac{-1}{10}\ln\dfrac{2}{3}$

$\therefore T = 20+75e^{\frac{1}{10}\ln\frac{2}{3}t}$

$t=15 \therefore T = 20+75e^{\frac{1}{10}\ln\frac{2}{3}(15)}$

$T = 60.82\ldots = 61^0 C$

(iii) $35 = 20+75e^{\frac{1}{10}\ln\frac{2}{3}t}$

$$t = \frac{\ln\frac{1}{5}}{\frac{1}{10}\ln\frac{2}{3}} = 39.69.. = 40\text{ min}$$

(iv) $\dfrac{dT}{dt} = 75\times\frac{1}{10}\ln\frac{2}{3}\,e^{\frac{1}{10}\ln\frac{2}{3}t}$

$t = 40 \therefore -0.600.. = -0.6$ deg *per* min

Question 7(b) (i) (ii) (iii) (iv)

Criteria	Marks
• (b) (i) one for the correct answer	1
• (b) (ii) one for the correct answer	1
• (b) (iii) one for $\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\cdot\frac{n}{n}\cdot\frac{(n-1)}{n}+\cdots+\frac{1}{r!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-r+1}{n}+\cdots+\frac{1}{n^n}$, one for $n\to\infty$, $\left(1+\frac{1}{n}\right)^n \to 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots$, one for simplification	3
• (b) (iv) one for $(1+1+\frac{1}{2!})+\frac{1}{3!}+\frac{1}{4!}+\cdots\cdots < (1+1+\frac{1}{2!})+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots\cdots$, and one for simplification	2

(b) (i)

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_rx^r + \cdots + x^n$$
$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots + x^n$$

(ii) $\left(1+\frac{1}{n}\right)^k = 1 + k\cdot\frac{1}{n} + \frac{k(k-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{1}{n^k}$

If k is fixed and as $n \to \infty$, then $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \cdots \frac{1}{n^k} \to 0$ and so $\left(1+\frac{1}{n}\right)^k \to 1$

(iii) $\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}\cdot\frac{1}{n^r} + \cdots + \frac{1}{n^n}$

$$= 1 + 1 + \frac{1}{2!}\cdot\frac{n}{n}\cdot\frac{(n-1)}{n} + \cdots + \frac{1}{r!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-r+1}{n} + \cdots + \frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\cdot 1\cdot\left(1-\frac{1}{n}\right) + \cdots\cdots + \frac{1}{r!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) + \cdots + \frac{1}{n^n}$$

As $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots$

$\therefore$ As $n \to \infty$, $\left(1+\frac{1}{n}\right)^n$ approaches the sum of an infinite series and the sum is clearly greater than 2.

Now $n! = 1\cdot 2\cdot 3\cdots\cdots(n-1)\,n$ there are n terms and in

$2^{n-1} = 2\bullet 2\bullet 2\bullet 2 \cdots\cdots\cdots\bullet 2$ there are $(n-1)$ terms

$\therefore n! > 2^{n-1}$ for all except $n \le 2$ when $n! = 2^{n-1}$

Hence $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all integral $n \ge 3$

(iv) From this $\frac{1}{3!} < \frac{1}{2^2}$, $\frac{1}{4!} < \frac{1}{2^3}$, $\frac{1}{5!} < \frac{1}{2^4}, \cdots$

$\therefore (1 + 1 + \frac{1}{2!}) + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots < (1 + 1 + \frac{1}{2!}) + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\cdots$

$= 1 +$ (limiting sum of a geometric series where $a = 1$ and $r = \frac{1}{2}$)

$$= 1 + \frac{1}{1-\frac{1}{2}} = 3$$

$\therefore \quad (1 + 1 + \frac{1}{2!}) + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots < 3$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = k$, where $2 < k < 3$

Mathematics Extension 1

2005 TRIAL HSC Examination Mapping Grid

Question	Marks	Content	Syllabus outcomes	Targeted Performance bands
1(a)	1	Trigonometric Functions	H5	1-2
1(b)	2	Logarithmic and Exponential Functions	H5	1-2
1(c)	2	Inverse Functions and the Inverse Trigonometric Functions	HE4	1-2
1(d)	2	Inverse Functions and the Inverse Trigonometric Functions	HE4	2-3
1(e)	2	Trigonometric Functions	H5	1-2
1(f)	3	Methods of Integration	HE6	2-3
2(a)	2	Logarithmic and Exponential Functions	H3	1-2
2(b)	2	Polynomials	PE3	2-3
2(c)	3	Circle Geometry	PE3	2-3
2(d)	3	Methods of Integration	HE6	2-3
2(e)	2	Further Trigonometry	PE6	2-3
3(a)(i)	1	Permutations, Combinations and Further Probability	PE3	2-3
3(a)(ii)	1	Permutations, Combinations and Further Probability	PE3,HE3	2-3
3(a)(iii)	1	Permutations, Combinations and Further Probability	PE3,HE3	3-4
3(b)(i)	1	Iterative methods for numerical estimation	HE7	2-3
3(b)(ii)	3	Iterative methods for numerical estimation	HE7	2-3
3(c)(i)	1	Harder Applications of Maths (2 unit)	H5	2-3
3(c)(ii)	2	Harder Applications of Maths (2 unit)	H5	2-3
3(c)(iii)	1	Harder Applications of Maths (2 unit)	H5	
3(c)(iv)	1	Harder Applications of Maths (2 unit)	H5	
4(a)(i)	4	Induction	HE2	2-3
4(a)(ii)	1	Induction	HE2	2-3
4(b)(i)	2	Polynomials	PE3	3-4
4(b)(ii)	1	Polynomials	PE3	2-3
4(b)(iii)	1	Polynomials	PE3	2-3
4(c)(i)	1	Simple Harmonic motion	HE3,HE5	2-3
4(c)(ii)	1	Simple Harmonic motion	HE3,HE5	2-3
4(c)(iii)	1	Simple Harmonic motion	HE3,HE5	2-3
5(a)(i)	1	Further Probability	HE3	2-3
5(a)(ii)	1	Further Probability	HE3	2-3