2004 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension II

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Answer each question in a SEPARATE Writing Booklet

Total marks – 120
- Attempt Questions 1 – 8
- All questions are of equal value

Question 1 (15 marks) Use a separate page/booklet **Marks**

(a) Find : $\int x\sqrt{3x-1}\,dx$ **3**

(b) By using the substitution $t = \tan\dfrac{\theta}{2}$, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\sin\theta}$$ **3**

(c) (i) Split into partial fractions: $\dfrac{8}{(x+2)(x^2+4)}$ **2**

(ii) Hence evaluate: $\displaystyle\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$ **3**

(d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$, $(n \geq 2)$

(i) Show that $I_n = (n-1)\,I_{n-2} - (n-1)\,I_n$ **2**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ **2**

Question 2 (15 marks) Use a separate page/booklet **Marks**

(a) If $z = 3 + 2i$, plot on an Argand diagram

(i) z and $\bar{z}$ **1**

(ii) iz **1**

(iii) $z(1 + i)$ **1**

(b) (i) Find all pairs of integers a and b such that $(a + ib)^2 = 8 + 6i$ **1**

(ii) Hence solve: $z^2 + 2z(1 + 2i) - (11 + 2i) = 0$ **2**

(c) (i) If $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, *find* z^6 **2**

(ii) Plot on an argand diagram, all complex numbers that are the solutions of $z^6 = 1$ **2**

(d) Sketch the locus of the following. Draw separate diagrams.

(i) $\arg(z - 1 - 2i) = \frac{\pi}{4}$ **1**

(ii) $z\bar{z} - 3(z + \bar{z}) \leq 0$ **2**

(iii) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ **2**

Question 3 (15 marks) Use a separate page/booklet **Marks**

(a) For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(i) Find the eccentricity. **1**

(ii) Find the coordinates of the foci S and S'. **1**

(iii) Find the equations of the directricies. **1**

(iv) Sketch the curve $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **1**

(v) Show that the coordinates of any point P can be represented by $(5\cos\theta, 4\sin\theta)$ **2**

(vi) Show that $PS + PS'$ is independent of the position of P on the curve. **2**

(vii) Show that the equation of the normal at the point P on the ellipse is $5x\sin\theta - 4y\cos\theta - 9\sin\theta\cos\theta = 0$ **2**

(viii) If the normal meets the major axis at L and the minor axis at M, prove that $\frac{PL}{PM} = \frac{16}{25}$ **2**

(ix) Show that the normal bisects $\angle SPS'$ **3**

Question 4 (15 marks) Use a separate page/booklet **Marks**

(a) The depth of water in a harbour on a particular day is $8 \cdot 2\ m$ at low tide and $14 \cdot 6\ m$ at high tide. Low tide is at 1:05 pm and high tide is at 7:20 pm.

The captain of a ship drawing $13 \cdot 3m$ water wants to leave the harbour on that afternoon. Find between what times he can leave. **5**

(b) If a > 0, b > 0 and c > 0, show that

(i) $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$ **2**

(ii) $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$ **2**

(iii) $(a+b+d)(b+c+d)(c+a+d)(a+b+c) \geq 81abcd$ **2**

(c) Using mathematical induction prove that $(1+x)^n - nx - 1$ is divisible by x^2. **4**

Question 5 (15 marks) Use a separate page/booklet **Marks**

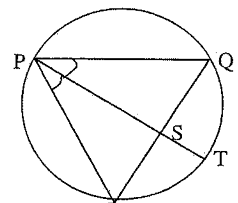
(a) A concrete beam of length $15m$ has plane sides. Cross-sections parallel to the ends are rectangular. The beam measures $4m$ by $3m$ at one end and $8m$ by $6m$ at the other end.

(i) Find an expression for the area of a cross-section at a distance x metres from the smaller end. **3**

(ii) Find the volume of the beam. **2**

(b) Find the volume of the solid generated by rotating the area bounded by the curve $y = \log_e x$, the x - axis and the line $x = 4$. Use the method of *cylindrical shells.* **4**

(c)



In the diagram, the bisector of the angle RPQ meets RQ in S and the circum-circle of the triangle PQR in T.

(i) Prove that the triangles PSQ and PRT are similar. **1**

(ii) Show that $PQ \times PR = PS \times PT$ **1**

(iii) Prove that $PS^2 = PQ \times PR - RS \times SQ$ **1**

(d) Use the expansion $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \text{-------} + {}^nC_nx^n$ to show

(i) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \text{-------------} + {}^nC_n = 2^n$ **1**

(ii) ${}^nC_1 + 2\ {}^nC_2 + 3\ {}^nC_3 + \text{---------} + n\ {}^nC_n = n \cdot 2^{n-1}$ **1**

(iii) ${}^nC_2 + 2\ {}^nC_3 + 3\ {}^nC_4 \text{-------} + (n-1)\ {}^nC_n = n \cdot 2^{n-1} - 2^n + 1$ **1**

Question 6 (15 marks) Use a separate page/booklet **Marks**

(a) A point is moving in a circular path about O.

(i) Define the angular velocity of the point with respect to O, at any time t. **1**

(ii) Derive expressions for the tangential and normal accelerations of the point at any time t. **4**

(b) A light inextensible string OP is fixed at the end O and is attached at the other end P to a particle of mass m which is moving uniformly in a horizontal circle whose centre is vertically below and distant x from O.

(i) Prove that the period of this motion is $2\pi\sqrt{\frac{x}{g}}$, where g is the acceleration due to gravity. **3**

(ii) What is the effect on the motion of the particle if the mass is doubled? **1**

(iii) If the number of revolutions per second is increased from 2 to 3, find the change in x. (Take g = 10 m/s^2) **2**

(c) The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets a directrix at Q. S is the corresponding focus.

(i) Find the equation of the tangent at P. **1**

(ii) Find the coordinates of Q. **1**

(iii) Show that PQ subtends a right angle at S. **2**

Question 7 (15 marks) Use a separate page/booklet **Marks**

(a) Given $y=\frac{x^3}{x^2-4}$

(i) Find the coordinates of all stationary points. **2**

(ii) Find the points of intersection with the coordinate axes and the position of all asymptotes. **1**

(iii) Hence sketch the curve $y=\frac{x^3}{x^2-4}$ **1**

(b) Use the graph $y=\frac{x^3}{x^2-4}$ to find the number of roots of the equation $x^3-k(x^2-4)=0$ for varying value of k. **2**

(c) Sketch the following curves:

(i) $y=\log_e(x+1)$ **1**

(ii) $y=\log_e|x+1|$ **1**

(iii) $y=|\log_e(x+1)|$ **1**

(iv) $y=\frac{1}{\log_e(x+1)}$ **2**

(d) A sociologist believes that the fraction $y(t)$ of a population who have heard a rumour after t days can be modelled by a continuous function given by $y(t)=\frac{y_0e^{kt}}{(1-y_0)+y_0e^{kt}}$, $t\geq 0$, where y_0 is the fraction, $0\leq y_0 < 1$ for all $t\geq 0$, who have heard the rumour at time $t=0$ and k is a positive constant.

(i) Show that $y_0\leq y(t)<1$ for all $t\geq 0$. **1**

(ii) Find the rate of change of y with respect to t. **1**

(iii) If $k = 0{\cdot}2$ and $y_o = 0{\cdot}1$, show that $y(5) = \dfrac{e}{e+9}$ **1**

(iv) Give an interpretation of the above results (i), (ii) and (iii) in terms of the sociological model. **1**

Question 8 (15 marks) Use a separate page/booklet **Marks**

(a) (i) Find a polynomial $p(x)$ with real coefficients having $3i$ and $1+2i$ as zeros. **2**

(ii) Find all zeros of the equation $6x^4 - 7x^3 - 28x^2 + 35x - 10 = 0$. **3**

(b) A body is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth.

(i) Prove that the speed v at any position x is given by $v^2 = u^2 + 2gR^2\left(\dfrac{1}{x} - \dfrac{1}{R}\right)$ **3**

(ii) Prove that the greatest height H above the Earth's surface is given by $H = \dfrac{u^2R}{2gR - u^2}$ **2**

(iii) Show that the body will escape from the Earth if $u \geq \sqrt{2gR}$ **1**

(iv) Find the minimum speed in km/s with which the body must be initially projected from the surface of the Earth so as to never return. (Take $R = 6400\,km,\ g = 10\,m/s^2$) **1**

(v) If $u = \sqrt{2gR}$, prove that the time taken to reach a height $3R$ above the surface of the Earth is $\dfrac{14}{3}\sqrt{\dfrac{R}{2g}}$. **3**

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Mathematics Extension II

- **Solutions including marking scale**
- **Mapping grid**

Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of past HSC papers. Moreover, some questions have been adapted from previous HSC examinations as well as from trial examinations from a variety of schools, in an attempt to provide students with exposure to a broad range of possible questions.

However, there is no guarantee whatsoever that the 2004 HSC examination will have similar content, style or format. This paper is intended only as a trial for the HSC examination or as revision leading up the examination.

ANSWERS QUESTION 1

Question 1 (a)

Criteria	Marks
• One mark for changing the variable, one for integration and one for simplification	3

Answer:

$I = \int x\sqrt{3x-1}\,dx$

Let $3x-1 = u^2$ then $3dx = 2\,u\,du$

$x = \frac{1}{3}(1+u^2)$ and $dx = \frac{2}{3}u\,du$

$I = \int \frac{1}{3}(1+u^2)\cdot u \cdot \frac{2}{3}u\,du$

$= \frac{2}{9}\int\left[(1+u^2)u^2\right]du$

$= \frac{2}{9}\left(\frac{u^3}{3}+\frac{u^5}{5}\right)+c$

$= \frac{2}{9}\left[\frac{(3x-1)^{\frac{3}{2}}}{3}+\frac{(3x-1)^{\frac{5}{2}}}{5}\right]+c$

$= \frac{2}{9}(3x-1)^{\frac{3}{2}}\left[\frac{1}{3}+\frac{(3x-1)}{5}\right]+c$

$= \frac{2}{9}(3x-1)^{\frac{3}{2}}\left(\frac{9x+2}{15}\right)+c$

$= \frac{2}{135}(3x-1)^{\frac{3}{2}}(9x+2)+c$

Question 1(b)

Criteria	Marks
• One mark for $\int_0^1 \frac{dt}{t^2+t+1}$, one for $\frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right]$ and one for simplification.	3

Answer:

Given $t = \tan\frac{\theta}{2}$

$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

$= \frac{1}{2}(1+t^2)\,d\theta$

or $d\theta = \frac{2}{1+t^2}\,dt$

Also $\sin\theta = \frac{2t}{1+t^2}$

$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\sin\theta}$

$= \int_0^1 \frac{2}{1+t^2}\times\frac{1}{2+\frac{2t}{1+t^2}}\,dt$

$= 2\int_0^1 \frac{1}{1+t^2}\times\frac{1+t^2}{2+2t^2+2t}\,dt$

$= \int_0^1 \frac{dt}{t^2+t+1}$

$= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]_0^1$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right]$

$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{3\sqrt{3}}$

$= \frac{\sqrt{3}\pi}{9}$

Question 1(c) (i)

Criteria	Marks
• One mark for writing the equations and one for solving them	2

Answer:

Let $\dfrac{8}{(x+2)(x^2+4)} = \dfrac{a}{x+2} + \dfrac{bx+c}{x^2+4}$

then $8 = a(x^2+4) + (bx+c)(x+2)$

Equating the coefficients of x^2, x and the constant term, we have

$a + b = 0$ -------(1) or $a = -b$

$2b + c = 0$ ------(2)

$4a + 2c = 8$ -----(3)

or $-2b + c = 4$ ----(3)′

From eqns (2) and (3)′ we have $c = 2$ and $b = -1$

$\therefore\ a = 1 \quad \therefore\ \dfrac{8}{(x+2)(x^2+4)} = \dfrac{1}{x+2} + \dfrac{2-x}{x^2+4}$

Question 1 (c) (ii)

Criteria	Marks
• One mark for writing as three definite integrals, one for evaluating the integrals and one for simplification..	3

Answer:

$$\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)} = \int_0^2 \frac{dx}{x+2} + \int_0^2 \frac{(2-x)dx}{x^2+4}$$

$$= \int_0^2 \frac{dx}{x+2} + \int_0^2 \frac{2\,dx}{x^2+4} - \int_0^2 \frac{x\,dx}{x^2+4}$$

$$= \left[\ln(x+2) + \frac{2}{2}\tan^{-1}\frac{x}{2} - \frac{1}{2}\ln(x^2+4)\right]_0^2$$

$$= \left[\ln 4 + \tan^{-1} 1 - \frac{1}{2}\ln 8\right] - \left[\ln 2 + \tan^{-1} 0 - \frac{1}{2}\ln 4\right]$$

$$= \left[\ln 4 + \frac{\pi}{4} - \frac{1}{2}\ln 8 - \ln 2 + 0 + \frac{1}{2}\ln 4\right]$$

$$= 2\ln 2 + \frac{\pi}{4} - \frac{3}{2}\ln 2 - \ln 2 + \ln 2$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{4} \quad or \quad \ln\sqrt{2} + \frac{\pi}{4}$$

Question 1(d) (i)

Criteria	Marks
• One mark for having to $I_n = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x\,dx$ and one for the final answer.	2

Answer:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx, \quad (n \ge 2)$$

Using integration by parts,

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} x\ \frac{d}{dx}(\sin x)\,dx$$

$$= \left[\cos^{n-1} x\ \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,(n-1)\cos^{n-2} x(-\sin x)\,dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,(1-\cos^2 x)\,dx$$

$$= (n-1)\left\{\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx - \int_0^{\frac{\pi}{2}} \cos^n x\,dx\right\}$$

$$= (n-1)\,[I_{n-2} - I_n]$$

Question 1(d) (ii)

Criteria	Marks
• One mark for finding $I_6 = \frac{5}{6}\cdot\frac{3}{4}\cdot I_2$ and one for the final answer.	2

Answer:

We use the this formula repeatedly to evaluate

$\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ which is I_6

$\therefore I_6 = 5[I_4 - I_6]$ or $6I_6 = 5I_4$

or $I_6 = \frac{5}{6}I_4$ and $I_4 = 3[I_2 - I_4]$ or $4I_4 = 3I_2$

$I_6 = \frac{5}{6}I_4$ or $I_4 = \frac{3}{4}I_2$

$= \frac{5}{6}\cdot\frac{3}{4}\cdot I_2$

$$I_2 = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$$

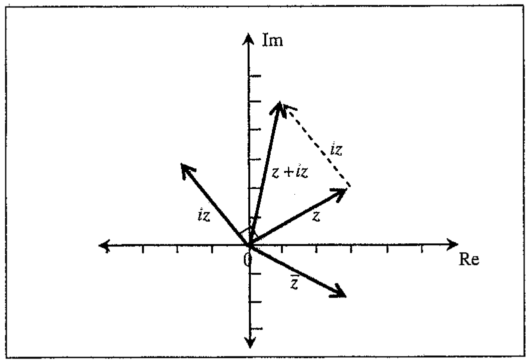
$$\therefore\ I_6 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \frac{5\pi}{32}$$

ANSWERS QUESTION 2

Question 2 (a)

Criteria	Marks
• One mark for plotting z and its conjugate , one for iz ensuring that angle between the vectors is 90^0 and one for the sum .	3

Answer:



Question 2 (b) (i) (ii)

Criteria	Marks
• One for the square root, one for correct application of the quadratic formula and one for final answers.	3

Answer:

(i) Let $(a+ib)^2=8+6i$ or $a^2+2abi-b^2=8+6i$ $\therefore a^2-b^2=8$ and $2ab=6$ or $ab=3$ By inspection we get $a=\pm 3$, $b=\pm 1$ or $\sqrt{8+6i}=\pm(3+i)$ (ii) $z^2+2z(1+2i)-(11+2i)=0$	(ii) contd. $z=\dfrac{-2(1+2i)\pm\sqrt{4(1+2i)^2+4(11+2i)}}{2}$ $=\dfrac{-2-4i\pm\sqrt{4(1+4i-4)+44+8i}}{2}$ $=\dfrac{-2-4i\pm\sqrt{32+24i}}{2}=\dfrac{-2-4i\pm 2\sqrt{8+6i}}{2}$ $=\dfrac{-2-4i\pm 2(3+i)}{2}=2-i$ or $-4-3i$

Question 2 (c) (i) and (ii)

Criteria	Marks
• One mark for writing the value of z^6, one for writing the roots, one for showing that the roots lie on a unit circle, one for showing the roots are separated by 60^0	4

Answer:

(i) $z^6=\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)^6$

$=\left(\cos\dfrac{6\pi}{3}+i\sin\dfrac{6\pi}{3}\right)$ By DeMoivre's theorem

$=\cos 2\pi+i\sin 2\pi=1$

$z^6=1=\cos 2\pi k+i\sin 2\pi k$ where k = 0, 1, 2, 3, 4, and 5

Again by DeMoivre's theorem,

$z=\cos\dfrac{2\pi k}{6}+i\sin\dfrac{2\pi k}{6}=\cos\dfrac{\pi k}{3}+i\sin\dfrac{\pi k}{3}$

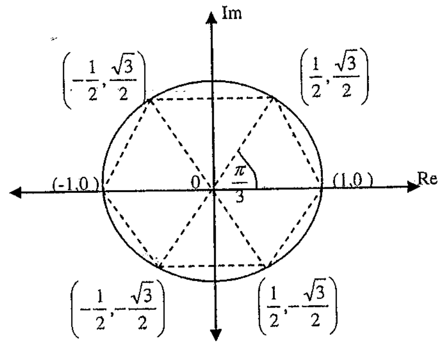
We get the roots z_1, z_2, z_3, z_4, z_5 and z_6 by substituting the values 0,1,2,3,4 and 5 repectively

$z_1=1$; $z_2=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

$z_3=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}=-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$ $z_4=\cos\pi+i\sin\pi=-1$

$z_5=\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}=-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$ $z_6=\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}=\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

(ii)
The roots lie on **a circle of unit radius and each separated by** 60^0.

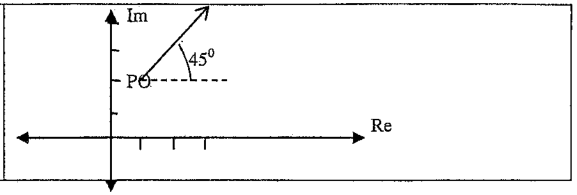


Question 2 (d) (i)

Criteria	Marks
• One mark for the complete answer.	1

Answer:

The locus is a half ray at P (1,2), making 45^0 to the *x- axis*, the point P is **not included.**



Question 2 (d) (ii)

Criteria	Marks
• One for deriving equation of circle and one for enunciating the locus	2

Answer:

If $z = x + iy$, then $\bar{z} = x - iy$

$z\bar{z} - 3(z+\bar{z})$

$= (x+iy)(x-iy) - 3(x+iy+x-iy)$

$= x^2 + y^2 - 6x$

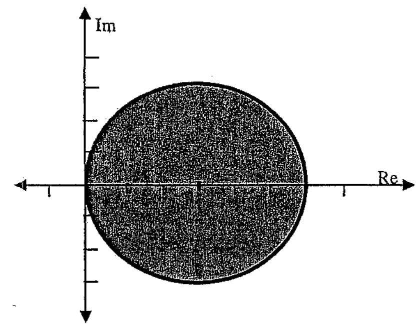
$= (x-3)^2 + y^2 - 9$

$\therefore\ z\bar{z} - 3(z+\bar{z}) \le 0$ is same as

$(x-3)^2 + y^2 - 9 \le 0$

or $(x-3)^2 + y^2 \le 9$

The locus is all points that are **on and inside** the **circle of radius 3 units** and **centre** at **(3, 0)**



Question 2 (d) (iii)

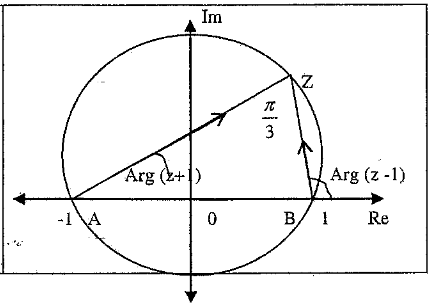
Criteria	Marks
• One mark for for the diagram and one for stating the locus,	2

Answer:

$$Arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$$

The locus is the **major arc of the circle** on the **chord joining** $(-1,0)$ **and** $(1,0)$ which **subtends 60^0 at the circumference.**

The points A $(-1,0)$ and B $(1,0)$ are **not** on the locus.



ANSWERS QUESTION 3

Question 3 (a) (i) to (iv)

Criteria	Marks
• One mark for the correct answers in parts (i), (ii), (iii) and (iv)	4

Answer:

(i) In $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $a = 5$ and $b = 4$

We know that $b^2 = a^2(1-e^2)$

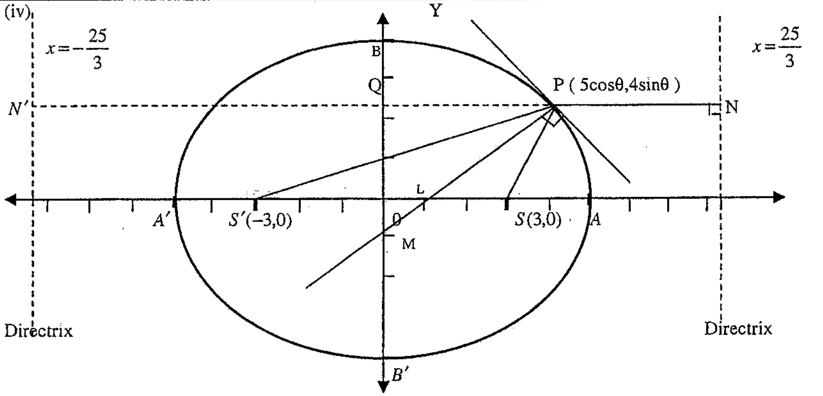
$\therefore\ 16 = 25(1-e^2)$ or $e^2 = 1 - \frac{16}{25}$ i.e. $e = \frac{3}{5}$

(ii) Coordinates of foci are $S\,(ae, 0)$ and $S'(-ae, 0)$

$\therefore S$ *is* $(3,0)$ and S' *is* $(-3,0)$

(iii) Equations of directricies are $(x = \pm\frac{a}{e})$ $x = \pm\frac{5}{3/5} = \pm\frac{25}{3}$

(iv)



Question 3 (a) (v) (vi)

Criteria	Marks
• (v) One mark for substitution and one for simplification.	2
• (vi) One mark for $PS=\frac{3}{5}\left(\frac{25}{3}-5\cos\theta\right)=5-3\cos\theta$ and one for simplification	2

Answer:

(v) $\frac{x^2}{25}+\frac{y^2}{16}=1$

Substituting $(5\cos\theta, 4\sin\theta)$ in L.H.S we get

$\frac{25\cos^2\theta}{25}+\frac{16\sin^2\theta}{16}=\cos^2\theta+\sin^2\theta$

$=1=$ R.H.S

Since $(5\cos\theta, 4\sin\theta)$ satisfies $\frac{x^2}{25}+\frac{y^2}{16}=1$

any point P on the ellipse can be represented by it.

(vi) $\frac{PS}{PN}=e$ (Definition of ellipse)

$\therefore PS=e\cdot PN$ and PN= QN – QP = $\frac{25}{3}-5\cos\theta$

So $PS=\frac{3}{5}\left(\frac{25}{3}-5\cos\theta\right)=5-3\cos\theta$

And $\frac{PS'}{PN'}=e$ or $PS'=e\cdot PN'=5+3\cos\theta$

$\therefore PS+PS'=(5-\cos\theta)+(5+\cos\theta)=10\ (=2a)$

Which is independent of the position of P.

Question 3 (a) (vii) (viii) (ix)

Criteria	Marks
• (vii) One mark for Gradient of normal at P $=\frac{5\sin\theta}{4\cos\theta}$ and one for simplification	2
• (viii) One mark for finding PL and one for simplification	2
• (ix) One mark for $\tan\angle S'PM=\left\lvert\dfrac{\frac{5\sin\theta}{4\cos\theta}-\frac{4\sin\theta}{5\cos\theta+3}}{1+\frac{5\sin\theta}{4\cos\theta}\times\frac{4\sin\theta}{5\cos\theta+3}}\right\rvert$ one mark for proving it equals $\frac{3\sin\theta}{4}$ and one for simplification.	3

(vii) Differentiating $\frac{x^2}{25}+\frac{y^2}{16}=1$ wrt x,

we have $\frac{2x}{25}+\frac{2y}{16}\frac{dy}{dx}=0$

The gradient at any point on the curve is given by

$\frac{dy}{dx}=-\frac{2x}{25}\times\frac{16}{2y}=-\frac{16x}{25y}$

Gradient at P $=-\frac{16\times5\cos\theta}{25\times4\sin\theta}=-\frac{4\cos\theta}{5\sin\theta}$

$\therefore$ Gradient of normal at P $=\frac{5\sin\theta}{4\cos\theta}$

Equation of normal at P is

$y-4\sin\theta=\frac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$

$4y\cos\theta-16\sin\theta\cos\theta=5x\sin\theta-25\sin\theta\cos\theta$

or $5x\sin\theta-4y\cos\theta-9\sin\theta\cos\theta=0$

(viii) This line meets the major axis ($y=0$) at

$L\left(\frac{9}{5}\cos\theta, 0\right)$

Similarly substituting $x=0$ in the normal we get

$M\left(0, \frac{-9}{4}\sin\theta\right)$

$\therefore$ PL $=\sqrt{\left(5\cos\theta-\frac{9}{5}\cos\theta\right)^2+(4\sin\theta)^2}$

$=\sqrt{\left(\frac{16}{5}\cos\theta\right)^2+16\sin^2\theta}$

$=\frac{4}{5}\sqrt{16\cos^2\theta+25\sin^2\theta}$

and PM $=\sqrt{(5\cos\theta)^2+\left(4\sin\theta+\frac{9}{4}\sin\theta\right)^2}$

$=\sqrt{25\cos^2\theta+\left(\frac{25}{4}\sin\theta\right)^2}$

$=\frac{5}{4}\sqrt{16\cos^2\theta+25\sin^2\theta}$

$\therefore \frac{PL}{PM}=\frac{\frac{4}{5}\sqrt{16\cos^2\theta+25\sin^2\theta}}{\frac{5}{4}\sqrt{16\cos^2\theta+25\sin^2\theta}}=\frac{16}{25}$

(ix) Gradient of PS $=\frac{4\sin\theta}{5\cos\theta-3}$

Gradient of PM $=\frac{5\sin\theta}{4\cos\theta}$

$\therefore \tan\angle SPM=\left\lvert\dfrac{\frac{5\sin\theta}{4\cos\theta}-\frac{4\sin\theta}{5\cos\theta-3}}{1+\frac{5\sin\theta}{4\cos\theta}\times\frac{4\sin\theta}{5\cos\theta-3}}\right\rvert$

$=\left\lvert\frac{25\sin\theta\cos\theta-15\sin\theta-16\sin\theta\cos\theta}{20\cos^2\theta-12\cos\theta+20\sin^2\theta}\right\rvert$

$=\left\lvert\frac{9\sin\theta\cos\theta-15\sin\theta}{20-12\cos\theta}\right\rvert$

$=\left\lvert\frac{3\sin\theta(3\cos\theta-5)}{4(5-3\cos\theta)}\right\rvert=\frac{3\sin\theta}{4}$

Gradient of $PS'=\frac{4\sin\theta}{5\cos\theta+3}$

$\therefore \tan\angle S'PM=\left\lvert\dfrac{\frac{5\sin\theta}{4\cos\theta}-\frac{4\sin\theta}{5\cos\theta+3}}{1+\frac{5\sin\theta}{4\cos\theta}\times\frac{4\sin\theta}{5\cos\theta+3}}\right\rvert$

$=\left\lvert\frac{25\sin\theta\cos\theta+15\sin\theta-16\sin\theta\cos\theta}{20\cos^2\theta+12\cos\theta+20\sin^2\theta}\right\rvert$

$=\left\lvert\frac{9\sin\theta\cos\theta+15\sin\theta}{20+12\cos\theta}\right\rvert$

$=\left\lvert\frac{3\sin\theta(3\cos\theta+5)}{4(5+3\cos\theta)}\right\rvert=\frac{3\sin\theta}{4}$

$\therefore \angle SPM=\angle S'PM$

ANSWERS QUESTION 4

Question 4 (a)

Criteria	Marks
• One mark for finding period, one for finding n, one for finding α, one for finding t and one for the final answer.	5

Answer:

Period T = 2 x [7:20 – 1:05] = 2 x 375 = 750 min
Amplitude = a = 0·5 (14·6 – 8·2) = 3·2 m
Since the motion is simple harmonic,

$\ddot{x} = -n^2 x \qquad [\ T = \frac{2\pi}{n}$ or $n = \frac{2\pi}{750} = \frac{\pi}{375}\ ]$

Solution of this equation is

$x = a\cos(nt+\alpha)$, where $0 \le \alpha \le 2\pi$

$= 3\cdot2\cos(nt+\alpha)$,

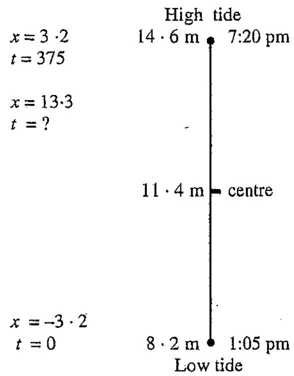
Considering the initial conditions,
When $t = 0$; $x = -3\cdot2$

$-3\cdot2 = 3\cdot2\cos(0+\alpha)$,

$\therefore \cos\alpha = -1$ or $\alpha = \pi$

$\therefore x = 3\cdot2\cos(nt+\pi)$

$= -3\cdot2\cos nt$, since $\cos(\theta+\pi) = -\cos\theta$,



Min. depth required = 13·3
i.e. x = 13·3 – 11·4 = 1·9

$\therefore 1\cdot9 = -3\cdot2\cos nt$

$\therefore nt = \cos^{-1}\left(-\frac{19}{32}\right) = \pi - \cos^{-1}\left(\frac{19}{32}\right)$

or $t = \frac{1}{n}\left[\pi - \cos^{-1}\frac{19}{32}\right] = \frac{375}{\pi}\left[\pi - \cos^{-1}\left(\frac{19}{32}\right)\right]$

≈ 263 min. or 4 h 23 min.

T – t = 750 – 263 = 487 min = 8 h 7 min
Hence the ship can leave the harbour between
1:05 + 4:23 = 5:28 pm and 1: 05 + 8:07 = 9:12 pm

Question 4 (b) (i) (ii)

Criteria	Marks
One for showing $a^2+b^2 \ge 2ab$, one showing $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ge 0$ One for correct expansion and showing $a^3+b^3+c^3 \ge 3abc$ and one for proving the inequality $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$	4

Answer:

i) If $a > 0, b > 0$ and $c > 0$, then

$(a-b)^2 \ge 0 \qquad (0$ if $a = b)$

i.e. $a^2 - 2ab + b^2 \ge 0$

$\therefore a^2 + b^2 \ge 2ab$

Similarly $b^2 + c^2 \ge 2bc$

and $c^2 + a^2 \ge 2ca$

Adding these we get,

$2a^2 + 2b^2 + 2c^2 \ge 2ab + 2bc + 2ca$

or $a^2 + b^2 + c^2 \ge ab + bc + ca$

or $a^2 + b^2 + c^2 - ab - bc - ca \ge 0$

ii) Now $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ge 0$

since a, b and c are each >0

$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= a^3 + ab^2 + ac^2 - a^2b - abc - ca^2 + ba^2 + b^3 + bc^2 - ab^2$
$- b^2c - abc + ca^2 + cb^2 + c^3 - abc - bc^2 - ac^2$

$= a^3 + b^3 + c^3 - 3abc$

or $a^3 + b^3 + c^3 \ge 3abc$

substituting a for a^3, b for b^3 and c for c^3, we get

$a + b + c \ge 3\, a^{\frac{1}{3}} b^{\frac{1}{3}} c^{\frac{1}{3}}$

or $\frac{a+b+c}{3} \ge (abc)^{\frac{1}{3}}$ or $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$

Question 4 (b) (iii)

Criteria	Marks
• One mark for writing the three inequalities and one for writing the product.	2

Answer:

Now $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$ from above

Similarly we can show $\frac{a+b+d}{3} \ge \sqrt[3]{abd}$, $\frac{b+c+d}{3} \ge \sqrt[3]{bcd}$ and $\frac{c+a+d}{3} \ge \sqrt[3]{cad}$

By multiplying we have,

$$\frac{(a+b+c)(a+b+d)(b+c+d)(c+a+d)}{81} \ge \sqrt[3]{a^3b^3c^3d^3}$$

or $(a+b+c)(a+b+d)(b+c+d)(c+a+d) \ge 81abcd$

Question 4 (c)

Criteria	Marks
• One mark for each step.	4

Answer:

Step 1.
When n = 2,
$(1+x)^2-2x-1=1+2x+x^2-2x-1=x^2$ which is divisible by x^2. ∴ True when $n=2$

Step 2.
Let it be true for $n=k$
i.e. $(1+x)^k - xk - 1$ is divisible by x^2
Let $(1+x)^k - xk - 1 = M x^2$ where M is a polynomial in x.
or $(1+x)^k = Mx^2 + xk + 1$

Step 3.
Now to show that it is true for $n=k+1$
i.e. $(1+x)^{k+1}-(k+1)x-1$ is divisible by x^2

Now $(1+x)^{k+1}-(k+1)x-1$
$=(1+x)(1+x)^k-(k+1)x-1$
$=(1+x)(Mx^2+xk+1)-(k+1)x-1$
$=Mx^2+xk+1+Mx^3+kx^2+x-kx-x-1$
$=Mx^2+Mx^3+kx^2$
$=x^2(M+Mx+k)$ which is divisible by x^2

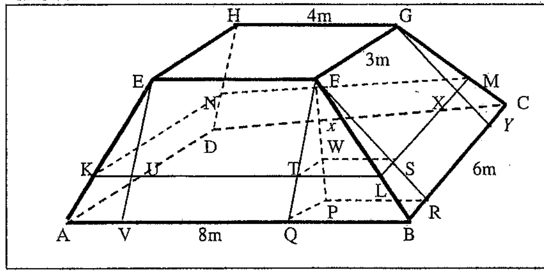
Step4.
If $(1+x)^n-xn-1$ is divisible by x^2 for $n=k$, then it is divisible by x^2 for $n=k+1$. But it is divisible by x^2 for $n=2$. ∴ it is divisible for x^2 for $n=2+1=3$, 4 ------ and so on.

ANSWERS QUESTION 5

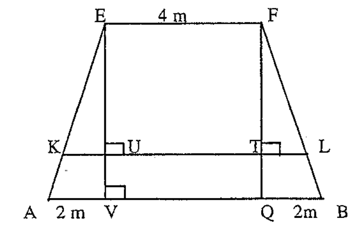
Question 5 (a) (i) (ii)

Criteria	Marks
• One mark for finding KL, one for find LM and one for $A(x)$. • One mark for showing the definite integral and one for evaluating it.	5

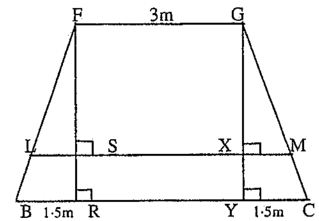
Answer:



In the figure KLMN is the cross-section and its distance from the top is x metres. FW = x metres



Front elevation



Side elevation

(i) If A (x) be the area of KLMN,
then A (x) = KL × LM
In the front elevation shown EV and FQ are perpendicular to AB.

∴ $AV = QB = \left(\frac{8-4}{2}\right) = 2\,m$

Let FP be the perpendicular from F to the end ABCD meeting the plane KLMN in W,
Then FW = x metres and FP = 15 m
Δ s TFW and QFP are similar

∴ $\frac{FT}{FQ}=\frac{FW}{FP}=\frac{x}{15}$

Δ s FTL and FQB are also similar

∴ $\frac{TL}{QB}=\frac{FT}{FQ}=\frac{x}{15}$

∴ $\frac{TL}{2}=\frac{x}{15}$ or $TL=\frac{2x}{15}$ metres

By symmetry $KU=TL=\frac{2x}{15}$ metres

∴ $LS=\frac{x}{15}\times BR=\frac{x}{15}\times 1\cdot 5=\frac{1\cdot 5\,x}{15}=\frac{x}{10}$ metres

Since $LS=XM$, $XM=\frac{x}{10}$ metres

∴ $LM=\left(3+\frac{2x}{10}\right)=\left(3+\frac{x}{5}\right)$ metres

Thus $A(x)=KL\times LM=\left(4+\frac{4x}{15}\right)\left(3+\frac{x}{5}\right)$

$=\{12+\frac{4x}{5}+\frac{12x}{15}+\frac{4x^2}{75}\}=\{12+\frac{24x}{15}+\frac{4x^2}{75}\}\ m^2$

(ii) Volume of the beam

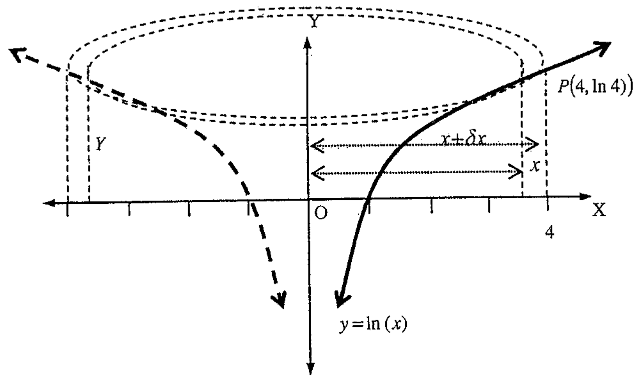
$=\lim_{\partial x\to 0}\sum_{x=0}^{15}A(x)\cdot\partial x=\int_0^{15}\left(12+\frac{24}{15}x+\frac{4x^2}{75}\right)dx$

$KL = KU + UT + TL = \left(\frac{2x}{15}+4+\frac{2x}{15}\right) = \left(\frac{4x}{15}+4\right)$ m In the side elevation shown, by symmetry BR = YC = 1·5 m By similar triangles, $\frac{LS}{BR} = \frac{FS}{FR} = \frac{FW}{FP} = \frac{x}{15}$	$= \left[12x + \frac{12x^2}{15} + \frac{4x^3}{225}\right]_0^{15} =$ $= (12\times15) + \frac{12}{15}\times15^2 + \frac{4}{225}\times15^3$ $= 420\, m^3$

Question 5 (b)

Criteria	Marks
• One mark for the volume of elementary shell, one for the definite integral showing the total volume, one for the first step in evaluation and one for the completion.	4

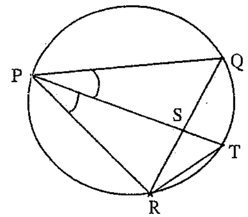
Answer:



The volume ∂v generated when a strip of area between the ordinates at x, $x+\partial x$ is rotated about Y-axis is approximately that of a hollow cylindrical shell of inner circumference $2\pi x$, inner height y and wall thickness ∂x. $\therefore \partial v = 2\pi x y \partial x$ Total volume $V = \underset{\partial x \to 0}{Lt} \sum_{x=1}^{4} 2\pi xy\ \partial x$ $y = \ln x$ $\therefore\ V = \int_1^4 2\pi x \cdot \ln x\ dx = 2\pi \int_1^4 x \ln x\ dx$	$= \pi \int_1^4 \ln x \frac{d}{dx}(x^2) dx =$ $\pi\left[\left(x^2 \ln x\right)_1^4 - \int_1^4 x^2 \cdot \frac{1}{x} dx\right]$ $= \pi\left[16\ln 4 - \left(\frac{x^2}{2}\right)_1^4\right] = \pi\left[16\ln 4 - \left(8 - \frac{1}{2}\right)\right]$ $= \pi\left(16 \ln 4 - \frac{15}{2}\right)$ $unit^3 = 46{\cdot}1$ (1dp) u^3

Question 5 (c) (i) (ii) (iii)

Criteria	Marks
• One mark for each of the parts	3



(i) Join RT. Consider the triangles PSQ and PRT $\angle PQS = \angle PTR$ (angles in the same segment) $\angle QPS = \angle RPT$ (given) ΔPSQ and ΔPRT are similar (AAA) (ii) $\therefore$ their corresponding sides are proportional $\frac{PQ}{PT} = \frac{PS}{PR}$ or $PQ \cdot PR = PS \cdot PT$	(iii) Now $PS \cdot PT = PQ \cdot PR$ $PS(PS + ST) = PQ \cdot PR$ or $PS^2 + PS \cdot ST = PQ \cdot PR$ or $PS^2 = PQ \cdot PR - PS \cdot ST$ $= PQ \cdot PR - RS \cdot SQ$ (Chords PT and RQ intersect in S and hence $PS \cdot ST = RS \cdot SQ$)

Question 5 (d) (i) (ii) (iii)

Criteria	Marks
• One mark for each of the parts.	3

Answer:

(i) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + \text{------------} + {}^nC_nx^n$

putting $x = 1$, we get

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 \text{------------} + {}^nC_n$ ------------ (1)

(ii) Differentiating with respect to x both sides of the binomial expansion, we get

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + 3\,{}^nC_3x^2 + 4\,{}^nC_4x^3 + \text{------------} + n\,{}^nC_nx^{n-1}$

substituting 1 for x, we have

$n\,2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + 4\,{}^nC_4 + \text{------------} + n\,{}^nC_n$ ------------ (2)

(iii) subtracting (1) from (2), we get

$n\,2^{n-1} - 2^n = -{}^nC_0 + {}^nC_2 + 2\,{}^nC_3 + 3\,{}^nC_4 + \text{--------} + (n-1)\,{}^nC_n$

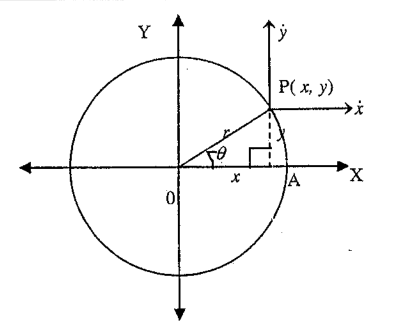
or $n\,2^{n-1} - 2^n + 1 = {}^nC_2 + 2\,{}^nC_3 + 3\,{}^nC_4 + \text{--------} + (n-1)\,{}^nC_n$ since ${}^nC_0 = 1$

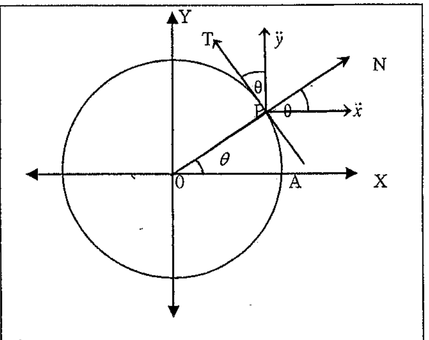
ANSWERS QUESTION 6

Question 6 (a) (i) (ii)

Criteria	Marks
• (i) One mark for defining angular velocity • (i) One mark for deriving $\ddot{x}$, one for $\ddot{y}$, one for tangential acceleration and one for normal acceleration	5

Answer:





(i) Let r be the radius of the circle centre O.
If the point moves from A to P in time t, where $\angle AOP = \theta$, then the angular velocity ω of P is defined as **the rate of change of θ with respect to time** t i.e. $\frac{d\theta}{dt} = \omega$

(ii) The velocity of P at every point on the circle on its path is tangential to the circle.
If arc AP = s, then $s = r\theta$

and $\frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$

From the figures, $\frac{x}{r} = \cos\theta$ or $x = r\cos\theta$

$\frac{y}{r} = \sin\theta$ or $y = r\sin\theta$

$\dot{x} = \frac{dx}{dt} = \frac{d}{dt}(r\cos\theta) = \frac{d}{d\theta}(r\cos\theta) \times \frac{d\theta}{dt}$

$= -r\sin\theta\frac{d\theta}{dt} = -r\sin\theta \cdot \omega$

$\dot{y} = \frac{dy}{dt} = \frac{d}{dt}(r\sin\theta) = \frac{d}{d\theta}(r\sin\theta) \times \frac{d\theta}{dt}$

$= r\cos\theta \cdot \omega$

Corresponding accelerations are given by

$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}(-r\sin\theta \cdot \omega)$

$= -r\sin\theta\frac{d\omega}{dt} + \omega \cdot \frac{d}{d\theta}(-r\sin\theta) \cdot \frac{d\theta}{dt}$

$= -r\sin\theta\frac{d\omega}{dt} - \omega^2 r\cos\theta$

$= -r(\sin\theta\frac{d\omega}{dt} + \omega^2\cos\theta)$

$\ddot{y} = \frac{d^2y}{dt^2} = \frac{d}{dt}(r\,\omega\cos\theta)$

$= r\frac{d\omega}{dt}\cos\theta + r\,\omega\frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt}$

$= r\dot{\omega}\cos\theta - r\omega^2\sin\theta$

$= r(\dot{\omega}\cos\theta - \omega^2\sin\theta)$

These accelerations are **resolved** in the direction of the **tangent PT** and along the **normal PN**.
Tangential acceleration

$= \ddot{y}\cos\theta - \ddot{x}\cos(\frac{\pi}{2} - \theta) = \ddot{y}\cos\theta - \ddot{x}\sin\theta$

$= r\cos\theta(\dot{\omega}\cos\theta - \omega^2\sin\theta)$

$+ r\sin\theta(\sin\theta\frac{d\omega}{dt} + \omega^2\cos\theta)$

$= r\dot{\omega}\cos^2\theta - r\omega^2\sin\theta\cos\theta$

$+ r\dot{\omega}\sin^2\theta + r\omega^2\sin\theta\cos\theta$

$= r\dot{\omega}(\cos^2\theta + \sin^2\theta) = r\ddot{\theta}$

Normal acceleration = $\ddot{y}\cos\left(\frac{\pi}{2} - \theta\right) + \ddot{x}\cos\theta$

$= \ddot{y}\sin\theta + \ddot{x}\cos\theta$

$= r\sin\theta\left(\dot{\omega}\cos\theta - \omega^2\sin\theta\right)$

$- r\cos\theta\left(\dot{\omega}\sin\theta + \omega^2\cos\theta\right)$

$= r\dot{\omega}\sin\theta\cos\theta - r\omega^2\sin^2\theta$

$- r\dot{\omega}\sin\theta\cos\theta - r\omega^2\cos^2\theta$

$= r\left[(\dot{\omega}\sin\theta\cos\theta - \dot{\omega}\sin\theta\cos\theta) - \omega^2(\sin^2\theta + \cos^2\theta)\right]$

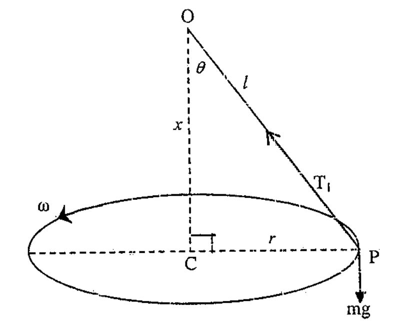
$= -r\omega^2$ i.e it is directed towards centre.

Question 6 (b) (i) (ii) (iii)

Criteria	Marks
• One mark for resolving forces at P, one for deriving ω and one for the period of motion. • One mark for stating no effect, one for finding x_1 and x_2 and one for finding the difference.	6

Answer:

(i)



Let ω be the angular velocity of the particle P about C. The forces acting on the particle are its weight mg and the tension T_1 in the string.
Resolving forces at P :
Horizontally (towards the centre C),

$mr\omega^2 = T_1\cos\left(\frac{\pi}{2} - \theta\right) = T_1\sin\theta$

Vertically
$mg = T_1\cos\theta$
From the above equations we get on division ,

$\frac{mr\omega^2}{mg} = \frac{T_1\sin\theta}{T_1\cos\theta}$ i.e. $\frac{r\omega^2}{g} = \tan\theta$

But $\tan\theta = \frac{r}{x}$ ∴ $\frac{r}{x} = \frac{r\omega^2}{g}$

or $\omega = \sqrt{\frac{g}{x}}$

The time for 1 complete revolution i.e. the period of motion

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{x}}} = 2\pi\sqrt{\frac{x}{g}}$

(ii) In the division of the two equations of motion, the mass is cancelled out and hence there is **no effect** on the motion if there's any change in the mass of the particle.

(iii) From above $\omega^2 = \frac{g}{x}$ $\quad \therefore x = \frac{g}{\omega^2}$

Let ω_1 and ω_2 be the angular velocities of P in the two situations in the problem.

Then $\omega_1 = 2$ revolutions / s $= 4\pi$ radians/s

$\omega_2 = 3$ revolutions / s $= 6\pi$ radians /s

Using ($x = \frac{g}{\omega^2}$) we can write

$$x_1 = \frac{g}{\omega_1^2} = \frac{g}{16\pi^2}$$

and $x_2 = \frac{g}{\omega_2^2} = \frac{g}{36\pi^2}$

$$x_1 - x_2 = \frac{g}{\pi^2}\left(\frac{1}{16} - \frac{1}{36}\right) = \frac{5g}{\pi^2} \approx 0\cdot 035m$$

The particle rises by about **3·5 cm**

Question 6 (c) (i) (ii) (iii)

Criteria	Marks
• (i) One mark for correct answer. (ii) One mark for correct answer. (iii) One mark for gradient of m_2, one for simplification.	4

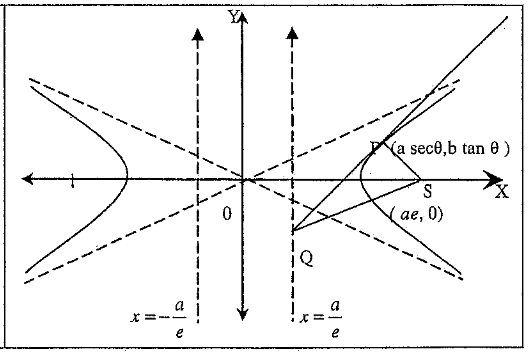
Answer:

(i)

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Differentiating wrt x we get

$$\frac{2x}{a^2} - \frac{2y\frac{dy}{dx}}{b^2} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Gradient at P $= \frac{b^2 a \sec\theta}{a^2 b \tan\theta} = \frac{b}{a\sin\theta}$



Equation of tangent at P is given by

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

$$ay\sin\theta - ab\tan\theta\sin\theta = bx - ab\sec\theta$$

$$bx - ay\sin\theta = ab(\sec\theta - \tan\theta\sin\theta)$$

$$= ab\left(\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}\right)$$

$$= ab\cos\theta$$

(ii) This meets the directrix at $x = \frac{a}{e}$

$$\therefore \frac{ba}{e} - ay\sin\theta - ab\cos\theta = 0$$

or $y = \frac{b(1 - e\cos\theta)}{e\sin\theta}$

$$\therefore Q \text{ is } \left(\frac{a}{e}, \frac{b(1 - e\cos\theta)}{e\sin\theta}\right)$$

(iii)

If the gradient of QS is m_1, then

$$m_1 = \frac{\frac{b(1 - e\cos\theta)}{e\sin\theta}}{\frac{a}{e} - ae} = \frac{b(1 - e\cos\theta)}{a(1 - e^2)\sin\theta}$$

If the gradient of PS is m_2, then

$$m_2 = \frac{b\tan\theta}{a\sec\theta - ae} = \frac{b\sin\theta}{a(1 - e\cos\theta)}$$

Now $m_1 m_2 = \frac{b(1 - e\cos\theta)}{a(1 - e^2)\sin\theta} \times \frac{b\sin\theta}{a(1 - e\cos\theta)}$

$$= \frac{b^2}{a^2(1 - e^2)} = -1 \text{ since } b^2 = a^2(e^2 - 1)$$

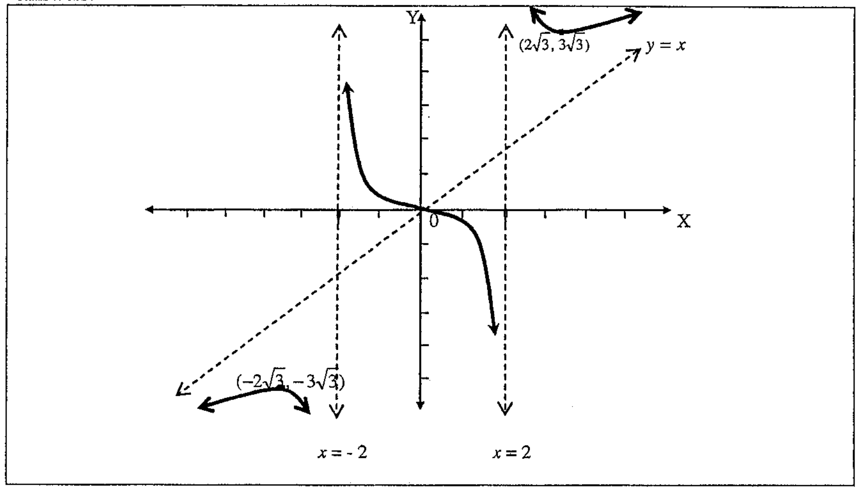
$\therefore$ PS and QS are perpendicular and hence PQ subtends a right angle at S.

ANSWERS QUESTION 7

Question 7 (a) (i) (ii) (iii) and (b)

Criteria	Marks
• One mark for vertical and one for the oblique asymptotes. One for finding the stationary points and one for the sketch. • One mark for solutions when $k < -3\sqrt{3}$ and $k > 3\sqrt{3}$ and one mark for the other values of k	6

Answers:



(i) Given $f(x)=\dfrac{x^3}{x^2-4} = x+\dfrac{4x}{(x-2)(x+2)}$

Vertical asymptotes are $x=\pm 2$

As $x\to\pm\infty$, $\dfrac{4x}{(x-2)(x+2)}\to 0$ and therefore $f(x)\to x$. The line $y=x$ is an asymptote to the the curve

$$f'(x)=\frac{(x^2-4)(3x^2)-(x^3)(2x)}{(x^2-4)^2}$$

$$=\frac{x^2(3x^2-12-2x^2)}{(x^2-4)^2}=\frac{x^2(x^2-12)}{(x^2-4)^2}$$

For stationary points $f'(x)=0$

i.e. $x^2(x^2-12)=0$

$\therefore x=0$ or $\pm\sqrt{12}$ $\quad(\pm 2\sqrt{3})$

When x = $2\sqrt{3}$, $y=\dfrac{(2\sqrt{3})^3}{8}=3\sqrt{3}$

When x = $-2\sqrt{3}$, $y=\dfrac{(-2\sqrt{3})^3}{8}=-3\sqrt{3}$

At $(2\sqrt{3}, 3\sqrt{3})$, $f'(x)$ changes sign from $-$ to $+$ and hence it is a point of minima.

At $(-2\sqrt{3}, -3\sqrt{3})$, $f'(x)$ changes sign from $+$ to $-$ and hence it is a point of maxima.

$f(x)=\dfrac{x^3}{x^2-4}$ and $f(-x)=\dfrac{-x^3}{x^2-4}=-f(x)$

$\therefore f(x)$ is an odd function and it has point symmetry about O.

(ii) From graph

(iii) Refer to graph

(b) $x^3-k(x^2-4)=0$ or $k=\dfrac{x^3}{(x^2-4)}$

Considering the graphs of $y=k$ and $y=\dfrac{x^3}{(x^2-4)}$. For $k>3\sqrt{3}$ and $k<-3\sqrt{3}$, there are **three** solutions **each**.

For $k=\pm 3\sqrt{3}$, there are **two** solutions **each.**

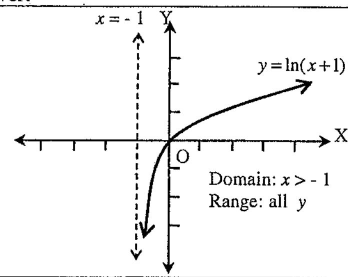
For $-3\sqrt{3}<k<3\sqrt{3}$, there is **one** solution on each side.

Question 7(c) (i) (ii) (iii) (iv)

Criteria	Marks
• One mark for each sketch correctly showing asymptotes and the intercepts on the axes. • For (iv) to get one more, points of intersection of $y=\ln(x+1)$ and $y=[\ln(x+1)]^{-1}$ have to be shown and that (-1,0) does not lie on the reciprocal graph must be mentioned.	5

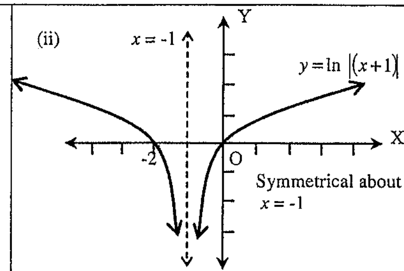
Answer:

(i)



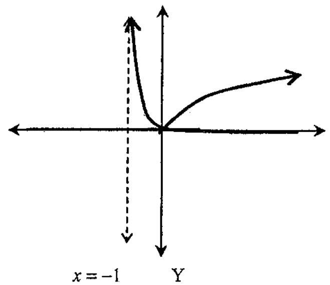
Domain: $x>-1$
Range: all y

(ii)



Symmetrical about $x=-1$

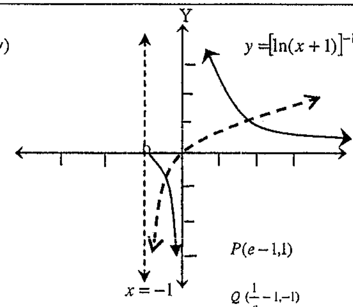
(iii)



$y=|\ln(x+1)|$

Domain: x > -1
Range: $y\geq 0$

(iv)



$y=\ln(x+1)$ and $y=[\ln(x+1)]^{-1}$ intersect at $P(e-1,1)$ and $Q(\frac{1}{e}-1,-1)$.

$[\ln(x+1)]^{-1}\to 0$ as $x\to -1^-$ so **(-1, 0)** is **not** on the curve.

Question 7 (d) (i) (ii) (iii) (iv)

Criteria	Marks
• One mark for each part correctly answered.	4

Answer:

(i)

Given $y(t)=\dfrac{y_0e^{kt}}{(1-y_0)+y_0e^{kt}}, t\geq 0$

and $0\leq y_0<1, \quad k>0$

Now $y(t)-y_0=\dfrac{y_0e^{kt}}{(1-y_0)+y_o\,e^{kt}}-\dfrac{y_0}{(1-y_0)+y_0}$

$=\dfrac{y_0e^{kt}}{(1-y_0)+y_o\,e^{kt}}-y_0$

$=y_0\left(\dfrac{e^{kt}-(1-y_0)-y_0\,e^{kt}}{(1-y_0)+y_0e^{kt}}\right)$

$=y_0\left[\dfrac{e^{kt}(1-y_0)-(1-y_0)}{(1-y_0)+y_0\,e^{kt}}\right]$

$=y_0(1-y_0)\left[\dfrac{e^{kt}-1}{(1-y_0)+y_0\,e^{kt}}\right]\geq 0$

since $y_0\geq 0$, $e^{kt}\geq 1$ as $kt\geq 0$ and $1-y_0>0$

$\therefore \quad y(t)\geq y_0$ ------------ (1)

Also, $1-y_0>0$

So $(1-y_0)+y_0\,e^{kt}>y_0e^{kt}$

$\therefore\ 1>\dfrac{y_0\,e^{kt}}{(1-y_0)+y_0\,e^{kt}}$ i.e. $1>y(t)$ --------- (2)

It follows from (1) and (2) ,

$y_0\leq y(t) < 1$

(ii) $y'(t)=\dfrac{\left[(1-y_0)+y_0e^{kt}\right]ky_0e^{kt}-ky_0e^{kt}y_0e^{kt}}{\left[(1-y_0)+y_0e^{kt}\right]^2}$

$=\dfrac{ky_0e^{kt}\left[(1-y_0)+y_0e^{kt}-y_0e^{kt}\right]}{\left[(1-y_0)+y_0e^{kt}\right]^2}$

$=\dfrac{ky_0e^{kt}(1-y_0)}{\left[(1-y_0)+y_0e^{kt}\right]^2}$ which is positive

(iii) $k=0\cdot 2,\ y_0=0\cdot 1$

$y(5)=\left[\dfrac{0\cdot 1e}{0\cdot 9+0\cdot 1\,e}\right]=\left[\dfrac{\frac{1}{10}e}{\frac{9}{10}+\frac{1}{10}e}\right]=\dfrac{e}{9+e}$

$\approx 0\cdot 23$

(iv) Interpretation:

(i) Means that not everyone in the population hears the rumour.

(ii) Means that the fraction of people who have heard the rumour is increasing with time.

(iii) Means that after 5 days nearly 23% ofthe population has heard the rumour. 10%were told it at first.

ANSWERS QUESTION 8

Question 8 (a) (i) (ii)

Criteria	Marks
• One mark for the for the factors and one for the product. • One each for the two divisions and for the last two zeroes.	5

Answer:

(i) Since $p(x)$ has real coefficients, the conjugates of $-3i$ and $1+2i$ are also zeros of $p(x)$

Hence by factor theorem $p(x)$ has factors $[x-(-3i)]$, $[x-(3i)]$, $[x-(1+2i)]$ and $[x-(1-2i)]$

$\therefore$ A polynomial of **lowest** degree is

$p(x)=[x-(-3i)]\,[x-(3i)]\,[x-(1+2i)]\,[x-(1-2i)]$

$=(x^2+9)(x^2-2x+5)$

$=x^4-2x^3+14x^2-18x+45$

(ii) $6x^4-7x^3-28x^2+35x-10=0$

If p/q is a root , then p must be a factor of -10 and q must be a factor of 6.

Hence p must be one of the integers $\pm 1, \pm 2, \pm 5, \pm 10$ and q must be one the integers $\pm 1, \pm 2, \pm 3, \pm 6$.

$\therefore$ The rational roots of the equation can be $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \ \pm 2, \pm\frac{2}{3}, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{3}$

$\pm\frac{5}{6}$, ± 10 and $\pm\frac{10}{3}$

Now if we choose $x=\frac{1}{2}$, the LHS $=6\left(\frac{1}{2}\right)^4-7\left(\frac{1}{2}\right)^3-28\left(\frac{1}{2}\right)^2+35\left(\frac{1}{2}\right)-10=0$

$\therefore\ x-\frac{1}{2}$ is a factor. Similarly we find $x=\frac{2}{3}$ satisfies the equation and hence $x-\frac{2}{3}$ is a factor also.

Dividing $p(x)$ by successively by $\frac{1}{2}$ and $\frac{2}{3}$ we get (using synthetic division)

$\frac{1}{2}$	6	-7	- 28	35	-10
		3	- 2	-15	10
	6	- 4	-30	20	0

When $6x^4-7x^3-28x^2+35x-10$ is divided by $x-\frac{1}{2}$ we get

$6x^3-4x^2-30x+20$ as quotient and 0 as remainder.

$\frac{2}{3}$	6	- 4	- 30	20
		4	0	- 20
	6	0	-30	0

The quotient when $6x^3 - 4x^2 - 30x + 20$ is divided by $x - \frac{2}{3}$ is $6x^2 - 30$

Hence the remaining roots must be the roots of the equation $6x^2 - 30 = 0$ which are $\sqrt{5}$ *and* $-\sqrt{5}$.

$\therefore$ The roots of the given equation are $\frac{1}{2}, \frac{2}{3}, \sqrt{5}$ *and* $-\sqrt{5}$.

Question 8 (b) (i)

Criteria	Marks
• One mark for equations of motion, one for the integration, one for finding the value of the constant and for the final expression for the speed.	3

Answer:

(i)
Taking the origin at the centre of the Earth and the upward direction as the positive direction of X – axis, equation of motion is

$m\ddot{x} = -\frac{mk}{x^2}$ or $\ddot{x} = -\frac{k}{x^2}$

On the surface of the Earth, $x = R$, $-g = -\frac{k}{R^2}$

or $k = R^2 g$

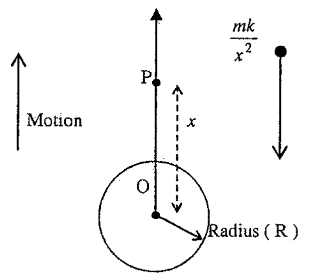
Now $v\frac{dv}{dx} = -\frac{k}{x^2}$ i.e. $\frac{d}{dx}\left[\frac{1}{2}v^2\right] = \frac{-R^2 g}{x^2}$

or $\frac{1}{2}v^2 = \frac{R^2 g}{x} + C$

Initially, $v = u$ and $x = R$

$\therefore \frac{1}{2}u^2 = \frac{R^2 g}{R} + C$ or $C = \frac{1}{2}u^2 - gR$

or $\frac{1}{2}v^2 = \frac{R^2 g}{x} + \frac{1}{2}u^2 - gR$



or $v^2 = \frac{2R^2 g}{x} + u^2 - 2gR$

or $v^2 = u^2 + 2R^2 g\left[\frac{1}{x} - \frac{1}{R}\right]$ which gives the velocity of the body at a distance x from centre.

Question 8 (b) (ii) (iii) (iv) (v)

Criteria	Marks
• One mark for finding x and one for finding height above the Earth's surface. • One mark for showing $u \geq \sqrt{2gR}$ • One mark for finding the min. speed for escaping gravitation. • One mark for finding dt, one for writing the definite integral and one for evaluating the integral.	7

(ii) when the body is at the highest point, $v = 0$ and hence $0 = \frac{2R^2 g}{x} + u^2 - 2gR$

$\frac{2R^2 g}{x} = 2gR - u^2$ or $x = \frac{2R^2 g}{2gR - u^2}$

$\therefore$ The greatest height H above the Earth's surface is given by

$H = \frac{2gR^2}{2gR - u^2} - R = \frac{2gR^2 - 2gR^2 + Ru^2}{2gR - u^2}$

$= \frac{Ru^2}{2gR - u^2}$

(iii) The greatest height H reached by the body will be infinite if $2gR - u^2 = 0$ or $u^2 = 2gR$ or $u = \sqrt{2gR}$. The body will escape from the Earth if $u \geq \sqrt{2gR}$

(iv) The minimum escape velocity is given by $u = \sqrt{2gR} = \sqrt{2 \times 0{\cdot}01 \times 6400}$

$= 11{\cdot}3\ km/s$ correct to 1 dec.

(v) From above, when $u = \sqrt{2gR}$, $v^2 = \frac{2R^2 g}{x}$

or $v = \frac{\sqrt{2gR^2}}{\sqrt{x}}$ i.e. $\frac{dx}{dt} = \frac{\sqrt{2gR^2}}{\sqrt{x}}$

or $dt = \frac{\sqrt{x}}{\sqrt{2gR^2}}dx$

The time for the body to reach a height of $3R$ above the Earth (then $x = 4R$) from initial time t =0 (when $x = R$) is given by

$t = \int_R^{4R} \frac{x^{\frac{1}{2}}}{\sqrt{2gR^2}}dx = \frac{1}{\sqrt{2gR^2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_R^{4R}$

$= \frac{1}{\sqrt{2gR^2}}\left[\frac{2}{3} \times 4^{\frac{3}{2}} R^{\frac{3}{2}} - \frac{2}{3}R^{\frac{3}{2}}\right]$

$= \frac{2}{3} \times \frac{1}{\sqrt{2gR^2}} \times 7R^{\frac{3}{2}} = \frac{14}{3}\sqrt{\frac{R}{2g}}$