2009 Trial Higher School Certificate

Trialmaths Enterprises

Mathematics Extension II

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 120
- Attempt Questions 1 – 8
- All questions are of equal value

Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of past HSC papers. Moreover, some questions have been adapted from previous HSC examinations as well as from trial examinations from a variety of schools, in an attempt to provide students with exposure to a broad range of possible questions.

Start each question on a SEPARATE page or in a SEPARATE booklet.

Question 1 (15 marks) Use a separate page/booklet **Marks**

(a) Find $\int \frac{\sin 2x}{3+\sin^2 x}dx$ **2**

(b) Find $\int \frac{1}{e^x + e^{-x}}dx$. **2**

(c) Evaluate $\int_{\sqrt{3}}^{3} \frac{1}{\sqrt{(x^2-1)}}dx$. **3**

(d) Evaluate $\int_{0}^{\pi/3} \frac{1}{1-\sin x}dx$. **4**

(e) Evaluate $\int_{0}^{4} \frac{x^2+4x+5}{(x+1)(x+3)}dx$. **4**

Question 2 (15 marks) Use a separate page/booklet **Marks**

(a) Find real x and y such that $(x+iy)^2 = 3+4i$. **2**

(b) Find $|z|$ and $\arg z$ when $z = -\sqrt{3} - i$. **2**

(c) (i) Express $1+i$ and $1-i$ in modulus/argument form. **2**

(ii) Hence evaluate $(1+i)^{40} + (1-i)^{40}$. **2**

(d) (i) Solve $z^5 = -1$. **2**

(ii) Hence show that $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$. **3**

(e) Indicate on an Argand diagram the region which contains the point P representing z when $|z| \le |z-4|$ and $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$. **2**

Question 3 (15 marks) Use a separate page/booklet **Marks**

(a) Sketch (showing critical points) the graph of: $y = 3(x + \sqrt{x})$ **2**

(b) If $f(x) = x^2 - 16$, sketch the following graphs on separate axes showing all relevant points.

(i) $y = f(x)$ **1**

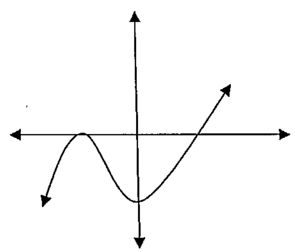
(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = -\dfrac{16}{f(x)}$ **2**

(iv) $y = |f(x)| + 6$ **2**

(c) Sketch the graph $y^2 = x^2(1-x^2)$ **2**

(d) The graph of $y = x^3 + 3x^2 - 4$ is sketched below



Sketch the curves $y = |x^3 + 3x^2 - 4|$ and $y = \ln|x^3 + 3x^2 - 4|$ on separate axes. **2**

(e) Prove $\dfrac{\sin B + \sin C}{\sin B - \sin C} = \tan\left(\dfrac{B+C}{2}\right)\cot\left(\dfrac{B-C}{2}\right)$ **2**

Question 4 (15 marks) Use a separate page/booklet **Marks**

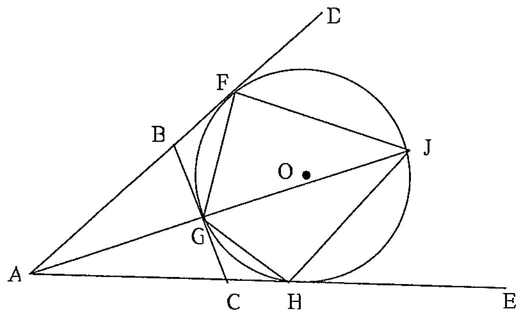
(a) A particle A_1 of mass m kg is dropped from point C and falls towards point D, which is directly underneath C. At the instant when A_1 is dropped, a second particle A_2, also of mass m kg, is projected upwards from D towards C with an initial velocity equal to twice the terminal velocity of A_1. Each particle experiences a resistance of magnitude mkv as it moves, where $v\,ms^{-1}$ is the velocity and k is a constant.

(i) Show that the terminal velocity of A_1 is $\frac{g}{k}$, where g is acceleration due to gravity. **2**

(ii) For particle A_2, show that $t = \frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$, where $v\,ms^{-1}$ is the velocity after t seconds. **3**

(iii) Suppose the particles collide at the instant when A_1 has reached 30% of its terminal velocity. Find the velocity of A_2 when they collide. Leave your answer in terms of g and k. **3**

(b) In the following diagram, the lines AH, AF and BC are all tangents to the circle.



(i) Prove that the triangles AFG and AJF are similar. **1**

(ii) Prove $(\text{AF})^2 = (\text{AJ})\times(\text{AG})$ **1**

(iii) Prove $\text{GJ} = \frac{(\text{AF})\times(\text{AH})}{(\text{AG})} - (\text{AG})$ **2**

(c) At what speed should a car travel round a bend of radius 70 m which is banked at an angle of 11^0. **3**

Question 5 (15 marks) Use a separate page/booklet **Marks**

(a) (i) If $I_n = \int_1^e (\ln x)^n\,dx$ for $n \ge 0$, show that $I_n = e - nI_{n-1}$ for $n \ge 1$. **3**

(ii) Hence evaluate I_4. **2**

(b) A committee of three is to be chosen at random from 4 women and n men $(n \ge 2)$.

(i) Find the number of possible committees containing exactly one woman. **1**

(ii) Find the number of possible committees containing exactly two women. **1**

(iii) Deduce that the probability P of the committee containing either one or two women is $P = \frac{12n}{(n+4)(n+3)}$ **1**

(c) The cubic equation $x^3 - x^2 + 4x - 2 = 0$ has roots α, β and γ.

(i) Find the equation whose roots are α^2, β^2 and γ^2. **2**

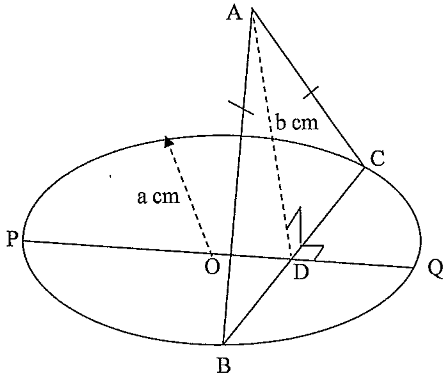
(ii) Find the value of $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$. **3**

(d) Given that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1-x)$ are acute, show that: $\sin(\sin^{-1}x - \cos^{-1}x) = 2x^2 - 1$. **2**

Question 6 (15 marks) Use a separate page/booklet **Marks**

(a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated about the line $y=c\ (c>b)$. Find the volume of the solid generated by using the method of cylindrical shells. **4**

(b) The base of the solid shown in the diagram is a circle of radius a cm.



Each cross section of the solid formed by a plane perpendicular to the fixed diameter PQ is an isosceles triangle of height b cm. One such cros section is shown by ΔABC in the above diagram, where AB=AC, BC is in the base of the solid and AD= b cm.

Find the volume of this solid by first drawing a diagram indicating your origin and other relevant information. **3**

(c) Prove by the method of mathematical induction that $x^3-4x^2+4x+3\geq 0$ where x is a positive integer. **3**

(d) Four men and two women are to be seated around a six seated rectangular table, as shown in the following diagram.

In how many ways can they be arranged so that two particular men, A and B, sit together and a women E, does not sit directly opposite man A? **2**

(e) If $P(x)=2x^4+9x^3+6x^2-20x-24$ has a triple root, factorise $P(x)$ into its linear factors. **3**

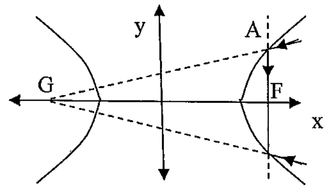
Question 7 (15 marks) Use a separate page/booklet **Marks**

(a) (i) Show that the equation $4x^2-y^2-16hx+2hy+15h^2-4a^2=0$, where h and a are positive constants, represents a hyperbola. **2**

(ii) If the tangent to this hyperbola at the point (p, q) is parallel to the straight line $y=(e^2-1)x$, where e is the eccentricity of the hyperbola, show that $p\text{-}q=h$. **3**

(b) A beam of light aimed at the focus G = *(-ae,0)* of a hyperbolic reflector whose equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, will be reflected through the focus F=*(ae,0)*. The figure shows the case where the light passes through the point A, vertically above F.

Not to scale



For the hyperbola given in part (a) above, $4x^2-y^2-16hx+2hy+15h^2-4a^2=0$, a similar beam of light aimed at its focus G_1, is reflected along the corresponding line A_1F_1, perpendicular to the x-axis.

Find the equation of the line A_1G_1, along which the light was initially travelling. **3**

(c) Given that the equation of the tangent at the point (x_1,y_1) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$, find the equation of the tangent in terms of a and b and its gradient m. **3**

(d) For the hyperbola $4x^2-y^2-16hx+2hy+15h^2-4a^2=0$, a tangent with positive gradient passes through the point $\left(\frac{2a}{3}+2h,h\right)$.

Show that the size of the angle, α, between this tangent and the asymptote with positive gradient, may be expressed as $\alpha=\tan^{-1}\left[\frac{2(41-15\sqrt{5})}{139}\right]$ **4**

Question 8 (15 marks) Use a separate page/booklet **Marks**

(a) (i) Prove that $a^2+b^2 \ge 2ab$ for all real values of a and b. **1**

(ii) Hence prove that $a^2+b^2+c^2 \ge ab+bc+ca$ for all real values of a, b and c. **2**

(iii) Use the factorisation $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ to deduce that $a^3+b^3+c^3 \ge 3abc$ for all <u>positive</u> values of a, b and c. **2**

(iv) Hence prove that $\frac{x_1+x_2+x_3}{3} \ge \sqrt[3]{x_1x_2x_3}$ for positive numbers x_1, x_2 and x_3. **1**

(b) Starting from the formula for $\sin 2x$ and $\cos 2x$

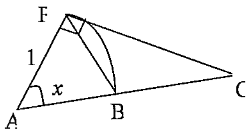
(i) Show that $\sin\frac{\alpha}{2^n}\cos\frac{\alpha}{2^n}=\frac{1}{2}\sin\frac{\alpha}{2^{n-1}}$ **2**

(ii) Show that $\sqrt{\frac{1}{2}+\frac{1}{2}\cos\alpha}=\cos\frac{\alpha}{2}$ **1**

(iii) By considering the area of the triangle APB, the circular sector APB and the triangle APC, show that

$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$, where x is in radians.

Note that the radius of the sector is 1. **3**



(iv) Hence, or otherwise, show that $\lim_{n\to\infty}\left[\frac{\frac{1}{2^n}}{\sin\left(\frac{\alpha}{2^n}\right)}\right]=\frac{1}{\alpha}$ **3**

End of Exam

STANDARD INTEGRALS

$\int x^n dx$	$=\frac{1}{n+1}x^{n+1}, n\ne -1;\ x\ne 0,$ if $n<0$
$\int \frac{1}{x} dx$	$=\ln x,\ x>0$
$\int e^{ax} dx$	$=\frac{1}{a}e^{ax},\ a\ne 0$
$\int \cos ax dx$	$=\frac{1}{a}\sin ax,\ a\ne 0$
$\int \sin ax dx$	$=-\frac{1}{a}\cos ax,\ a\ne 0$
$\int \sec^2 ax dx,$	$=\frac{1}{a}\tan ax,\ a\ne 0$
$\int \sec ax \tan ax dx$	$=\frac{1}{a}\sec ax,\ a\ne 0$
$\int \frac{1}{a^2+x^2} dx$	$=\frac{1}{a}\tan^{-1}\frac{x}{a},\ a\ne 0$
$\int \frac{1}{\sqrt{a^2-x^2}} dx$	$=\sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$
$\int \frac{1}{\sqrt{x^2-a^2}} dx$	$=\ln(x+\sqrt{x^2-a^2}),\ x>a>0$
$\int \frac{1}{\sqrt{x^2+a^2}} dx$	$=\ln(x+\sqrt{x^2+a^2})$

NOTE : $\ln x=\log_e x,\quad x>0$

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EXT II

Mathematics

- **Solutions including marking scale**
- **Mapping grid**

These suggested solutions and marking schemes are issued as a guide only. Individual teachers may find many acceptable responses and employ different marking schemes.

ANSWERS QUESTION 1

Question 1 (a)

Criteria
• One for $\sin 2x = 2\sin x\cos x$, one for simplification.

Answer:

Now $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$ with $f(x) = \sin^2 x + 3$

$$\int \frac{\sin 2x}{3+\sin^2 x}dx = \int \frac{2\sin x\cos x}{3+\sin^2 x}dx = \ln(\sin^2 x + 3) + c$$

Question 1 (b)

Criteria
• One for substitution, one for simplification

Answer:

Using the substitution $e^x = u, du = e^x dx$

$$\int \frac{1}{e^x + e^{-x}}dx = \int \frac{e^x}{e^{2x}+1}dx$$

$$= \int \frac{1}{u^2+1}du = \tan^{-1} u + c = \tan^{-1} e^x + c$$

Question 1 (c)

Criteria
• One for integral, one for substitution, one for simplification.

Answer:

Using $\int \frac{1}{\sqrt{(x^2-a^2)}}dx = \ln\left|x + \sqrt{(x^2-a^2)}\right| + c$ with $a = 1$,

$$\int_{\sqrt{3}}^{3} \frac{1}{\sqrt{(x^2-1)}}dx = \ln\left|x+\sqrt{x^2-1}\right|_{\sqrt{3}}^{3} = \ln(3+\sqrt{8}) - \ln(\sqrt{3}+\sqrt{2}) = \ln(\frac{3+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}).$$

Question 1 (d)

Criteria
• One for using T results, one for limits, one for integral, one for simplification.

Answer:

Let $t=\tan\frac{x}{2}, 0<x<\frac{\pi}{3}, 0<t<\frac{1}{\sqrt{3}}, x=2\tan^{-1}t, dx=\frac{2}{1+t^2}dt$.

Now $\sin x=\frac{2\tan(x/2)}{1+\tan^2(x/2)}$,

$$\int_0^{\pi/3}\frac{1}{1-\sin x}dx=\int_0^{\pi/3}\frac{1}{1-\frac{2\tan(x/2)}{1+\tan^2(x/2)}}dx=\int_0^{1/\sqrt{3}}\frac{1}{1-\frac{2t}{1+t^2}}\times\frac{2}{1+t^2}dt=\int_0^{1/\sqrt{3}}\frac{1+t^2}{1+t^2-2t}\times\frac{2}{1+t^2}dt$$

$$=2\int_0^{1/\sqrt{3}}\frac{1}{(1-t)^2}dt=\left[\frac{2}{1-t}\right]_0^{1/\sqrt{3}}=\frac{2}{1-1/\sqrt{3}}-2=\frac{2\sqrt{3}}{\sqrt{3}-1}-2=\frac{2}{\sqrt{3}-1}$$

$=\sqrt{3}+1$

Question 1 (e)

Criteria
• One for division, one for finding a and b, one for integral, one for simplification

Answer:

$\frac{x^2+4x+5}{(x+1)(x+3)}=1+\frac{2}{(x+1)(x+3)}$.

$\frac{2}{(x+1)(x+3)}=\frac{a}{(x+1)}+\frac{b}{(x+3)}$

$2=a(x+3)+b(x+1)$.

Let $x=-1:\rightarrow 2=2a\Rightarrow a=1$.

Let $x=-3:\rightarrow 2=-2b\Rightarrow b=-1$.

$$\int_0^4\frac{x^2+4x+5}{(x+1)(x+3)}dx$$

$$=\int_0^4 1dx+\int_0^4\frac{2}{(x+1)(x+3)}dx$$

$$=[x]_0^4+\int_0^4\frac{1}{x+1}dx-\int_0^4\frac{1}{x+3}dx$$

$$=4+[\ln|x+1|]_0^4-[\ln|x+3|]_0^4$$

$=4+\ln 5-\ln 1-(\ln 7-\ln 3)$.

$=4+\ln\frac{15}{7}$

ANSWERS QUESTION 2

Question 2 (a)

Criteria
• One for equating real and imaginary parts, one for simplification.

Answer:

$(x+iy)^2=3+4i$

$\left(x^2-y^2\right)+(2xy)i=3+4i$

Equating real parts:

$x^2-y^2=3$

Equating imaginary parts

$2xy=4$

$\therefore x^4-x^2y^2=3x^2$ and $x^2y^2=4$

$\therefore x^4-3x^2-4=0$

$\left(x^2-4\right)\left(x^2+1\right)=0$

$\therefore x=\pm 2,\ y=\pm 1$

Question 2 (b)

Criteria
• One each for modulus and argument.

Answer:

$z=-\sqrt{3}-i=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$

$=2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$.

$|z|=2, \arg z=-\frac{5\pi}{6}$

Question 2 (c) (i) (ii)

Criteria
• (i) One each for finding modulus and argument of 1+i and 1-i (ii) one for using de Moivre's theorem, one for simplification

Answer:

$z=1+i$. $z=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$1-i=\bar{z}=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$.

Using *de* Moivre's theorem $z^{40}=2^{20}\text{cis}(10\pi)$,

$(\bar{z})^{40}=2^{20}\text{cis}(-10\pi)$.

$z^{40}+(\bar{z})^{40}=z^{40}+\overline{\left(z^{40}\right)}=2\text{Re}\left(z^{40}\right)=2^{21}\cos(10\pi)$.

$\therefore(1+i)^{40}+(1-i)^{40}=2^{21}\cos(10\pi)$.

$=2^{21}$

Question 2 (d) (i) (ii)

Criteria
• (i) One for modulus and argument, one for final answer. (ii) One for factors, one for $\left(z-\text{cis}\frac{3\pi}{5}\right)\left(z-\text{cis}\left(-\frac{3\pi}{5}\right)\right)=z^2-2z\cos\frac{3\pi}{5}+1$ one for simplification

Answer:

(i) $|-1|=1$ and $\arg(-1)=\pi$.

Therefore the complex 5th roots of -1 are $\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$, and -1. Note the roots are equally spaced around a unit circle in an Argand diagram by an angle of $\frac{2\pi}{5}$ and modulus of 1.

(ii) $z^5+1=(z+1)\left(z-\text{cis}\frac{\pi}{5}\right)\left(z-\text{cis}\left(-\frac{\pi}{5}\right)\right)\left(z-\text{cis}\frac{3\pi}{5}\right)\left(z-\text{cis}\left(-\frac{3\pi}{5}\right)\right)$.

Note

$$\left(z-\text{cis}\frac{\pi}{5}\right)\left(z-\text{cis}\left(-\frac{\pi}{5}\right)\right)=\left(\left(z-\cos\frac{\pi}{5}\right)-i\sin\frac{\pi}{5}\right)\left(\left(z-\cos\frac{\pi}{5}\right)+i\sin\frac{\pi}{5}\right)=\left(z-\cos\frac{\pi}{5}\right)^2+\left(\sin\frac{\pi}{5}\right)^2=$$

$$=z^2-2z\cos\frac{\pi}{5}+1$$

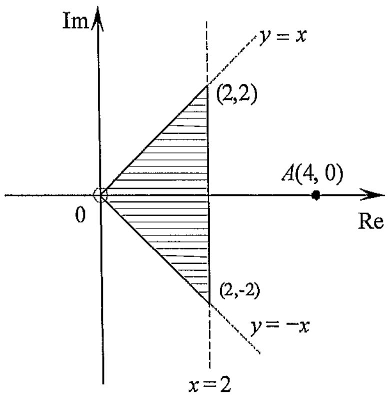
and $\left(z-\text{cis}\frac{3\pi}{5}\right)\left(z-\text{cis}\left(-\frac{3\pi}{5}\right)\right)=z^2-2z\cos\frac{3\pi}{5}+1$.

$\therefore z^5+1=(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$.

Question 2 (e)

Criteria
• One for positions of $y=\pm x$ and coordinates of (2,2) and (2,-2), one for shading.

Answer:



$|z|=|z-4|$ is the perpendicular bisector of OA.

$\arg z=\frac{\pi}{4}$ is the ray $y=x,\ x>0$.

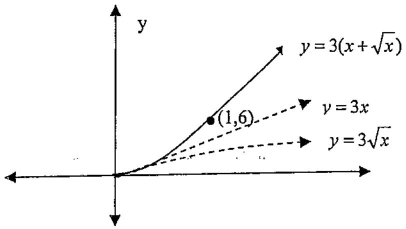
$\arg z=-\frac{\pi}{4}$ is the ray $y=-x,\ x>0$.

ANSWERS QUESTION 3

Question 3 (a)

Criteria
• One each for position and shape of curve.

Answer:



x

Question 3 (b) (i)

Criteria
• One for correct shape and position of curve.

Answer:

$y=x^2-16$

X intercepts when $y=0$

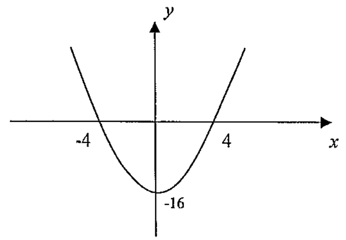
$x^2-16=0$

$x^2=16$

$x=\pm4$

Y intercept when $x=0$

$y=-16$



Question 3 (b) (ii)

Criteria
• One each for position and shape of curve

Answer:

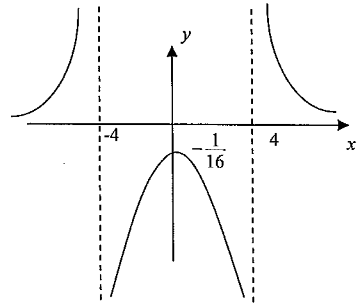
$y=\frac{1}{f(x)}=\frac{1}{x^2-16}$

Asymptotes: $x=\pm4$

When $x<-4$ and $x>4$, $y=\frac{1}{f(x)}>0$

For $-4<x<4$ $y=\frac{1}{f(x)}<0$

When $x=0$, $y=-\frac{1}{16}$



Question 3 (b) (iii)

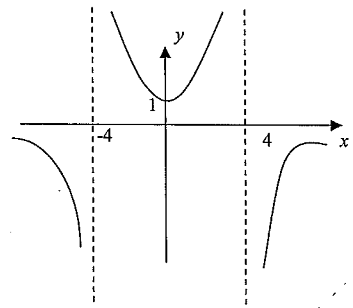
Criteria
• One each for position and shape of curve

Answer:

At x =0

$y=-\frac{1}{16}\times-16=1$

Note: when multiplying by a negative number, the graph is reflected about the x axis.

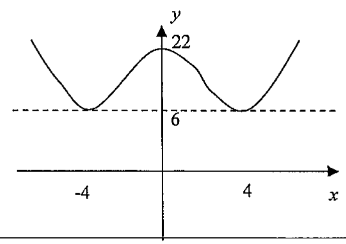


Question 3 (b) (iv)

Criteria
• One each for position and shape of curve

Answer:

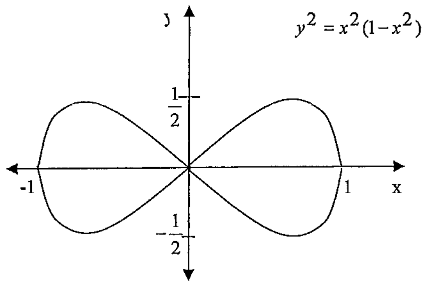
$y=|f(x)|+6$



Question 3 (c)

Criteria
• One each for position and shape of curve

Answer:

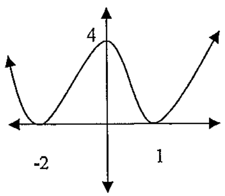


Question 3 (d)

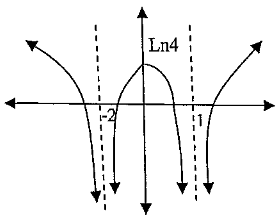
Criteria
• One mark for each curve

Answer:

$y=\left|x^3+3x^2-4\right|$



$y=\ln\left|x^3+3x^2-4\right|$



Question 3 (e)

Criteria
• One for $\dfrac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}$, one for simplification

(e) To prove $\dfrac{\sin B+\sin C}{\sin B-\sin C}=\tan\left(\dfrac{B+C}{2}\right)\cot\left(\dfrac{B-C}{2}\right)$

$$\text{L.H.S}=\frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}$$

$$=\tan\left(\frac{B+C}{2}\right)\cot\left(\frac{B-C}{2}\right)$$

$=\text{RHS}$

ANSWERS QUESTION 4

Question 4 (a)(i)

Criteria
• One for $\ddot{x}=0$ and, one for simplification.

Answer: For A_1: $\ddot{x}=g-kv$. Terminal velocity is achieved when $\ddot{x}=0$ i.e. when $0=g-kv$.

$\therefore$ terminal velocity $=\dfrac{g}{k}$

Question 4 (a) (ii)

Criteria
• One for integration, one for constant, one for simplification.

Answer:

For A_2:

$$\frac{dv}{dt}=-g-kv$$

$$t=\int\frac{-1}{g+kv}dv$$

$$=\frac{-1}{k}\ln(g+kv)+c_1$$

$When\, t=0, v=\dfrac{2g}{k}$

$$\therefore c_1=\frac{1}{k}\ln(3g)$$

$$\therefore t=\frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$$

Question 4 (a) (iii)

Criteria
• One for $t=\dfrac{1}{k}\ln\left(\dfrac{g}{g-kv}\right)$, one for $\dfrac{1}{k}\ln\dfrac{10}{7}=\dfrac{1}{k}\ln\left(\dfrac{3g}{g+kv}\right)$, one for simplification.

Answer:

For A_1

$$\frac{dv}{dt}=g-kv$$

$$t=\int\frac{1}{g-kv}dv$$

$$=-\frac{1}{k}\ln(g-kv)+c_2$$

$when\, t=0, v=0\Rightarrow c_2=\dfrac{1}{k}\ln g$

$$t=\frac{1}{k}\ln\left(\frac{g}{g-\frac{3g}{10}}\right)=\frac{1}{k}\ln\frac{10}{7}$$

$$\frac{1}{k}\ln\frac{10}{7}=\frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$$

$t=\frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$

When the particles collide, A_1 has a velocity

$\frac{3g}{10k}$

$\frac{10}{7}=\frac{3g}{g+kv}$

$10g+10kv=21g \Rightarrow 10kv=11g \Rightarrow v=\frac{11g}{10k}$

At the instant the particle collide, A_2 has a velocity

$\frac{11g}{10k}ms^{-1}$

Question 4 (b) (i)

Criteria
• One for final answer.

Answer

In Δ's AFG, AJF

$\angle AFG=\angle AJF$ (angle between tangent and chord= angle in alternate segment)

$\angle FAG=\angle FAJ$ (common angle)

Third angles are equal

$\therefore$ Δ AFG is similar to ΔAJF (AAA)

Question 4 (b) (ii) and (iii)

Criteria
• (ii) One for final answer. • (iii) One for $\Rightarrow AJ=\frac{AF^2}{AG}$, one for simplification.

Answer

(ii)

$\frac{AF}{AG}=\frac{AJ}{AF}$ (corresponding sides of similar triangles in the same ratio)

$\therefore AF^2=AG\times AJ$

(iii)

Now GJ=AJ-AG also $AF^2=AG\times AJ \Rightarrow AJ=\frac{AF^2}{AG}$

$\therefore GJ=\frac{AF^2}{AG}-AG$

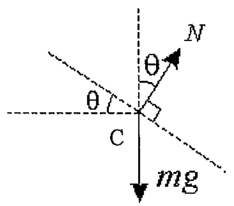
Now AF=AH (Tangent from a common point are equal)

$\therefore GJ=\frac{AF\times AH}{AG}-AG$

Question 4 (c)

Criteria
• One for equation A, one for equation B, one or simplification.

Forces on the car C.



Let R be the radius of a bend. The car has no tendency to slip. Therefore:

The vertical components sum to zero

$N\cos\theta=mg$. (A)

The horizontal components sum to

$\frac{mv^2}{R}$ Thus $N\sin\theta=\frac{mv^2}{R}$. (B)

(B) / (A),

$\tan\theta=\frac{v^2}{Rg}$

$v=(Rg\cdot\tan\theta)^{1/2}$

$R=70, g=9.8, \theta=11°\Rightarrow v=11.54ms^{-1}$.

ANSWERS QUESTION 5

Question 5 (a) (i) (ii)

Criteria
• One for using integration by parts, one for reduction formula, one for simplification.

Answer:

(i)

$Let\ u=(\ln x)^n$

$\frac{du}{dx}=n(\ln x)^{n-1}\times\frac{1}{x}$

$du=n(\ln x)^{n-1}\times\frac{1}{x}dx$

$Let dv=dx$

$\therefore v=x$

Using integration by parts

$\int u.dv=uv-\int v.du$

$\therefore \int_1^e(\ln x)^n dx=\left[(\ln x)^n\times x\right]_1^e-\int_1^e xn(\ln x)^{n-1}\times\frac{1}{x}dx$

$=e-nI_{n-1}$

(ii)

$I_0=\int_1^e 1dx=e-1$.

$\therefore I_4=e-4I_3=e-4(e-3I_2)$

$=-3e+12(e-2I_1)$

$=9e-24(e-I_0)$

$=-15e+24(e-1)$

$=9e-24$.

Question 5 (b) (i) (ii) (iii)

Criteria
• (i) (ii) (iii) One for correct answer.

Answer:

(i) For a committee to contain 1 woman, 1 woman must be chosen from the four possible, and the two men from the n possible.

The number of ways is ${}^4C_1\times{}^nC_2=\frac{4n(n-1)}{2}=2n(n-1)$

(ii) ${}^4C_2\times{}^nC_1=6n$

(iii) The total number of possible committees is ${}^{(n+4)}C_3$ probability is $\frac{2n(n-1)+6n}{(n+4)(n+3)\left(\frac{n+2}{6}\right)}=\frac{12n(n+2)}{(n+4)(n+3)(n+2)}$

Question 5 (c) (i) (ii)

Criteria
• (i) One for $x^{\frac{1}{2}}(x+4)=x+2$, one for simplification (ii One for expansion of $\alpha^2+\beta^2+\gamma^2$, one for $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-4(\alpha+\beta+\gamma)$, one for simplification.

Answer:

(c) (i) Since α^2 is a root, then

$x=\alpha^2$ so $\alpha=x^{1/2}\Rightarrow\left(x^{\frac{1}{2}}\right)^3-\left(x^{\frac{1}{2}}\right)^2+4x^{\frac{1}{2}}-2=0$

$x^{\frac{1}{2}}(x+4)=x+2$

$x\left(x^2+8x+16\right)=x^2+4x+4$

$x^3+8x^2+16x-x^2-4x-4=0$

$x^3+7x^2+12x-4=0$

(ii) Since $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

then $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=(\alpha\beta+\alpha\gamma+\beta\gamma)^2$
$-2((\alpha\beta)(\alpha\gamma)+(\alpha\beta)(\beta\gamma)+(\alpha\gamma)(\beta\gamma))$
$-(A)$

$=(\alpha\beta+\alpha\gamma+\beta\gamma)^2$
$-2\left(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2\right)$

$=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(2\alpha+2\beta+2\gamma)$

(since $\alpha\beta\gamma$ is equal to 2) $-(B)$

$=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-4(\alpha+\beta+\gamma)$

$=4^2-4(1)$ since $\alpha\beta+\alpha\gamma+\beta\gamma=4,\ \alpha+\beta+\gamma=1$

$=12$

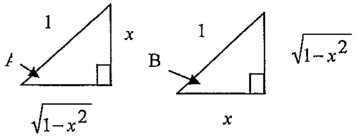
Question 5 (d)

Criteria
• One for finding cosA and sinB in terms of x and one for simplification.

Answer:

Let $A=\sin^{-1}x$ and $B=\cos^{-1}x$

$\therefore x=\sin A$ and $x=\cos B$



$\sqrt{1-x^2}=\cos A$ and $\sqrt{1-x^2}=\sin B$

$$\begin{aligned}\therefore \text{LHS} &= \sin(\sin^{-1}x-\cos^{-1}x)\\ &= \sin(A-B)\\ &= \sin A\cos B-\cos A\sin B\\ &= x.x-\sqrt{1-x^2}\sqrt{1-x^2}\\ &= x^2-(1-x^2)\\ &= 2x^2-1\end{aligned}$$

ANSWERS QUESTION 6

Question 6 (a)

Criteria
• One for v= $4\pi\int_{-b}^{b}(c-y)x\,dy$,one for v = $4\pi\int_{-b}^{b}(c-y)\times\frac{a}{b}\sqrt{b^2-y^2}\,dy$, one for v = $=\frac{4\pi a}{b}\left(\int_{-b}^{b}c\sqrt{b^2-y^2}\,dy-\int_{-b}^{b}y\sqrt{b^2-y^2}\,dy\right)$, one for simplification.

Answer:

$$V=\lim_{\delta y\to 0}\sum_{y=-b}^{b}2\pi(c-y)\times 2x\delta y=4\pi\int_{-b}^{b}(c-y)x\,dy=4\pi\int_{-b}^{b}(c-y)\times\frac{a}{b}\sqrt{b^2-y^2}\,dy,\text{since }\frac{x^2}{a^2}=1-\frac{y^2}{b^2}=\frac{b^2-y^2}{b^2}$$

$$=\frac{4\pi a}{b}\left(\int_{-b}^{b}c\sqrt{b^2-y^2}\,dy-\int_{-b}^{b}y\sqrt{b^2-y^2}\,dy\right)=\frac{4\pi a}{b}\left(c\times\frac{1}{2}\pi b^2-0\right)=2\pi^2abc$$

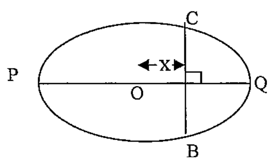
Note: $\int_{-b}^{b}\sqrt{b^2-y^2}\,dy=\frac{1}{2}\pi b^2$, since it represents a semi – circle

and $\int_{-b}^{b}y\sqrt{b^2-y^2}\,dy=0$, since it is an odd function

Question 6 (b)

Criteria
• One for area of triangle ABC, one for integral, one for simplification.

Answer:



O is the origin

$BC=2\sqrt{a^2-x^2}$

Area of ΔABC

$=\frac{1}{2}\times 2\sqrt{a^2-x^2}\times b$

$=b\sqrt{a^2-x^2}$

$$\begin{aligned}v &= b\int_{-a}^{a}\sqrt{a^2-x^2}\,.dx\\ &= b\times\text{area of semi-circle of radius a}\\ &= b\times\frac{\pi a^2}{2}\\ &= \frac{1}{2}\pi a^2b\ \text{unit}^3\end{aligned}$$

Question 6 (c)

Criteria
• One for testing x=1, one for $k(k^2-4k+4)+3\ge 0$, one for LHS= $(k+1)(k-1)^2+3$ one for conclusion

Answer:

$x^3-4x^2+4x+3\ge 0$

For $x=1$, LHS $=1-4+4+3=4\ge 1$

$\therefore$ True for $x=1$

Assume true for $x=k$

$\therefore k^3-4k^2+4k+3\ge 0$

$\Rightarrow k(k^2-4k+4)+3\ge 0$

$\Rightarrow k(k-2)^2+3\ge 0$

When $x=k+1$

LHS $=(k+1)(k+1-2)^2+3$

$=(k+1)(k-1)^2+3$

Now, $(k-1)^2\ge 0$

and $k+1>0$

$\therefore (k+1)(k-1)^2+3\ge 0$

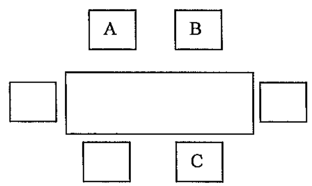
Since true when $x=1$, must also be true when $x=2$, 3 and so on. So the statement is true for all positive integer values of x.

Question 6 (d) (e)

Criteria
• (d) One for determining number of ways B and C can sit, one for simplification
• (e) One for second derivative, one for x = -1/4 or -2, one for simplification

Answer:

(d) Sit A down first. B can sit in two positions. Women C can sit in 3 positions. Remaining 3 people can be seated in 3! ways.



Thus total number of ways= 2(3)(3!)=36

(e) $P(x)=2x^4+9x^3+6x^2-20x-24$ has a triple root

$P'(x)=8x^3+27x^2+12x-20$ has a double root

$P''(x)=24x^2+54x+12$ has a 1 fold root.

$\therefore\ 24x^2+54x+12=0$

$4x^2+9x+2=0$

$(4x+1)(x+2)=0$

$\therefore x=-\frac{1}{4}$ or -2

Substituting $x=-2$,

$P(-2)=2(-2)^4+9(-2)^3+6(-2)^2-20(-2)-24=0$

$\therefore x=-2$ is the triple root and by inspection,

$P(x)=(x+2)^3(2x-3)$

ANSWERS QUESTION 7

Question 7 (a) (i) (ii) (b)

Criteria
• (i) One for completing the square, one for simplification (ii) One for gradient of tangent $=\frac{4(p-2h)}{q-h}$, one for finding $(e^2-1)=\frac{4(p-2h)}{q-h}$, one for simplification. (b) One for finding focus at F_1, one for finding coordinates at A_1, one for simplification

Answer

(a)

(i) $4x^2-16hx-y^2+2hy=4a^2-15h^2$

$4\left(x^2-4hx\right)-\left(y^2-2hy\right)=4a^2-15h^2$

completing the square

$4\left(x^2-4hx+4h^2\right)-\left(y^2-2hy+h^2\right)$

$=4a^2-15h^2+16h^2-h^2$

$4(x-2h)^2-(y-h)^2=4a^2$

$(x-2h)^2-\frac{(y-h)^2}{4}=a^2$

$\frac{(x-2h)^2}{a^2}-\frac{(y-h)^2}{4a^2}=1$

$\therefore$ of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\therefore$ a hyperbola

(ii) To find the gradient of the tangent

d.w.r.t.x

$\frac{2(x-2h)}{a^2}-\frac{2(y-h)}{4a^2}\times\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{4(x-2h)}{y-h}$

$\therefore$ at (p,q) gradient of tangent is $\frac{4(p-2h)}{q-h}$

given $y=\left(e^2-1\right)x$ $\therefore$ gradient is $\left(e^2-1\right)$

$\left(e^2-1\right)=\frac{4(p-2h)}{q-h}$....$A$

now $e^2=1+\frac{b^2}{a^2}$ for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$e^2-1=\frac{4a^2}{a^2}=4$

sub into A

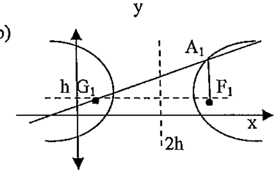
$4=\frac{4(p-2h)}{q-h}$

$4q-4h=4p-8h$

$q-h=p-2h$

$p-q=h$

(b)



$\frac{(x-2h)^2}{a^2}-\frac{(y-h)^2}{4a^2}=1$....$B$

is a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{4a^2}=1$ translated 2h to the right and h units up

The focus of $\frac{x^2}{a^2}-\frac{y^2}{4a^2}=1$ is at $(\pm\sqrt{5}a,0)$

the foci at translated hyperbola is at $F_1\left(\sqrt{5}a+2h,h\right)$ and $G_1\left(-\sqrt{5}a+2h,h\right)$

At A_1, $x=\sqrt{5}+2h$

sub into B

$\frac{\left(\sqrt{5}a+2h-2h\right)^2}{a^2}-\frac{(y-h)^2}{4a^2}=1$

$5-\frac{(y-h)^2}{4a^2}=1$

$16a^2=(y-h)^2$

$y=4a+h$

$\therefore A_1$ has coordinates $\left(\sqrt{5}a+2h,4a+h\right)$

gradient A_1G_1 is $\frac{4a+h-h}{\left(\sqrt{5}a+2h\right)-\left(-\sqrt{5}a+2h\right)}=\frac{2}{\sqrt{5}}$

Thus equation of A_1G_1 is

$y-h=\frac{2}{\sqrt{5}}\left(x-\left(-\sqrt{5}a+2h\right)\right)$

$\sqrt{5}(y-h)=2x+2\sqrt{5}a-4h$

$\therefore\ 0=2x-\sqrt{5}y+2\sqrt{5}a+\left(\sqrt{5}-4\right)h$

Question 7 (c) (d)

Criteria
• (c) One finding equations at A and B, one for finding equation at E, one for simplification. (d) One for finding equation of tangent to hyperbola, one for $5a^2m^2=36a^2$, one for gradient of asymptote =2, one for simplification.

Answer

(c)

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

$b^2xx_1-a^2yy_1=a^2b^2$

$y=\left(\frac{b^2x_1}{a^2y_1}\right)x-\frac{b^2}{y_1}$....$A$

is the equation in $y=mx+c$ form

$m=\frac{b^2x_1}{a^2y_1}$....B

squaring both sides

$m^2=\frac{b^4x_1^2}{a^4y_1^2}$

$b^4x_1^2=m^2a^4y_1^2$....C

point x_1 on the hyperbola $\Rightarrow\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$

$\frac{x_1^2}{a^2}=1+\frac{y_1^2}{b^2}$

$x_1^2=a^2\left(1+\frac{y_1^2}{b^2}\right)$....$D$

sub D into C

$b^4a^2\left(1+\frac{y_1^2}{b^2}\right)=m^2a^4y_1^2$

$b^4a^2+b^2a^2y_1^2=m^2a^4y_1^2$

$y_1^2\left(m^2a^2-b^2\right)=b^4$

$y_1^2=\frac{b^4}{m^2a^2-b^2}$, where $m^2a^2-b^2>0$

$\therefore y_1=\pm\frac{b^2}{\sqrt{m^2a^2-b^2}}$....$E$

sub B and E into A

$y=mx\pm\frac{b^2}{\frac{b^2}{\sqrt{m^2a^2-b^2}}}$

$\therefore y=mx\pm\sqrt{m^2a^2-b^2}$

If (x_1,y_1) is in quadrants 1 and 2, then y − intercept is negative

$\therefore y=mx-\sqrt{m^2a^2-b^2}$

If (x_1,y_1) is in quadrants 3 and 4, then y − intercept is positive

$\therefore y=mx+\sqrt{m^2a^2-b^2}$

(d)

$\frac{(x-2h)^2}{a^2}-\frac{(y-h)^2}{4a^2}=1$

equation of tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ when (x_1,y_1)

is in 1st quadrant is

$y=mx-\sqrt{m^2a^2-b^2}$

$\therefore$ equation of tangent to the hyperbola is

$y-h=m(x-2h)-\sqrt{m^2a^2-b^2}$

sub $\left(\frac{2a}{3}+2h,h\right)$

$\frac{2am}{3}-\sqrt{\left(m^2-4\right)a^2}=0$

$\frac{2am}{3}=\sqrt{\left(m^2-4\right)a^2}$

$4a^2m^2=9a^2\left(m^2-4\right)$

$5a^2m^2=36a^2$

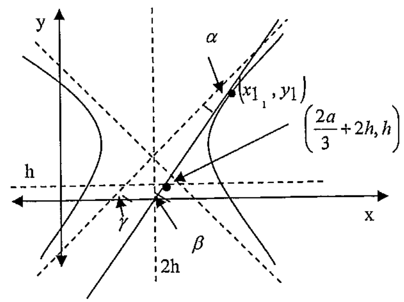
$m^2=\frac{36}{5}\Rightarrow m=\frac{6}{\sqrt{5}},m>0$

asymptote of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with positive gradient is $\frac{b}{a}$

$\therefore$ gradient of asymptote $=\frac{2a}{a}=2$

using notation on graph, $\tan\chi=2,\tan\beta=\frac{6}{\sqrt{5}}$

and $\alpha=\beta-\chi\rightarrow\tan(\beta-\chi)=\frac{\tan\beta-\tan\chi}{1+\tan\beta\tan\chi}$



$$\therefore \tan\alpha = \frac{\frac{6}{\sqrt{5}}-2}{1+2\times\frac{6}{\sqrt{5}}} = \frac{6-2\sqrt{5}}{12+\sqrt{5}} = \frac{6-2\sqrt{5}}{12+\sqrt{5}}\times\frac{12-\sqrt{5}}{12-\sqrt{5}}$$

$$= \frac{82-30\sqrt{5}}{139}$$

$$\therefore \tan\alpha = \frac{2\left(41-15\sqrt{5}\right)}{139}$$

$$\therefore \alpha = \tan^{-1}\frac{2\left(41-15\sqrt{5}\right)}{139}$$

ANSWERS QUESTION 8

Question 8 (a) (i) (ii)

Criteria
• (i) One for correct answer (ii) One for forming all three inequalities, one for simplification

Answer:

(i) LHS-RHS $= a^2+b^2-2ab = (a-b)^2$

≥ 0 for all values of a and b.

$\therefore a^2+b^2 \geq 2ab$

(ii)
From part (i), it follows that

$a^2+b^2 \geq 2ab$

$b^2+c^2 \geq 2bc$

$c^2+a^2 \geq 2ca$

for all real a, b and c.
Adding these three inequalities:

$2a^2+2b^2+2c^2 \geq 2ab+2bc+2ca$

$\therefore a^2+b^2+c^2 \geq ab+bc+ca$

for all real values of a, b and c.

Question 8 (b) (iii) (iv)

Criteria
• (iii) One for $a^2+b^2+c^2-ab-bc-ca \geq 0$, one for simplification. (iv) One for correct answer.

Answer:

i(iii) If a, b and c are positive, then $a+b+c>0$.

also, from part (ii), for all values of a, b and c,

$a^2+b^2+c^2-ab-bc-ca \geq 0$

$\therefore (a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) \geq 0 \quad \therefore a^3+b^3+c^3-3abc \geq 0$

$\therefore a^3+b^3+c^3 \geq 3abc$

(iv) In the result in part (iii),
replace a with $\sqrt[3]{x_1}$,
replace b with $\sqrt[3]{x_2}$ and
replace c with $\sqrt[3]{x_3}$.

$\therefore x_1+x_2+x_3 \geq 3\times\sqrt[3]{x_1}\times\sqrt[3]{x_2}\times\sqrt[3]{x_3}$

$\therefore \frac{x_1+x_2+x_3}{3} \geq \sqrt[3]{x_1x_2x_3}$

for positive values of x_1, x_2 and x_3.

Question 8 (b) (i)

Criteria
• One $\frac{1}{2}\sin\left(\frac{2\alpha}{2^n}\right)$, one for simplification

Answer:

$$\sin\frac{\alpha}{2^n}\cos\frac{\alpha}{2^n} = \frac{1}{2}\sin\left(\frac{2\alpha}{2^n}\right) = \frac{1}{2}\sin\left(\frac{\alpha}{2^{n-1}}\right)$$

Question 8 (b) (ii)

Criteria
• One for correct answer.

Answer:

Rearranging $\cos 2\theta = 2\cos^2\theta - 1$ *gives* $\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

$$\therefore \sqrt{\frac{1}{2}+\frac{1}{2}\cos\alpha} = \sqrt{\cos^2\frac{\alpha}{2}} = \cos\frac{\alpha}{2}$$

Question 8 (b) (iii)

Criteria
• One for area of triangle APB and sector APB, one for Area $\Delta APB \leq$ Area sector $APB \leq$ Area ΔAPC, one for simplification.

Answer:

$Area\,\Delta APB = \frac{1}{2}\times 1\times 1\times\sin x = \frac{1}{2}\sin x$

$Area\,\sec tor\ APB = \frac{1}{2}\times 1^2\times x = \frac{1}{2}x$

Area $\Delta APC = \frac{1}{2}\times 1\times\tan x = \frac{1}{2}\tan x$

Area $\Delta APB \leq$ Area sector $APB \leq$ Area ΔAPC

$\Rightarrow \frac{1}{2}\sin x \leq \frac{1}{2}x \leq \frac{1}{2}\tan x$

$\Rightarrow 1 \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x}$

$\Rightarrow 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$

Question 8 (b) (iv)

Criteria
• One for $\frac{1}{\alpha} \leq \frac{\frac{1}{2^n}}{\sin\left(\frac{\alpha}{2^n}\right)} \leq \frac{1}{\alpha\cos\left(\frac{\alpha}{2^n}\right)}$, one for as $n\to\infty$ $\frac{1}{\alpha\cos\left(\frac{\alpha}{2^n}\right)} \to \frac{1}{\alpha\cos 0} = \frac{1}{\alpha}$, one for simplification

Answer:

$$1 \leq \frac{\left(\frac{\alpha}{2^n}\right)}{\sin\left(\frac{\alpha}{2^n}\right)} \leq \frac{1}{\cos\left(\frac{\alpha}{2^n}\right)} \ \textit{replacing } x \textit{ with } \frac{\alpha}{2^n}$$

$$\frac{1}{\alpha} \le \frac{\frac{1}{2^n}}{\sin\left(\frac{\alpha}{2^n}\right)} \le \frac{1}{\alpha\cos\left(\frac{\alpha}{2^n}\right)}$$

$$\text{As n} \to \infty \frac{1}{\alpha\cos\left(\frac{\alpha}{2^{\text{n}}}\right)} \to \frac{1}{\alpha\cos 0} = \frac{1}{\alpha}$$

$$\therefore \text{ as } n \to \infty \quad \frac{1}{\alpha} \le \frac{\frac{1}{2^n}}{\sin\left(\frac{\alpha}{2^n}\right)} \le \frac{1}{\alpha}$$

$$\text{So } \lim_{n\to\infty} \frac{\frac{1}{2^n}}{\sin\left(\frac{\alpha}{2^n}\right)} = \frac{1}{\alpha}$$