TRINITY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

YEAR 12 MATHEMATICS

2002 APRIL EXAMINATION

ASSESSMENT WEIGHTING 30 %

TIME ALLOWED 3 HOURS
(Plus 5 minutes reading time)

Directions to Students

1) Attempt all questions.

2) Show all necessary working.

3) Begin each question in a new booklet.

4) Marks for each part question are indicated on the paper.

5) Board approved calculators are allowed.

6) ***Write your student number and your class on each booklet.***

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \ln\left\{x+\sqrt{(x^2-a^2)}\right\},\quad |x| > |a|$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \ln\left\{x+\sqrt{(x^2+a^2)}\right\}$$

NOTE : $\ln x = \log_e x;\quad x > 0$

Question 1

Marks

a) Express $\frac{5}{\sqrt{2}}$ with a rational denominator. 2

b) Solve for x: $x^2 < 25$ 2

c) Solve for x: $4^x = 8$ 2

d) Simplify: $4a - 2(3 - a)$ 2

e) If $x = 12$ is a solution of $kx + 24 = 0$, calculate the value of k. 2

f) Simplify $8^{\frac{-2}{3}} \times 49^{\frac{1}{2}}$. Give your answer as a fraction. 2

Question 2 (Start a new booklet)

a) Differentiate with respect to x: $y = \frac{1}{x^2}$ 2

b) Find a primitive function of $\sqrt{x}$ 2

c) Write down the exact value of $\tan 300°$ 2

d) Sketch the graph of $y = |x + 2|$ 2

e) How much will $600 grow to if invested at 8.5% compound interest for 5 years? 2

f) Sketch the curve $y = \frac{1}{x}$ 2

Question 3 (Start a new booklet)

Marks

a) A straight line has an x-intercept of -4 and a y-intercept of 2. Sketch this line. 1

b) Find the gradient of the line in a) above. 1

c) Calculate the area of the triangle formed by the line above and the x and y axes. 2

d) Find the equation of the line which is perpendicular to the line in a) and which passes through the point (0, 7) 2

e) Find the point of intersection of the line in a) and the line in d) 2

f) Simplify $\sqrt{72} + \sqrt{50}$ 2

g) Evaluate correct to 2 decimal places: $\frac{112}{\sqrt{8.1} \times 5.3}$ 2

Question 4 (Start a new booklet)

a) Consider the series: $5 + 11 + 17 + 23 + \text{----------} 1199$

i) How many terms are in this series? 2

ii) Find the sum of all the terms in this series. 2

b) Find the 10th term of the series; $-128, 64, -32, \_\_\_\_$ 2

c) A ball is dropped from a height of 30 metres, then on each bounce it reaches $\frac{2}{3}$ of its previous height. Find the total distance the ball travels before coming to rest. 2

d) Roberto invests $500 at the end of each month in a savings account which pays 12% p.a. compound interest. How much money will be in the account at the end of 20 years? Answer to nearest dollar. 4

Question 5 (Start a new booklet) **Marks**

a) If α *and* β are the roots of $2x^2-4x+1$, calculate the values of $\alpha^2+\beta^2$ and $\frac{1}{\alpha}+\frac{1}{\beta}$ — 3

b) Find the values of c which ensures that $x^2-2x-2c+3$ is positive definite. — 2

c) Solve for x: $x^6+9x^3+8=0$ — 2

d) If $x^2+2x+4 \equiv A(x-1)^2+B(x-1)+C$, calculate the values A, B, and C — 3

e) Find the value of k if the roots of $5x^2+x-2k=0$ are reciprocals of each other. — 2

Question 6 (Start a new booklet)

a) Find the equation of the locus of a point which moves such that it is equidistant from (0, 0) and (2, 2) — 3

b) For the given parabola; $(y+4)^2=-20(x-3)$, — 4

Find i) coordinates of vertex

ii) equation of directrix

iii) Sketch the curve.

c) Find the equation of the tangent to the parabola $y=x^2-2x+4$ at the point (3, 7) — 3

d) Find the equation of the parabola which has its vertex at (2 , 0) and the focus at (2, 2) — 2

Question 7 (Start a new booklet) **Marks**

a) For the function ; $y=x^3-3x^2-9x+2$

i) Determine the coordinates of the stationary points. You must show clearly the method by which you determine if these points are maximum or minimum values. — 4

ii) Find the coordinates of any points of inflexion. — 2

iii) Sketch the curve. — 2

iv) State the x values for which $\frac{dy}{dx}<0$ — 1

b) The gradient function of a curve is given by $\frac{dy}{dx}=x^{-2}$. — 3

If the curve passes through the point $\left(\frac{1}{2},4\right)$ find its equation.

Question 8 (Start a new booklet)

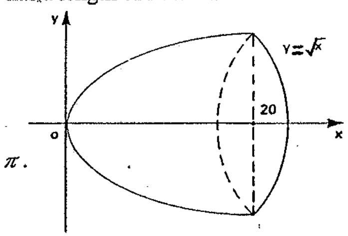
a) Sketch the curve $y=x^3$ for values of x between –2 and 2. — 1

b) On your sketch clearly shade the area bounded by this curve, the x-axis and the lines x = –1 and x =2. — 1

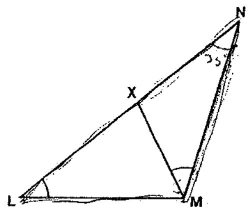
c) Calculate the area described in b) — 3

d) Calculate the value of $\int_1^3 \frac{1}{x}dx$ using Simpson's Rule with 5 function values. — 4

e) The shell shown in the diagram below has a length of 20cm and its shape is generated by revolving $y=\sqrt{x}$ about the x – axis. Calculate the volume. Answer as a multiple of π. — 3



Question 9 (Start a new booklet) Marks



5

a) In the diagram above $\angle NLM = \angle NMX = 40°$. $\angle LNM = 35°$

NL = 12, NM = 6

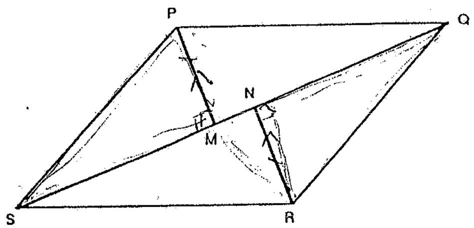
Copy the diagram on to your paper and

i) Prove ΔNLM *and* ΔNMX *are similar*

ii) Prove NX = 3

iii) Prove LM = 2MX

b)



In the diagram above, PQRS is a parallelogram. PM and RN are perpendicular to QS. 4

Prove that triangles PSM and QNR are congruent and hence show that PNRM is a parallelogram

c) Prove that the sum of the exterior angles of a triangle is 360 degrees. 3

Question 10 (Start a new booklet) Marks

a) The table below gives values of F(x) for $0 \le x \le 2$. 3

x	0	0.5	1	1.5	2
F (x)	2.32	4.61	9.28	10.33	6.42

Use the trapezoidal rule with 5 function values to determine the value of $\int_0^2 F(x)dx$ correct to 1 decimal place.

b) Given that f(x) = $x^2 + x$, find the values of "a" such that $f''(a) = f(a)$. 4

c) A triangle ABC is such that the sum of the lengths of its base and its perpendicular height is 50 cm. Calculate the maximum area of the triangle. 5

END OF EXAMINATION

Q1)

a) $\frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$ (1) (1)

b) $x^2 < 25$

$-5 < x < 5$ (2)

c) $4^x = 8$

$2^{2x} = 2^3$ (1)

$2x = 3$

$x = 1\frac{1}{2}$ (1)

d) $4a - 2(3-a)$

$= 4a - 6 + 2a$ (1)

$= 6a - 6$ (1)

e) $kx + 24 = 0$

$(x = 12)$ $12k + 24 = 0$ (1)

$k = -2$ (1)

f) $8^{-\frac{2}{3}} \times 49^{\frac{1}{2}}$

$= 2^{-2} \times 7$ (1)

$= \frac{7}{4}$ or $1\frac{3}{4}$ (1)

Q2)

a) $y = \frac{1}{x^2}$

$y = x^{-2}$ (1)

$\frac{dy}{dx} = \frac{-2}{x^3}$ (1)

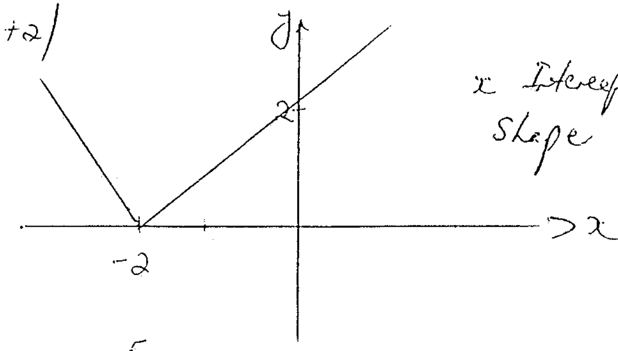
b) Primitive of $x^{\frac{1}{2}} = \frac{2x^{\frac{3}{2}}}{3} + c$ (2)

LESS 1 IF NO C

c) $\tan 300° = -\tan 60°$ (1)

$= -\sqrt{3}$ (1)

d) $y = |x+2|$

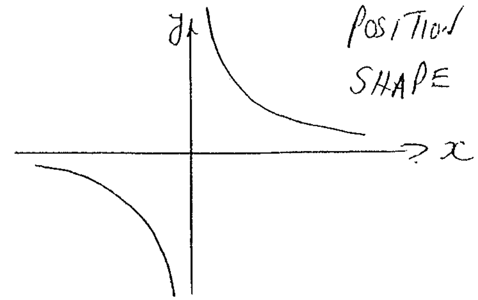


x Intercept (1)
Shape (1)

e) $A = 600(1.085)^5$ (1)

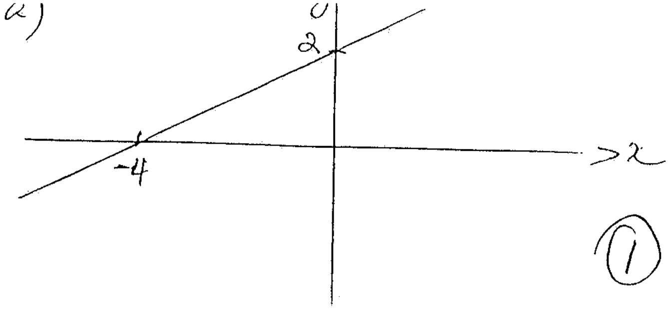
$A = \$902.19$ (1)

f) $y = \frac{1}{x}$



POSITION (1)
SHAPE (1)

Q3) a)



(1)

b) GRADIENT $= \frac{1}{2}$ (1)

c) Area $= \frac{1}{2} \times 2 \times 4$

$= 4$ units^2 (2)

d) Perpendicular gradient $= -2$

Equation $y = -2x + 7$ (2)

e) $\left.\begin{array}{l} y = \frac{1}{2}x + 2 \\ y = -2x + 7 \end{array}\right\}$ Solve simultaneously

$\frac{1}{2}x + 2 = -2x + 7$

$x + 4 = -4x + 14$

$5x = 10$

$x = 2$, Intersection $(2, 3)$ (2)

f) $\sqrt{72} + \sqrt{50} = 6\sqrt{2} + 5\sqrt{2}$

$= 11\sqrt{2}$ (2)

g) $\dfrac{11.2}{\sqrt{8.1} \times 5.3} = 7.43$ (2)

Q4) a) $5 + 11 + 17 + 23 + \dots 1199$

i) Arithmetic Series, $a = 5$

$d = 6$

$a + (n-1)d = 1199$ (1)

$5 + 6(n-1) = 1199$

$5 + 6n - 6 = 1199$

$6n = 1200$

$n = 200$ 200 Terms (1)

ii) $S_n = \frac{n}{2}(a + l)$ (1)

$= 100(5 + 1199)$

$= 100 \times 1204$ (1)

$= 120400$

b) $-128, 64, -32$ Geometric Series $a = -128$

$r = -\frac{1}{2}$

$T_{10} = ar^9$ (1)

$= -128 \times \left(-\frac{1}{2}\right)^9$

$= -128 \times \frac{-1}{512}$ (1)

$T_{10} = \frac{1}{4}$

c) Limiting Sum:

Total Distance $= 2 \times$ Limiting Sum $- 30$

$= 2 \times \frac{a}{1-r} - 30$ Recognise Limiting Sum (1)

$= 2 \times \frac{30}{\left(\frac{1}{3}\right)} - 30$ Answer (1)

$= 150$ Metres

d) Total $= 500\left[1 + 1.01 + 1.01^2 + \dots 1.01^{239}\right]$

$= 500(1.01^{240} - 1)$ = \$1491[illegible] (1)

Q5) a) $2x^2 - 4x + 1$

$\alpha + \beta = 2, \quad \alpha\beta = -\frac{1}{2}$ 1

$(\alpha^2 + \beta^2) = (\alpha+\beta)^2 - 2\alpha\beta$

$= 4 - 2 \times -\frac{1}{2}$

$= 5$ 1

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= -4$ 1

b) $x^2 - 2x - 2c + 3$

For Positive Definite $a > 0$ and $b^2 - 4ac < 0$ 1

$4 - 4(-2c + 3) < 0$

$4 + 8c - 12 < 0$

$8c < 8$

$c < 1$ 1

c) $x^6 + 9x^3 + 8 = 0$

Let $u = x^3$

$u^2 + 9u + 8 = 0$

$(u+8)(u+1) = 0$

$u = -8, \quad u = -1$ 1

$x^3 = -8, \quad x^3 = -1$

$\underline{x = -2}, \quad \underline{x = -1}$ 1

d)

$x^2 + 2x + 4 \equiv A(x-1)^2 + B(x-1)$

$\equiv Ax^2 - 2Ax + A + Bx - 1$

$\equiv Ax^2 + x(B - 2A) + A - 1$

$\therefore A = 1$ 1

$B - 2A = 2$

$\underline{B = 4}$ 1

$A - B + C = 4$

$\underline{C = 7}$ 1

e) $5x^2 + x - 2k = 0$

Root reciprocals

$\therefore$ Product Roots $= 1$ 1

$\frac{c}{a} = 1$

[illegible]

Q6) Let point on locus be $P(x, y)$

a) Then $\sqrt{x^2 + y^2} = \sqrt{(x-2)^2 + (y-2)^2}$ 1

$x^2 + y^2 = x^2 - 4x + 4 + y^2 - 4y + 4$ 1

$4x + 4y = 8$

$\underline{x + y = 2}$ 1

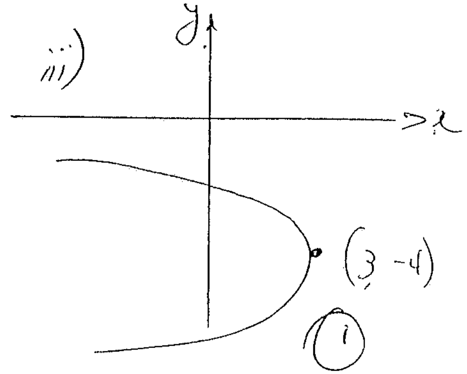
b) $(y+4)^2 = -20(x-3)$

i) VERTEX $(3, -4)$ 1

ii) DIRECTRIX: $x = 8$ 1

Recognise $a = 5$ 1

iii)



c) $y = x^2 - 2x + 4$

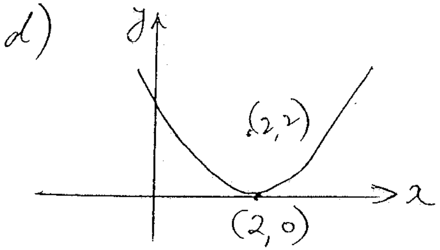
$\frac{dy}{dx} = 2x - 2$ 1

At $(3, 7)$ gradient Tangent $= 4$ 1

Equation Tangent: $y - 7 = 4(x - 3)$

$y = 4x - 5$ 1

d)



$a = 2$ 1

Equation: $(x-2)^2 = 8y$ 1

Q7) $y = x^3 - 3x^2 - 9x + 2$

a) $\frac{dy}{dx} = 3x^2 - 6x - 9 = 0$ for STAT. PTS

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, \quad x = -1$ (1)

$x = 3, \quad y = 27 - 27 - 27 + 2$

$y = -25$

$x = -1, \quad y = -1 - 3 + 9 + 2$

$y = 7$

$\frac{d^2y}{dx^2} = 6x - 6$

When $x = 3$, $f''(x) > 0$

$\therefore$ MINIMUM at $(3, -25)$ (1)

(1) MARK FOR TEST

When $x = -1$, $f''(x) < 0$

$\therefore$ MAXIMUM at $(-1, 7)$ (1)

ii) For inflexion, $f''(x) = 0$ (1)

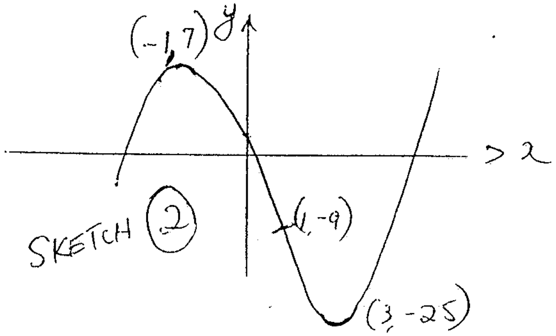
$6x - 6 = 0$

$x = 1$

Test: $x < 1$, $f''(x) < 0$; $x > 1$, $f''(x) > 0$ $\therefore$ Inflexion at $(1, -9)$ (1)

MUST DO TEST

iii)



SKETCH (2)

iv) $\frac{dy}{dx} < 0$ when $-1 < x < 3$ (1)

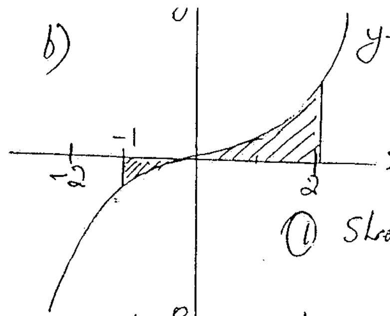
b) $\frac{dy}{dx} = x^{-2}$

$y = -\frac{1}{x} + c$ (1)

$(\frac{1}{2}, 4)$: $4 = -2 + c$

Equation: $y = -\frac{1}{x} + 6$ (1)

Q8) a) b)



Sketch (1)

(1) Shaded Area

c) $\text{Area} = \left| \int_{-1}^{0} x^3 \, dx \right|$ (1) $+ \int_0^2 x^3 \, dx$

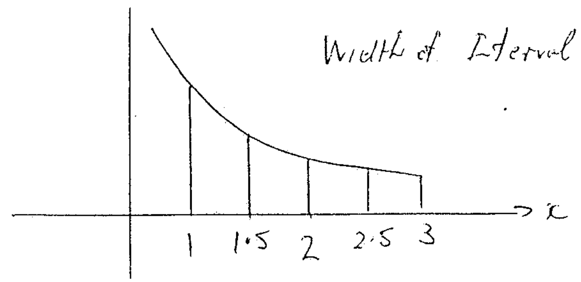
$= \left| \left[\frac{x^4}{4} \right]_{-1}^{0} \right| + \left[\frac{x^4}{4} \right]_0^2$ (1)

$= \left| 0 - \frac{1}{4} \right| + \left(\frac{16}{4} - 0 \right)$

$= \frac{1}{4} + 4$

Area $= 4\frac{1}{4}$ UNITS^2 (1)

d) $\int_1^3 \frac{1}{x} \, dx$ SIMPSON'S RULE 5 FUNCTION Values.



Width of Interval $= 0.5$ (1)

(1)

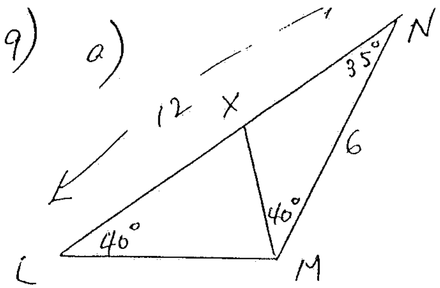
Area $\doteqdot \frac{0.5}{3} \left[f(1) + f(3) + 2f(2) + 4\left(f(1.5) + f(2.5)\right) \right]$

$\doteqdot \frac{0.5}{3} \left[1 + \frac{1}{3} + 2 \times \frac{1}{2} + 4\left(\frac{2}{3} + \frac{2}{5}\right) \right]$ (1)

$\doteqdot 1.1$ (1)

e) $Vol = \pi \int y^2 \, dx = \pi \int x \, dx$ (1) $= \pi \left[\frac{x^2}{2} \right]_0^{20}$ (1)

Q9) a)



NL = 12
NM = 6

DIAGRAM (1)

i) In Δ's NLM and NMX
$\angle LNM$ is Common
$\angle NLM = \angle NMX$ Given
$\therefore$ Triangles are Similar (2)

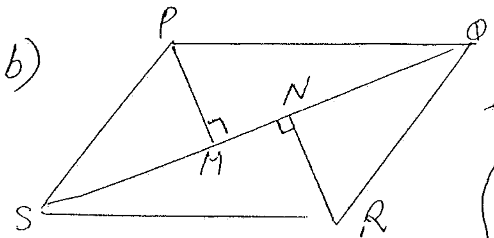
ii) From (i) $\frac{NM}{LN} = \frac{NX}{NM}$

$\frac{6}{12} = \frac{NX}{6}$

$\therefore NX = 3$ (1)

iii) Corresponding Sides are in equal ratio
LM and MX are corresponding sides
Sides in ratio 2 : 1
Then LM = 2 MX (1)

b)

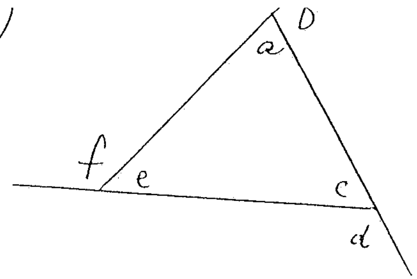


In Δ's PSM and QNR
PS = QR (opp Sides Parallelog
$\angle PSM = \angle NQR$ (Alt $\angle$'s, PS ∥
$\angle PMS = \angle QNR$ (Right Angles
Δ's Congruent AAS (2)

Now in PNRM, PM ∥ RN (Equal Alternate ang

... Congruent Δ's

c)



SUGGESTED PROOF

Now $a + b + c + d + e + f = 3 \times 180$
$= 540$ (1)

$(a + e + c) + b + d + f = 540$
$180 + b + d + f = 540$ (1)
$b + d + f = 360$ (1)

Obviously students can use their own notation Check method

Q 10) Trapezoidal Rule

a) Each strip is 0·5 units wide (1)

$$Area = \frac{0.5}{2}\left[2.32 + 6.42 + 2(4.61 + 9.28 + 10.33)\right]$$ (1)

$A = 14.295$ (1)

b) $f(x) = x^2 + x$

$f(a) = a^2 + a$

$f'(a) = 2a + 1$ (1)

$f''(a) = 2$ (1)

Given $f''(a) = f(a)$

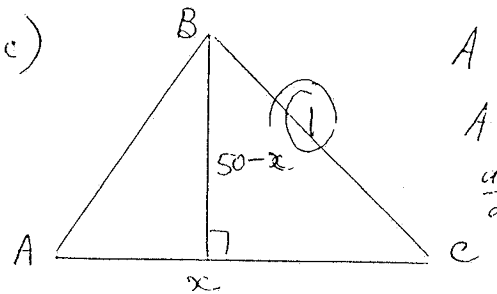
$2 = a^2 + a$ (1)

$a^2 + a - 2 = 0$

$(a+2)(a-1) = 0$

$a = 1, \ a = -2$ (1)

c)



$A = \frac{1}{2}x(50 - x)$ (1)

$A = 25x - \frac{x^2}{2}$

$\frac{dA}{dx} = 25 - x = 0$ for MAX OR MIN Value

$x = 25$ (1)

$\frac{d^2A}{dx^2} = -1$ (1) ∴ MAXIMUM VALUE

MUST DO TEST FOR

∴ MAX AREA = $\frac{1}{2} \times 25 \times 25$