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CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL



2017 Trial Examination

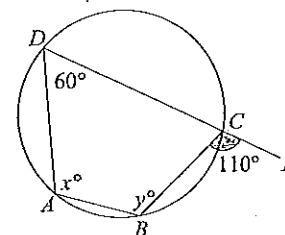
# FORM VI MATHEMATICS EXTENSION 1

Friday 4th August 2017

## SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

### QUESTION ONE



Suppose  $ABCD$  is a cyclic quadrilateral with  $DC$  produced to  $E$ . What are the values of  $x$  and  $y$ ?

- (A)  $x = 120, y = 110$
- (B)  $x = 110, y = 110$
- (C)  $x = 120, y = 120$
- (D)  $x = 110, y = 120$

### QUESTION TWO

Let  $A = (-3, 2)$  and  $B = (4, -7)$ . The interval  $AB$  is divided externally in the ratio  $5 : 3$  by the point  $P(x, y)$ . What is the value of  $x$ ?

- (A)  $14\frac{1}{2}$
- (B) 13
- (C)  $1\frac{3}{8}$
- (D)  $-13\frac{1}{2}$

#### General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

#### Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

#### Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

#### Collection

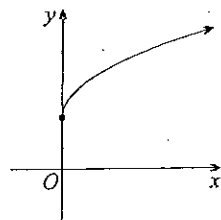
- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

#### Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 125 boys

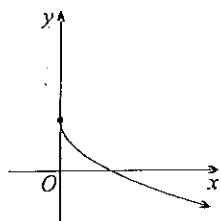
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FMW

QUESTION THREE

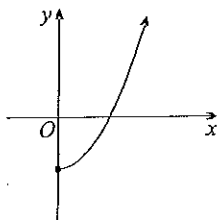


The diagram shows the graph of  $y = f(x)$ . Which diagram shows the graph of  $y = f^{-1}(x)$ ?

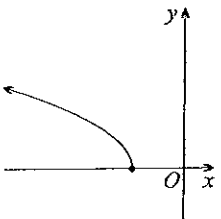
(A)



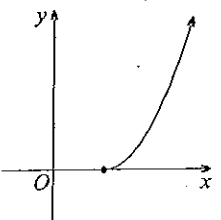
(B)



(C)



(D)



Examination continues overleaf ...

QUESTION FOUR

What is the derivative of  $\sin^{-1} 3x$ ?

(A)  $\frac{1}{3\sqrt{1-9x^2}}$

(B)  $\frac{-1}{3\sqrt{1-3x^2}}$

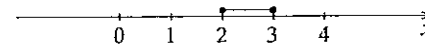
(C)  $\frac{3}{\sqrt{1-9x^2}}$

(D)  $\frac{3}{\sqrt{1-3x^2}}$

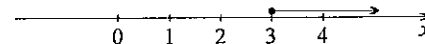
QUESTION FIVE

Which number line graph shows the correct solution to  $\frac{x}{x-2} \geq 3$ ?

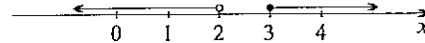
(A)



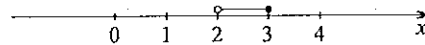
(B)



(C)



(D)



Examination continues next page ...

**QUESTION SIX**

What is the domain of the function  $y = 4 \sin^{-1} \frac{x}{3}$ ?

- (A)  $-3 \leq x \leq 3$
- (B)  $-\frac{1}{3} \leq x \leq \frac{1}{3}$
- (C)  $-2\pi \leq x \leq 2\pi$
- (D)  $-\frac{\pi}{8} \leq x \leq \frac{\pi}{8}$

**QUESTION SEVEN**

What is the maximum value of  $P = 6 \cos \theta + 4 \sin \theta$ ?

- (A) 10
- (B) 6
- (C)  $2\sqrt{13}$
- (D)  $2\sqrt{5}$

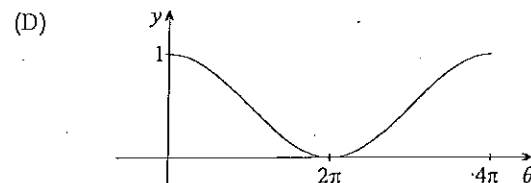
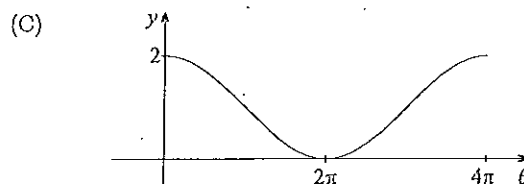
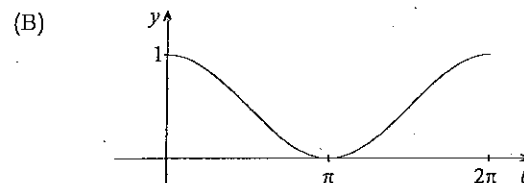
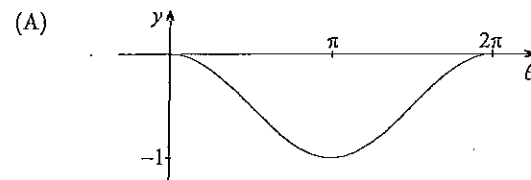
**QUESTION EIGHT**

A particle moves on a line so that its distance from the origin at time  $t$  seconds is  $x$  cm and its acceleration is given by  $\frac{d^2x}{dt^2} = 10 - 2x^3$ . If  $v$  represents the velocity of the particle, and the particle changes direction 1 cm on the negative side of the origin, which of the following equations is correct?

- (A)  $v^2 = 20x - x^4$
- (B)  $v^2 = 20x - x^4 + 21$
- (C)  $v = 10x - \frac{1}{2}x^4$
- (D)  $v = 10x - \frac{1}{2}x^4 + 11\frac{1}{2}$

**QUESTION NINE**

Which of the diagrams below best represents the graph of  $y = \cos^2 \frac{1}{2}\theta$ ?



**QUESTION TEN**

What is the coefficient of  $z^3$  in the expansion of  $(1 + z + z^2)^5$ ?

- (A) 10
- (B) 20
- (C) 30
- (D) 40

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet.

Marks

(a) Find the exact value of  $\sin \frac{\pi}{8} \cos \frac{\pi}{8}$ . 2

(b) Evaluate  $\sin^{-1}(\sin \frac{4\pi}{3})$ . 1

(c) Show that  $\lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{x} = \frac{1}{2}$ . 1

(d) Find the following integrals:

(i)  $\int \frac{4x}{16+x^2} dx$  1

(ii)  $\int \frac{3}{9+x^2} dx$  1

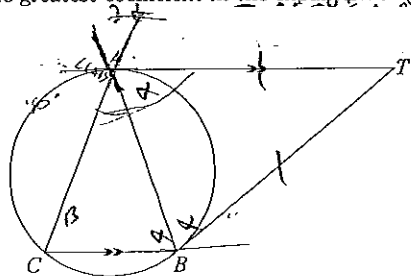
(iii)  $\int \frac{-1}{\sqrt{25+x}} dx$  1

(e) Write down a general solution of the equation  $\sin x = -\frac{1}{2}$ . 1

(f) If  $a$ ,  $b$  and  $c$  are the roots of the equation  $3x^3 + 4x^2 - 5x - 8 = 0$ , find the value of  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ . 2

(g) By expanding, find the greatest coefficient in the expansion of  $(4x+3)^4$ . 2

(h)



Tangents touching a circle at  $A$  and  $B$  respectively, intersect at  $T$ . Point  $C$  is on the circle and  $AT \parallel CB$ . Prove that  $AB=AC$ .

QUESTION TWELVE (15 marks) Use a separate writing booklet.

Marks

(a) An object is put in a freezer to cool. After  $t$  minutes, its temperature is  $T^\circ\text{C}$ . The freezer is at a constant temperature of  $-8^\circ\text{C}$ . The object's temperature  $T$  decreases according to the differential equation  $\frac{dT}{dt} = -k(T+8)$ , where  $k$  is a positive constant.

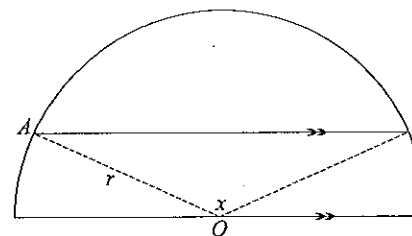
(i) Show that  $T = Ae^{-kt} - 8$ , where  $A$  is a constant, is a solution of the differential equation. 1

(ii) If the object cools from an initial temperature of  $40^\circ\text{C}$  to  $30^\circ\text{C}$  in half an hour, find the values of  $A$  and  $k$ . 2

(iii) When will the temperature of the object be  $0^\circ\text{C}$ ? Give your answer correct to the nearest hour. 1

(iv) Explain what will happen to  $T$  eventually. 1

(b)



The diagram above shows a semi-circle of radius  $r$  with centre  $O$ . Chord  $AB$  is drawn parallel to the base such that it divides the semi-circle into two parts of equal area. Chord  $AB$  subtends an angle of  $x$  radians at the centre  $O$ .

(i) Show that  $\sin x = x - \frac{\pi}{2}$ . 1

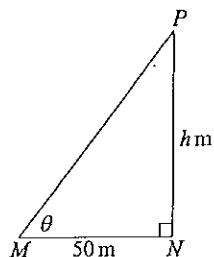
(ii) The equation has a root near  $x = 2$ . Use one application of Newton's method to find a better approximation for this root, writing your answer correct to three significant figures. 2

(c) (i) Use the substitution  $u = 3x + 1$  to show that  $\int_0^1 \frac{x}{(3x+1)^2} dx = \frac{2}{9} \ln 2 - \frac{1}{12}$ . 2

(ii) Hence find the volume of the solid formed when the region bounded by the curve 1

$y = \frac{6\sqrt{x}}{3x+1}$ , the  $x$ -axis and the line  $x = 1$  is rotated about the  $x$ -axis. Give your answer in exact form.

(d)



Bowie jumps out of a helicopter and by the time he reaches the position  $P$ ,  $h$  metres above the ground, he is falling at a constant rate of 150 kilometres per hour. Point  $N$  is on the ground directly below  $P$  and  $M$  lies 50 metres from  $N$ . The angle of elevation of  $P$  from  $M$  is  $\theta$  radians.

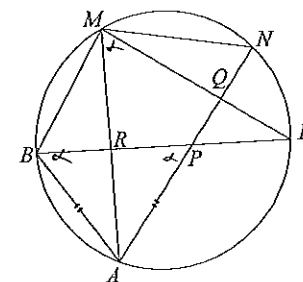
(i) Show that  $\frac{dh}{d\theta} = \frac{50}{\cos^2 \theta}$ . 1

(ii) Find the rate of decrease of the angle of elevation when Bowie reaches a height of 1200 metres. Give your answer in radians per second. 3

QUESTION FOURTEEN (15 marks) Use a separate writing booklet.

Marks

(a)

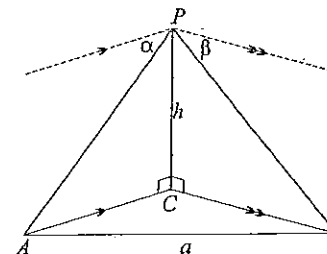


3

The diagram above shows a cyclic quadrilateral  $ABMN$ . Point  $P$  lies on  $AN$  such that  $AB = AP$  and  $BP$  produced meets the circle again at  $D$  and  $AM$  at  $R$ . The chord  $MD$  intersects  $AN$  at  $Q$ .

Copy the diagram and show that  $QPRM$  is a cyclic quadrilateral.

(b)



The diagram above shows two points  $A$  and  $B$  on level ground.  $B$  is  $a$  metres due east of  $A$ . A tower, of height  $h$  metres, is also on the same level ground and its bearing is  $N\theta E$  and  $N\phi W$  from  $A$  and  $B$  respectively. From the top of the tower  $P$ , the angle of depression of  $A$  is  $\alpha$  and of  $B$  is  $\beta$ .

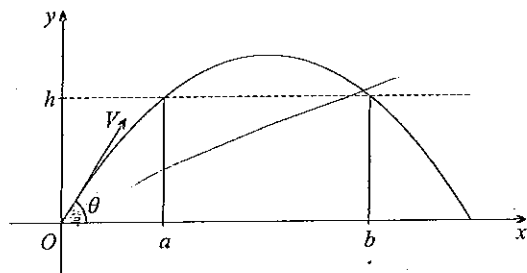
(i) Prove that  $h \sin(\theta + \phi) = a \cos \phi \tan \alpha$ . 2

(ii) Prove that  $h^2(\cot^2 \alpha - \cot^2 \beta) - 2ha \cot \alpha \sin \theta + a^2 = 0$ . 2

(c) If  $f^{(n)}(x)$  denotes the  $n$ th derivative of  $f(x) = \frac{1}{x}$ , prove by mathematical induction 3

that  $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$  for all positive integers  $n$ .

(d)



A particle is fired from  $O$  with initial velocity  $V$  m/s at an angle  $\theta$  to the horizontal. The particle just clears two thin vertical towers of height  $h$  metres at horizontal distances of  $a$  metres and  $b$  metres from  $O$ .

The equations of motion of the particle are  $x = Vt \cos \theta$  and  $y = Vt \sin \theta - \frac{1}{2}gt^2$ .  
(Do NOT prove these equations.)

- (i) Show that  $V^2 = \frac{a^2 g(1 + \tan^2 \theta)}{2(a \tan \theta - h)}$ . 2
- (ii) Hence show that  $\tan \theta = \frac{h(a + b)}{ab}$ . 2
- (iii) Hence show that  $\tan \theta = \tan \alpha + \tan \beta$ , where  $\alpha$  and  $\beta$  are the angles of elevation from  $O$  to the tops of the towers. 1

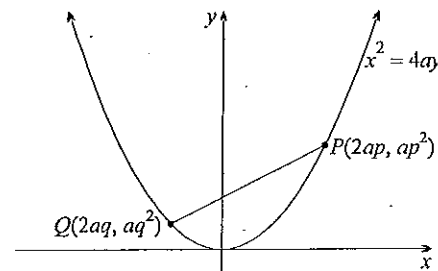
————— End of Section II —————

END OF EXAMINATION

QUESTION THIRTEEN (15 marks) Use a separate writing booklet.

Marks

(a)



The points  $P(2ap, ap^2)$  and  $Q(2aq, aq^2)$  lie on the parabola with equation  $x^2 = 4ay$ .

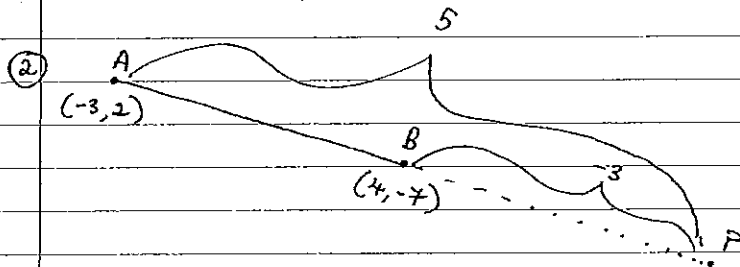
- (i) Find the coordinates of  $M$ , the midpoint of  $PQ$ . 1
  - (ii) Show that the equation of the chord  $PQ$  is  $y = \frac{1}{2}(p + q)x - apq$ . 1
  - (iii) If the chord always passes through the point  $(0, 2a)$ , find the equation of the locus of  $M$ . 2
- (b) A particle moves along a straight line and its displacement,  $x$  centimetres, from a fixed point  $O$  at a given time  $t$  seconds is given by  $x = 2 + \cos^2 t$ .
- (i) Show that its acceleration is given by  $\ddot{x} = 10 - 4x$ . 2
  - (ii) Explain why the motion is simple harmonic. 1
  - (iii) Find the centre, amplitude and period of the motion. 2
- (c) The polynomial  $P(x)$  is given by  $P(x) = x^3 - mx^2 + mx - 1$ , where  $m$  is a constant.
- (i) Show that  $(x - 1)$  is a factor of  $P(x)$ . 1
  - (ii) Hence find a quadratic factor of  $P(x)$ . 2
  - (iii) Hence find the set of values of  $m$  for which all the roots of the equation  $P(x) = 0$  are real. 2
  - (iv) If  $m = 3$ , the graph of  $y = P(x)$  is a transformation of the graph of  $y = x^3$ . Describe this transformation. 1

# Extension I TRIAL 2017 SOLUTIONS

Multiple Choice:

- ①  $x = 110$  (exterior angle of cyclic quadrilateral ABCD)  
 $y = 120$  (opposite angles of cyclic quadrilateral ABCD)

choose **D**

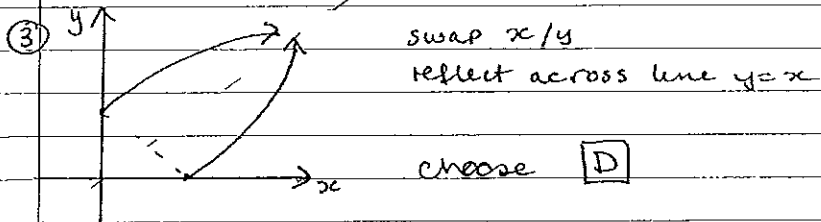


$$x = \frac{(-3)(-3) + 5(4)}{5-3}$$

$$= \frac{29}{2}$$

choose **A**

$$= 14\frac{1}{2}$$



④  $y = \sin^{-1}(3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \times 3$$

$$= \frac{3}{\sqrt{1-9x^2}}$$

choose **C**

⑤  $(x-2)^2 \times \frac{x}{x-2} > 3(x-2)^2$

$$x(x-2) > 3(x-2)^2$$

$$3(x-2)^2 - x(x-2) \leq 0$$

$$(x-2)(3(x-2) - x) \leq 0$$

$$(x-2)(2x-6) \leq 0$$

$$2(x-2)(x-3) \leq 0$$

choose **D**

⑥  $y = 4 \sin^{-1} \frac{x}{3}$

$$-1 \leq \frac{x}{3} \leq 1$$

choose **A**

$$-3 \leq x \leq 3$$

⑦  $R = \sqrt{6^2 + 4^2}$

$$= \sqrt{52}$$

choose **C**

$$= 2\sqrt{13}$$

⑧  $\frac{d^2x}{dt^2} = 10 - 2x^3$

$$\frac{d}{dx} \left( \frac{1}{2}v^2 \right) = 10 - 2x^3$$

$$\frac{1}{2}v^2 = 10x - \frac{1}{2}x^4$$

$$v^2 = 20x - x^4 + C$$

at  $x = -1$ ,  $v = 0$

$$0 = 20(-1) - (-1)^4 + C$$

$$C = 21$$

$$v^2 = 20x - x^4 + 21$$

choose **B**

9) using  $\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$   
 $= 2\cos^2 \frac{\theta}{2} - 1$

$$2\cos^2 \frac{\theta}{2} = \cos \theta + 1$$

$$\cos^2 \frac{\theta}{2} = \frac{1}{2} \cos \theta + \frac{1}{2}$$

So  $y = \cos^2 \frac{\theta}{2}$  has amplitude  $\frac{1}{2}$ , range  $0 \leq y \leq 1$   
 period  $2\pi$

choose [B]

10)  $(1+z+z^2)^5 = (1+z(1+z))^5$

now  $(1+z(1+z))^5 = {}^5C_0 + {}^5C_1 z(1+z) + {}^5C_2 z^2(1+z)^2 + {}^5C_3 z^3(1+z)^3$   
 $+ {}^5C_4 z^4(1+z)^4 + {}^5C_5 z^5(1+z)^5$

terms in  $z^3$  come from  ${}^5C_2 z^2(1+z)^2$  and  ${}^5C_3 z^3(1+z)$

term in  $z^3 = ({}^5C_2 \times 2 + {}^5C_3 \times 1) z^3$

coefficient =  $10 \times 2 + 10 \times 1$   
 $= 30$

choose [C]

## Section II

11) (a)  $\sin \frac{\pi}{8} \cos \frac{\pi}{8} = \frac{1}{2} \sin \frac{\pi}{4}$  ✓

$$= \frac{1}{2\sqrt{2}}$$
 ✓

$$= \frac{\sqrt{2}}{4}$$

(b)  $\sin^{-1}(\sin \frac{4\pi}{3}) = \sin^{-1}(-\frac{\sqrt{3}}{2})$  ✓  
 $= -\frac{\pi}{3}$

(c)  $\lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{\frac{x}{2}}$  ✓

$$= \frac{1}{2}$$

(d) (i)  $\int \frac{4x}{16+x^2} dx = 2 \int \frac{2x}{16+x^2} dx$

$$= 2 \ln(16+x^2) + C$$
 ✓

(ii)  $\int \frac{3}{9+x^2} dx = 3 \times \frac{1}{3} \tan^{-1} \frac{x}{3}$  ✓

$$= \tan^{-1} \frac{2x}{3} + C$$

(iii)  $\int \frac{-1}{\sqrt{25+x}} dx = -1 \int (25+x)^{-\frac{1}{2}} dx$

$$= -1 \times \frac{(25+x)^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= -2\sqrt{25+x} + C$$
 ✓

(e)  $\theta = n\pi + (-1)^n \sin^{-1}(-\frac{1}{2})$  ✓  
 $= n\pi + (-1)^n (-\frac{\pi}{6})$

(f)  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ac+ab}{abc}$  ✓

$$= \frac{-\frac{1}{5} + \frac{1}{3} + \frac{1}{4}}{-\frac{1}{60}}$$

$$= -\frac{17}{2}$$



$$(9) (4x+3)^4 = (4x)^4 + {}^4C_1 (4x)^3(3) + {}^4C_2 (4x)^2(3)^2 + {}^4C_3 (4x)(3)^3 + 3^4$$

$$= 256x^4 + 768x^3 + 864x^2 + 432x + 81$$

the greatest coefficient is 864

(h)  $\angle TAB = \angle ACB$  (angle in the alternate segment)  
 $\angle TAB = \angle ABC$  (alternate angles,  $AT \parallel CB$ )  
 $AC = AB$  (sides opposite equal angles)

15

$$(12) (a) (i) T = Ae^{-kt} - 8$$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T+8)$$

(ii) at  $t=0$ ,  $T=40$

$$40 = A - 8$$

$$A = 48$$

at  $t=30$ ,  $T=30$

$$30 = 48e^{-30k} - 8$$

$$\frac{38}{48} = e^{-30k}$$

$$-30k = \log_e \left( \frac{19}{24} \right)$$

$$k = -\frac{1}{30} \log_e \left( \frac{19}{24} \right)$$

$$= \frac{1}{30} \log_e \left( \frac{24}{19} \right)$$

(iii) if  $T=0$

$$0 = 48e^{-kt} - 8$$

$$\frac{8}{48} = e^{-kt}$$

$$-kt = \log_e \left( \frac{1}{6} \right)$$

$$t = -\frac{1}{k} \log_e \left( \frac{1}{6} \right)$$

$$= 230.091... \text{ minutes}$$

$$= 3.83... \text{ h}$$

$$= 4 \text{ h}$$

(iv) as  $t \rightarrow \infty$   
 $T \rightarrow -8$

$$(b) (i) \frac{1}{2} \pi^2 (x - \sin x) = \frac{1}{2} \times \frac{1}{2} \pi^2 \pi^2 \quad \checkmark$$

$$x - \sin x = \frac{\pi}{2}$$

$$\sin x = x - \frac{\pi}{2}$$

$$(ii) \text{ let } f(x) = \sin x - x + \frac{\pi}{2}$$

$$f'(x) = \cos x - 1 \quad \checkmark$$

$$z_1 = 2 - \frac{\sin 2 - 2 + \frac{\pi}{2}}{\cos 2 - 1}$$

$$= 2.339 \dots$$

$$\approx 2.34 \quad \checkmark$$

$$(c) \int_0^1 \frac{x}{(3x+1)^2} dx$$

$$\text{let } u = 3x+1$$

$$\frac{du}{dx} = 3$$

$$du = 3 dx$$

$$3x = u - 1$$

$$x = \frac{u-1}{3}$$

$$\text{if } x = 0$$

$$u = 1$$

$$\text{if } x = 1$$

$$u = 4$$

$$= \int_1^4 \frac{\frac{u-1}{3}}{u^2} \times \frac{du}{3}$$

some sensible attempt  $\checkmark$

$$= \frac{1}{9} \int_1^4 \frac{u-1}{u^2} du$$

$$= \frac{1}{9} \int_1^4 \frac{1}{u} - u^{-2} du$$

$$= \frac{1}{9} \left[ \ln u + \frac{1}{u} \right]_1^4$$

$$= \frac{1}{9} \left( \ln 4 + \frac{1}{4} - (\ln 1 + 1) \right)$$

$$= \frac{1}{9} \left( 2 \ln 2 - \frac{3}{4} \right)$$

$$= \frac{2}{9} \ln 2 - \frac{1}{12}$$

$$(ii) V = \pi \int_0^1 \left( \frac{6\sqrt{x}}{3x+1} \right)^2 dx$$

$$= \pi \int_0^1 \frac{36x}{(3x+1)^2} dx$$

$$\checkmark = 36\pi \times \left( \frac{2}{9} \ln 2 - \frac{1}{12} \right)$$

$$= \pi (8 \ln 2 - 3)$$

cubic units

$$(d) (i) \tan \theta = \frac{h}{50}$$

$$h = 50 \tan \theta$$

$$\frac{dh}{d\theta} = 50 \sec^2 \theta \quad \checkmark$$

$$= \frac{50}{\cos^2 \theta}$$

$$(ii) \frac{d\theta}{dt} = \frac{d\theta}{dh} + \frac{dh}{dt}$$

$$= \frac{\cos^2 \theta}{50} \times \frac{125}{3} \quad \checkmark$$

$$= \frac{1}{577} \times \frac{125}{3}$$

$$= \frac{5}{3462} \text{ radians/s} \quad \checkmark$$

$$150 \text{ km/h}$$

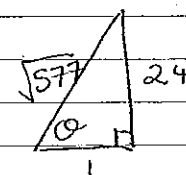
$$= \frac{150 \times 1000}{60 \times 60}$$

$$= \frac{125}{3} \text{ m/s}$$

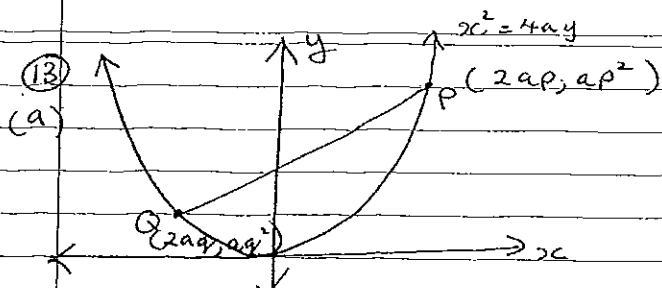
$$\text{if } h = 1200$$

$$\tan \theta = \frac{1200}{50}$$

$$= 24$$



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(i)  $M = \left( \frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2} \right)$   
 $= (a(p+q), a \frac{(p^2+q^2)}{2})$  ✓

(ii)  $m = \frac{ap^2 - aq^2}{2ap - 2aq}$   
 $= \frac{p^2 - q^2}{2(p-q)}$   
 $= \frac{(p-q)(p+q)}{2(p-q)}$   
 $= \frac{p+q}{2}, p \neq q$

equation is  $y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$  ✓  
 $y = \frac{1}{2}(p+q)x - apq$

(iii) if the chord passes through  $(0, 2a)$   
 $2a = -apq$   
 $pq = -2$  ✓

so  $x^2 = a^2(p+q)^2$   
 $= a^2(p^2 + q^2 + 2pq)$   
 $= a^2 \left( \frac{2y}{a} - 4 \right)$  ✓  
 $= 2a(y - 2a)$

(b) (i)  $x = 2 + \cos^2 t$   
 $\dot{x} = 2 \cos t \times -\sin t$   
 $= -2 \sin t \cos t$   
 $= -\sin 2t$  ✓  
 $\ddot{x} = -2 \cos 2t$   
 $= -2(2 \cos^2 t - 1)$   
 $= -4 \cos^2 t + 2$   
 $= -4(x-2) + 2$   
 $= 10 - 4x$  as required } ✓

OR  $x = 2 + \cos^2 t$   
 $= 2 + \frac{1}{2}(\cos 2t + 1)$   
 $= 2\frac{1}{2} + \frac{1}{2} \cos 2t$  ✓  
 $\dot{x} = -\sin 2t$   
 $\ddot{x} = -2 \cos 2t$   
 $= -2(x - 2\frac{1}{2}) \times 2$  } ✓  
 $= 10 - 4x$  as required

(ii)  $\ddot{x} = 10 - 4x$   
 $= -4(x - 2\frac{1}{2})$   
 which is of the form  $\ddot{x} = -n^2(x - x_0)$   
 (acceleration is proportional to displacement but in the opposite direction)

OR  $x = 2\frac{1}{2} + \frac{1}{2} \cos 2t$   
 which is just a transformation of  $x = \cos t$   
 so is simple harmonic

(iii) centre:  $x = 2\frac{1}{2}$  period =  $\frac{2\pi}{2} = \pi$   
 amplitude =  $\frac{1}{2}$  ✓ one correct  
 ✓ three correct

(c)  $P(x) = x^3 - mx^2 + mx - 1$

(i)  $P(1) = 1 - m + m - 1 = 0$

so  $(x-1)$  is a factor

(ii) 
$$\begin{array}{r} x^2 + (1-m)x + 1 \\ x-1 \overline{) x^3 - mx^2 + mx - 1} \\ \underline{x^3 - x^2} \phantom{+ mx - 1} \\ (1-m)x^2 + mx - 1 \phantom{- 1} \\ \underline{(1-m)x^2 - x + m} \phantom{- 1} \\ x - 1 \phantom{- 1} \\ \underline{x - 1} \\ 0 \end{array}$$

a solution by inspection is fine

the quadratic factor is  $x^2 + (1-m)x + 1$

(iii) for real roots we need

$(1-m)^2 - 4(1)(1) \geq 0$   
 $m^2 - 2m + 1 - 4 \geq 0$   
 $m^2 - 2m - 3 \geq 0$

$(m-3)(m+1) \geq 0$

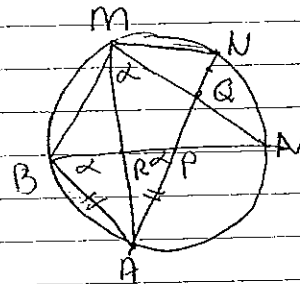


$m \leq -1$  or  $m \geq 3$

(iv) If  $m = 3$ ,  $P(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3$

This is the graph of  $y = x^3$  shifted 1/unit to the right.

(14) (a)



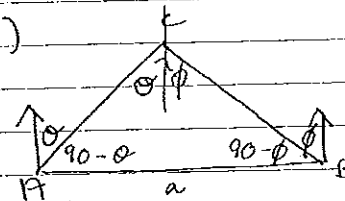
let  $\angle ABP = \alpha$

then  $\angle PPB = \alpha$  (angles opposite equal sides)

$\angle AMQ = \alpha$  (angles at the circumference on arc AN)

$Q, P, R, M$  is a cyclic quadrilateral (exterior angle equals opposite interior angles)

(b)



$\angle ACB = \theta + \phi$

(i) in  $\triangle ACB$ ,  $\frac{\sin(\theta + \phi)}{a} = \frac{\sin(90 - \phi)}{AC}$

in  $\triangle APC$ ,  $\tan \alpha = \frac{h}{AC}$   
 $AC = \frac{h}{\tan \alpha}$

so  $\frac{\sin(\theta + \phi)}{a} = \frac{\sin(90 - \phi)}{\frac{h}{\tan \alpha}}$

$$h \sin(\theta + \phi) = a \sin(90 - \phi) \tan \alpha$$

$$h \sin(\theta + \phi) = a \cos \phi \tan \alpha$$

as required

(ii) from  $\triangle APC$

$$\cot \alpha = \frac{AC}{h}$$

$$AC^2 = h^2 \cot^2 \alpha$$

from  $\triangle BPC$

$$\cot \beta = \frac{BC}{h}$$

$$BC^2 = h^2 \cot^2 \beta$$

In  $\triangle ABC$

$$\cos(90 - \theta) = \frac{a^2 + h^2 \cot^2 \alpha - h^2 \cot^2 \beta}{2 \times a \times h \cot \alpha}$$

$$\sin \theta = \frac{h^2 (\cot^2 \alpha - \cot^2 \beta) + a^2}{2ha \cot \alpha}$$

$$2ha \cot \alpha \sin \theta = h^2 (\cot^2 \alpha - \cot^2 \beta) + a^2$$

$$h^2 (\cot^2 \alpha - \cot^2 \beta) - 2ha \cot \alpha \sin \theta + a^2 = 0$$

(c) Step 1: let  $n=1$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$\text{now } f^{(1)}(x) = \frac{(-1)^1 1!}{x^{1+1}}$$

$$= -\frac{1}{x^2} \text{ as required}$$

the result is true for  $n=1$

step 2: suppose  $k$  is a positive integer for which the result is true

$$\text{that is } f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}$$

$$= (-1)^k k! x^{-(k+1)}$$

we now prove the result is true for  $n=k+1$ , that is we prove that

$$f^{(k+1)}(x) = \frac{(-1)^{k+1} (k+1)!}{x^{k+2}}$$

$$\text{now } f^{(k+1)}(x) = -(k+1) (-1)^k k! x^{-(k+1)-1} \text{ by } \frac{d}{dx}$$

$$= (-1)^{k+1} (k+1)! x^{-(k+2)}$$

$$= \frac{(-1)^{k+1} (k+1)!}{x^{k+2}} \text{ as required}$$

so by the principle of mathematical induction the result is true for all positive integers  $n$ .

$$(14) (d) \quad x = vt \cos \theta$$

$$y = vt \sin \theta - \frac{1}{2} g t^2$$

(i) at  $x = a$ ,  $y = h$

$$\text{so } a = vt \cos \theta$$

$$t = \frac{a}{v \cos \theta}$$

$$\text{and } h = \cancel{v} \times \frac{a}{\cancel{v} \cos \theta} \sin \theta - \frac{1}{2} g \left( \frac{a}{v \cos \theta} \right)^2 \quad \checkmark$$

$$h = a \tan \theta - \frac{\frac{1}{2} g a^2}{v^2 \cos^2 \theta}$$

$$\frac{g a^2 \sec^2 \theta}{2 v^2} = a \tan \theta - h$$

$$2 v^2 = \frac{g a^2 \sec^2 \theta}{a \tan \theta - h}$$

$$v^2 = \frac{g a^2 \sec^2 \theta}{2 (a \tan \theta - h)}$$

$$= \frac{g a^2 (1 + \tan^2 \theta)}{2 (a \tan \theta - h)} \quad (i) \quad \checkmark$$

$$(ii) \text{ Similarly, } v^2 = \frac{g b^2 (1 + \tan^2 \theta)}{2 (b \tan \theta - h)} \quad (ii)$$

equating (i) + (ii)

$$\cancel{g} \frac{b^2 (1 + \cancel{\tan^2} \theta)}{\cancel{2} (b \tan \theta - h)} = \cancel{g} \frac{a^2 (1 + \cancel{\tan^2} \theta)}{\cancel{2} (a \tan \theta - h)}$$

$$b^2 (b \tan \theta - h) = a^2 (b \tan \theta - h)$$

$$a b^2 \tan \theta - b^2 h = a^2 \tan \theta - a^2 h$$

$$a b^2 \tan \theta - b a^2 \tan \theta = b^2 h - a^2 h$$

$$ab \tan \theta (b - a) = h (b^2 - a^2)$$

$$ab \tan \theta = \frac{h (b - a) (b + a)}{\cancel{b - a}}$$

$$\tan \theta = \frac{h (b + a)}{ab}$$

$$(iii) \quad \tan \theta = \frac{hb}{ab} + \frac{ha}{ab}$$

$$= \frac{h}{a} + \frac{h}{b}$$

$$= \tan \alpha + \tan \beta \quad \text{as required} \quad \checkmark$$

