CSSA

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2017

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Thursday, 3 August 2017

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A formula Reference Sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2 - 6

10 marks

- Attempt Questions 1 - 10
- Allow 15 minutes for this section

Section II Pages 7 - 15

90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10.

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1 What is $-1+i\sqrt{3}$ expressed in modulus-argument form?

(A) $2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

(B) $2(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

(C) $\sqrt{2}(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

(D) $\sqrt{2}(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

2 What is the equation of the conic whose foci are $(\pm 3,0)$ and directrices are $x=\pm 12$?

(A) $\frac{x^2}{36}+\frac{y^2}{16}=1$

(B) $\frac{x^2}{36}+\frac{y^2}{27}=1$

(C) $\frac{x^2}{36}-\frac{y^2}{16}=1$

(D) $\frac{x^2}{36}-\frac{y^2}{27}=1$

3 Which of the following is an expression for $\int \frac{x^2-1}{x^2+1}dx$?

(A) $x+\tan^{-1}x+c$

(B) $x-\tan^{-1}x+c$

(C) $x+2\tan^{-1}x+c$

(D) $x-2\tan^{-1}x+c$

4 The polynomial equation $P(x)=0$ has roots α, β and γ.
What are the roots of $P(2x-3)=0$?

(A) $\alpha-\frac{3}{2}, \beta-\frac{3}{2}, \gamma-\frac{3}{2}$

(B) $2\alpha-3, 2\beta-3, 2\gamma-3$

(C) $\frac{\alpha}{2}+3, \frac{\beta}{2}+3, \frac{\gamma}{2}+3$

(D) $\frac{\alpha+3}{2}, \frac{\beta+3}{2}, \frac{\gamma+3}{2}$

5 Which of the following is an expression for $\int x\sin 2x\,dx$?

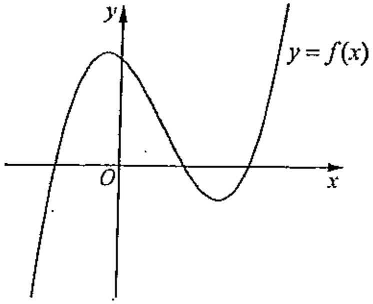
(A) $-\frac{x}{2}\cos 2x+\frac{1}{4}\sin 2x+c$

(B) $-\frac{x}{2}\cos 2x-\frac{1}{4}\sin 2x+c$

(C) $2x\cos 2x+4\sin 2x+c$

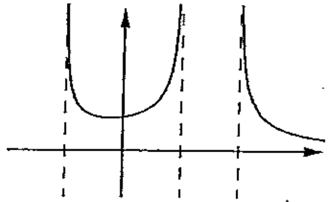
(D) $2x\cos 2x-4\sin 2x+c$

6 The graph of $y=f(x)$ is shown below.

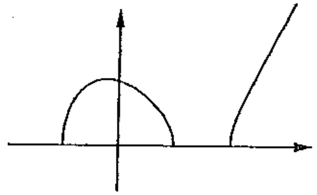


What is the best representation of the graph of $y=\frac{1}{\sqrt{f(x)}}$?

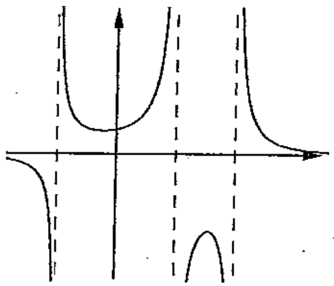
(A)



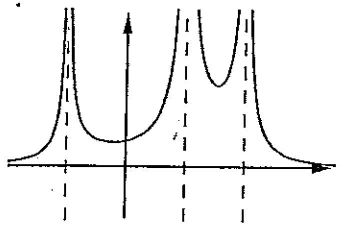
(B)



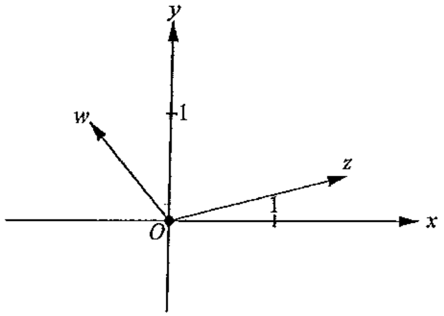
(C)



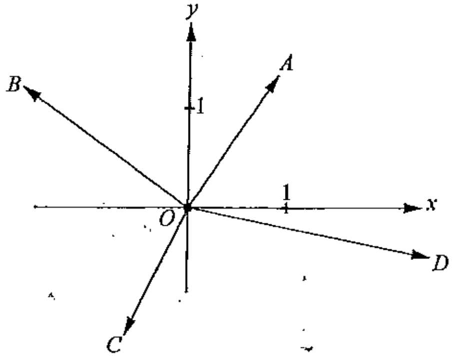
(D)



7 The diagram shows vectors representing the complex numbers z and w.



In the diagram below, which vector best represents the complex number $\frac{z}{w}$?



(A) OA

(B) OB

(C) OC

(D) OD

8 A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its velocity and k is a positive constant.

Which of the following is the correct expression for the distance, x, fallen from rest?

(A) $\frac{1}{2k}(\log_e \frac{v}{g-kv^2} + \log_e \frac{v}{g})$

(B) $\frac{1}{2k}(\log_e \frac{g}{g-kv^2})$

(C) $\frac{1}{2k}(\log_e \frac{v}{g-kv^2} - \log_e \frac{v}{g})$

(D) $\frac{1}{2k}(\log_e \frac{g-kv^2}{g})$

9 If four socks are chosen from a drawer containing five different pairs of socks, what is the probability that no socks match?

(A) $1\times\frac{8}{9}\times\frac{6}{8}\times\frac{4}{7}$

(B) $1\times\frac{1}{9}\times\frac{1}{8}\times\frac{1}{7}$

(C) $1\times\frac{9}{10}\times\frac{8}{9}\times\frac{7}{8}$

(D) $1\times\frac{8}{10}\times\frac{6}{9}\times\frac{4}{8}$

10 Find the number of solutions to the equation $\tan^{-1}x + \tan^{-1}2x = \tan^{-1}3$.

(A) 1

(B) 2

(C) 3

(D) 4

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find values for a and b so that $(a+ib)^2=16+30i$. **2**

(ii) Hence or otherwise, solve the equation $z^2-(1+i)z-(4+7i)=0$. **2**

(b) Find $\displaystyle\int_0^1 \frac{2e^x}{e^{2x}+2e^x+1}\,dx$. **3**

(c) Sketch the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$ showing its foci, directrices and asymptotes. **4**

(d) Let $P(x)=x^4+ax^3+36x^2-35x+b$, where a and b are real constants.

Given that $x=5$ and $x=\dfrac{1-i\sqrt{3}}{2}$ are zeros of $P(x)$,

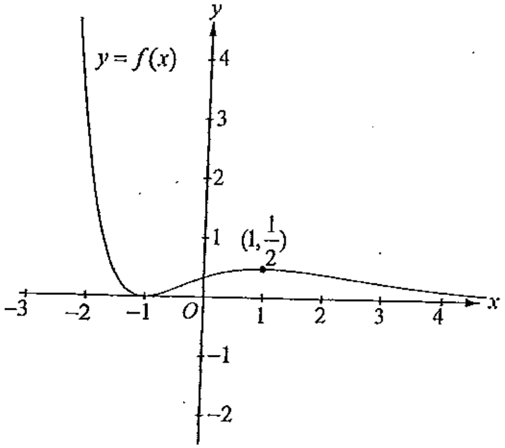
(i) explain why x^2-x+1 must be a factor of $P(x)$. **2**

(ii) find a and b. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y=f(x)$.



Draw separate one-third page diagrams for each of the following functions.

(i) $y=\dfrac{1}{f(x)}$ **2**

(ii) $y=\sqrt{f(x)}$ **2**

(iii) $y=x\,f(x)$ **2**

(b) (i) Find the roots of $z^5+1=0$ in modulus-argument form and show these roots on an Argand diagram, showing all essential features. **3**

(ii) Express z^5+1 as a product of real linear and quadratic factors. **2**

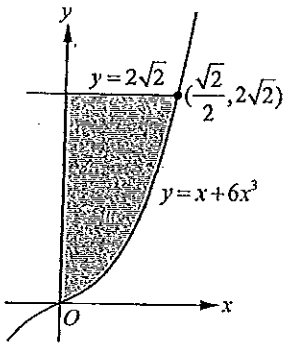
(c) (i) If x, y and z are positive numbers, show that $x^2+y^2+z^2\geq xy+xz+yz$. **2**

(ii) Hence, or otherwise, prove that $x^3+y^3+z^3\geq 3xyz$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) The region bounded by the curve $y = x + 6x^3$, the y-axis and the line $y = 2\sqrt{2}$ is rotated about the y-axis. **3**



Use the method of cylindrical shells to calculate the volume of the resultant solid. Give your answer correct to 3 significant figures.

(b) P represents a complex number z, where z satisfies the condition that $|z-2| = 2$.

(i) Sketch the locus of P on an Argand diagram. **1**

(ii) Hence or otherwise, find the value of k given that $\arg(z-2) = k(\arg(z^2-2z))$. **2**

(c) Let α, β and γ be the roots of $x^3 - 7x^2 + 18x - 7 = 0$.

(i) Find a cubic equation with integer coefficients that has roots $1+\alpha^2, 1+\beta^2$ and $1+\gamma^2$. **3**

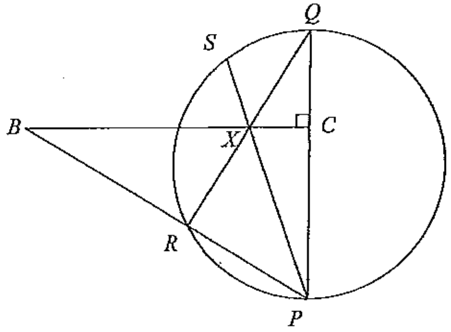
(ii) Hence, or otherwise, find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$. **1**

Question 13 continues on page 10

Question 13 (continued)

(d) In the diagram, PQ is a diameter of the circle. The chords QR and PS intersect at X. The point C lies on PQ such that XC is perpendicular to PQ. The point B is the intersection of PR produced and CX produced.

Copy or trace the diagram into your writing booklet.



(i) Show that $\angle PBC = \angle PQR$. **1**

(ii) Show that $SBRX$ is a cyclic quadrilateral. **2**

(iii) Show that the points B, S and Q are collinear. **2**

End of Question 13

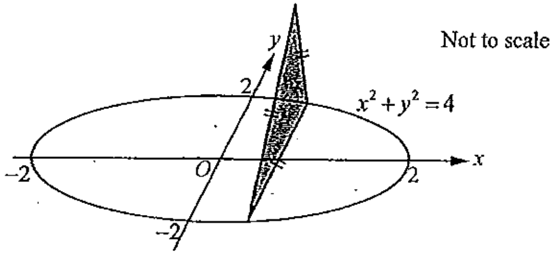
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^2 = (2\cos\theta)x - 1$ has complex roots α and β.

(i) Find α and β. **2**

(ii) Hence show that $\alpha^6 + \beta^6 = 2\cos 6\theta$. **2**

(b) The base of a solid is the region bounded by a circle whose equation is $x^2 + y^2 = 4$. **3**
Vertical cross-sections of the solid perpendicular to the x-axis are equilateral triangles as shown in the diagram.

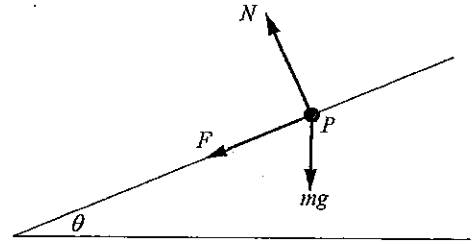


Find the volume of the solid.

Question 14 continues on page 12

Question 14 continued

(c) A car of mass m travels at a constant speed of v around a circular track of radius R. The track is banked at an angle θ to the horizontal. The car is subject to a gravitational force mg, a normal reaction force N and a frictional force F parallel to the track, as shown in the diagram.



(i) By resolving forces vertically and horizontally at P, show that **3**

$$F = m\left(\frac{v^2}{R}\cos\theta - g\sin\theta\right).$$

(ii) What speed must the driver maintain in order for the car to experience no sideways frictional force, given that $R = 130$ m, $\theta = 9°$ and $g = 10\,\text{ms}^{-2}$. **1**

(d) Let $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$ where $n \geq 0$.

(i) Show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ for $n \geq 2$. **3**

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx$. **1**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that $n! > 2^n$ for all integers $n \geq 4$. **3**

(b) You are given that $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$. **3**

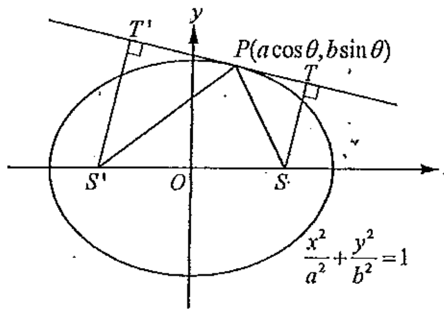
Hence, or otherwise, solve $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$ for $0 \leq x \leq 2\pi$.

(c) (i) Show that $x^{\frac{1}{2}} + (by)^{\frac{1}{2}} = b^{\frac{1}{2}}$ where $b > 0$ is a monotonic decreasing function for all values of x in its domain. **2**

(ii) Sketch the curve $x^{\frac{1}{2}} + (by)^{\frac{1}{2}} = b^{\frac{1}{2}}$ where $b > 0$. **1**

(iii) Hence sketch $|x|^{\frac{1}{2}} + (by)^{\frac{1}{2}} = b^{\frac{1}{2}}$ where b > 0. **1**

(d) Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.



Not to scale

(i) Show that the equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is $bx\cos\theta + ay\sin\theta - ab = 0$. **2**

(ii) Show that $b^2\cos^2\theta + a^2\sin^2\theta = a^2(1 - e^2\cos^2\theta)$. **1**

(iii) Perpendicular lines from the foci S and S' meet the tangent at P in the points T and T' respectively. **2**

Show that $ST \times S'T' = b^2$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A class consisting of $3x$ students is to be divided into three groups made up of $x-2, x$ and $x+2$ students.

(i) Show that the number of ways that this can be done is given by $\frac{(3x)!}{x!(x-2)!(x+2)!}$. **2**

(ii) Suppose that the three groups have been chosen. **2**
In how many ways can the $3x$ students be arranged around a circular table if the students in each group are to be seated together.

(b) A body whose mass is 1kg is projected vertically from a point on level ground with velocity of 50 ms^{-1}.

The forces acting on the body are gravity and air resistance of $\frac{v}{5}$ Newtons where v is the velocity of the body. Use $g = 10$ ms^{-2}.

(i) Show that the equation of motion of the body is $\ddot{x} = -\frac{(50+v)}{5}$. **1**

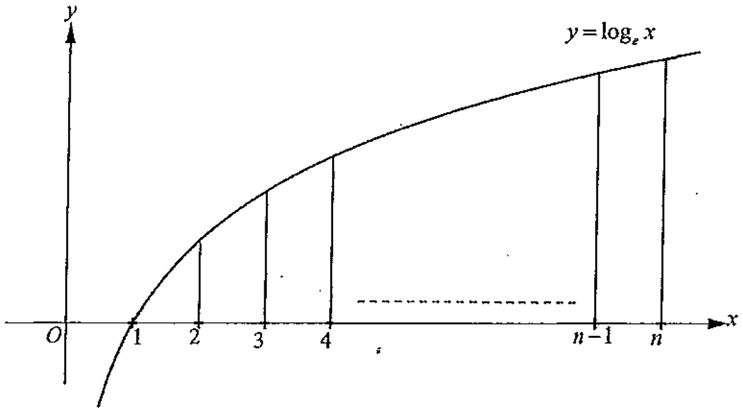
(ii) Find the maximum height reached by the body. **3**

(iii) Find the time taken for the body to reach its maximum height. **2**

Question 16 continues on page 15

Question 16 continued

(c) The diagram shows the graph of $y = \log_e x$ and $(n-1)$ strips of equal width from $x = 1$ to $x = n$.



(i) Show that $\frac{\log_e 1 + \log_e 2}{2} + \frac{\log_e 2 + \log_e 3}{2} + ... + \frac{\log_e (n-1) + \log_e n}{2} < \int_1^n \log_e x \, dx$ **2**

(ii) Hence deduce that $n! < \frac{en^{n+\frac{1}{2}}}{e^n}$. **3**

End of paper

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MATHEMATICS EXTENSION 2 - MARKING GUIDELINES

Section I

10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Solution	Mark
$z = -1 + i\sqrt{3}$ $\|z\| = 2$ $\arg(z) = \frac{2\pi}{3}$ $z = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ Hence (B)	1

6400-2

Question 2 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E2/3

Solution	Mark
Foci are between the directrices, so the conic is an ellipse. $ae = 3 \quad \to \quad a = \frac{3}{e}$ $\frac{a}{e} = 12$ $\therefore \frac{3}{e^2} = 12$ $e^2 = \frac{1}{4}$ $e = \frac{1}{2}$ $a = 6$ $b^2 = a^2(1-e^2)$ $= 36(1-\frac{1}{4})$ $= 27$ $\therefore$ Equation of conic is $\frac{x^2}{36} + \frac{y^2}{27} = 1$ Hence (B)	1

Question 3 (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E2/3

Solution	Mark
$\int \frac{x^2-1}{x^2+1} dx$ $= \int \frac{x^2+1-2}{x^2+1} dx$ $= \int (1 - \frac{2}{x^2+1}) dx$ $= x - 2\tan^{-1} x + c$ Hence (D)	1

Question 4 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E2/3

Solution	Mark
Put $2x - 3 = \alpha$ $x = \frac{\alpha + 3}{2}$ Hence (D)	1

Question 5 (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E3

Solution	Mark
$\int v\,du = uv - \int u\,dv$ $v = x \qquad du = \sin 2x$ $dv = 1 \qquad u = -\frac{1}{2}\cos 2x$ $\int x \sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$ $= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$ Hence (A)	1

Question 6 (1 mark)
Outcomes Assessed: E7
Targeted Performance Bands: E2/3

Solution	Mark
By inspection, (A)	1

Question 7 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2/3

Solution	Mark
$\arg(\frac{z}{w}) = \arg(z) - \arg(w)$ Hence, by inspection (C)	1

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Question 8 (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands: E3/4

Solution	Mark
The equation of motion is given by $m\ddot{x} = mg - mkv^2$. $\ddot{x} = g - kv^2$ $v\frac{dv}{dx} = g - kv^2$ $\frac{dv}{dx} = \frac{g - kv^2}{v}$ $\frac{dx}{dv} = \frac{v}{g - kv^2}$ $x = -\frac{1}{2k}\log_e(g - kv^2) + C$ When $x = 0, v = 0$, $C = \frac{1}{2k}\log_e g$ $\therefore x = \frac{1}{2k}\log_e g - \frac{1}{2k}\log_e(g - kv^2)$ $= \frac{1}{2k}\log_e(\frac{g}{g - kv^2})$ Hence (B)	1

Question 9(1 mark)
Outcomes Assessed: E2
Targeted Performance Bands: E3/4

Solution	Mark
Represent the pairs of socks by aa, bb,cc,dd,ee Select the first sock (say a) with probability 1 Probability that second sock is different $= \frac{8}{9}$ Probability that third sock is different $= \frac{6}{8}$ Probability that third sock is different $= \frac{4}{7}$ Hence (A)	1

Question 10 (1 mark)
Outcomes Assessed: E2
Targeted Performance Bands: E3/4

Solution	Mark
Let $A = \tan^{-1} x, B = \tan^{-1} 2x$ $\tan A = x$, $\tan B = 2x$ $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ $= \frac{x + 2x}{1 - x \times 2x}$ $= \frac{3x}{1 - 2x^2}$ $A + B = \tan^{-1}\left(\frac{3x}{1-2x^2}\right)$ $\therefore \tan^{-1} x + \tan^{-1} 2x = \tan^{-1}\left(\frac{3x}{1-2x^2}\right)$ $\tan^{-1} 3 = \tan^{-1}\left(\frac{3x}{1-2x^2}\right)$ $3 = \frac{3x}{1-2x^2}$ $1 - 2x^2 = x$ $2x^2 + x - 1 = 0$ $x = \frac{1}{2}, -1$ But $\tan^{-1}(-1)$ and $\tan^{-1}(-2)$ are negative and $\tan^{-1}(3)$ is positive so $x = -1$ cannot be a solution. Hence (A).	1

Section II
90 marks

Question 11 (15 marks)

11 (a) (i) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Equates correctly imaginary and real parts	1

Sample Answer:

$(a+ib)^2=16+30i$

$a^2+2abi-b^2=16+30i$

$a^2-b^2=16$

$ab=15 \rightarrow b=\dfrac{15}{a}$

$a^2-\dfrac{225}{a^2}=16$

$a^4-16a^2-225=0$

$(a^2-25)(a^2+9)=0$

$a=\pm 5$

$\therefore$ When $a=5,\ b=3$ and when $a=-5,\ b=-3$.

11 (a) (ii) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Substitutes correctly into quadratic equation	1

Sample Answer:

$z^2-(1+i)z-(4+7i)=0$

$z=\dfrac{(1+i)\pm\sqrt{[-(1+i)]^2+4(4+7i)}}{2}$

$=\dfrac{(1+i)\pm\sqrt{1+2i+i^2+16+28i}}{2}$

$=\dfrac{(1+i)\pm\sqrt{16+30i}}{2}$

$=\dfrac{(1+i)\pm(5+3i)}{2}$

$=\dfrac{6+4i}{2},\dfrac{-4-2i}{2}$

$\therefore$ Solutions are $3+2i$ and $-2-i$.

Question 11 (continued)
11 (b) (3 marks)
Outcomes Assessed: E8
Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	3
* Finds the correct integral	2
* Changes limits correctly from x to u	1

Sample Answer:

$$I=\int_0^1 \frac{2e^x}{e^{2x}+2e^x+1}dx$$
$$=\int_0^1 \frac{2e^x}{(e^x+1)^2}dx$$

Put $u=e^x+1$

$du=e^x\,dx$

When $x=1, u=e+1$

When $x=0, u=2$

$$\therefore I=2\int_2^{e+1}\frac{1}{u^2}du$$
$$=2\int_2^{e+1}u^{-2}\,du$$
$$=2\left[\frac{u^{-1}}{-1}\right]_2^{e+1}$$
$$=-2\left[\frac{1}{u}\right]_2^{e+1}$$
$$=-2\left(\frac{1}{e+1}-\frac{1}{2}\right)$$
$$=-2\left(\frac{2-(e+1)}{2(e+1)}\right)$$
$$=-2\left(\frac{1-e}{2(e+1)}\right)$$
$$=\frac{e-1}{e+1}$$

Question 11 (continued)
11 (c) (4 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2/3

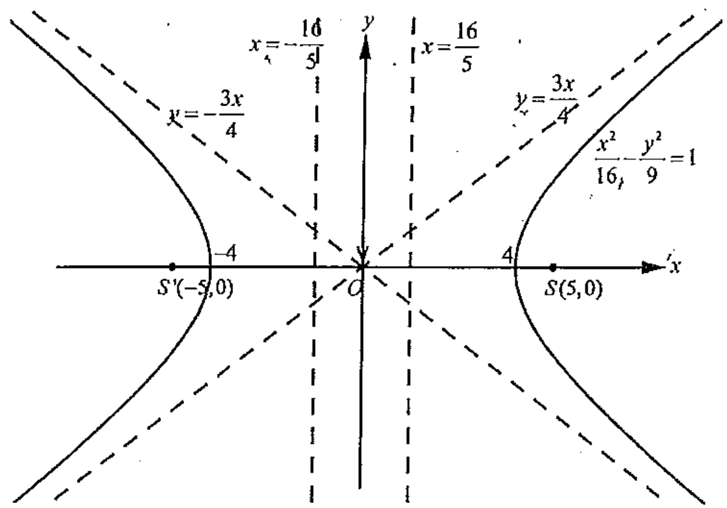
Criteria	Marks
* Correct solution	4
* Demonstrates significant progress towards answer	3
* Finds correct value of e along with either directrices or foci	2
* Finds correct value of e	1

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$
$$b^2=a^2(e^2-1)$$
$$e^2-1=\frac{9}{16}$$
$$e^2=\frac{25}{16}$$
$$e=\frac{5}{4}$$

Foci: $(\pm 5, 0)$

Directrices: $x=\pm\frac{16}{5}$

Asymptotes: $y=\pm\frac{3x}{4}$



Question 11 (continued)

11 (d)(i) (2 marks)

Outcomes Assessed: E4

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Correctly states either co-efficients of $P(x)$ are real or finds only the conjugate factor	1

Sample Answer:

Since the coefficients of $P(x)$ are real, then $x=\dfrac{1+\sqrt{3}i}{2}$ is also a root.

$$\text{Sum of roots} = \frac{1-\sqrt{3}i}{2}+\frac{1+\sqrt{3}i}{2} = 1$$

$$\text{Product of roots} = \frac{1-\sqrt{3}i}{2}\times\frac{1+\sqrt{3}i}{2} = 1$$

$\therefore$ Quadratic factor is x^2-x+1

11 (d)(ii) (2 marks)

Outcomes Assessed: E4

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Finds correctly either a or b	1

Sample Answer:

$$P(x)=(x-5)(x^2-x+1)(x-\alpha)$$
$$=x^4-(6+\alpha)x^3+6(1+\alpha)x^2-(5+6\alpha)x+5\alpha$$

Equate coefficients of x^2: $6(1+\alpha)=36$

$\alpha=5$

Equate coefficients of x^3: $-(6+\alpha)=a$

$a=-11$

Equate constants: $b=5\alpha$

$b=25$

Question 12

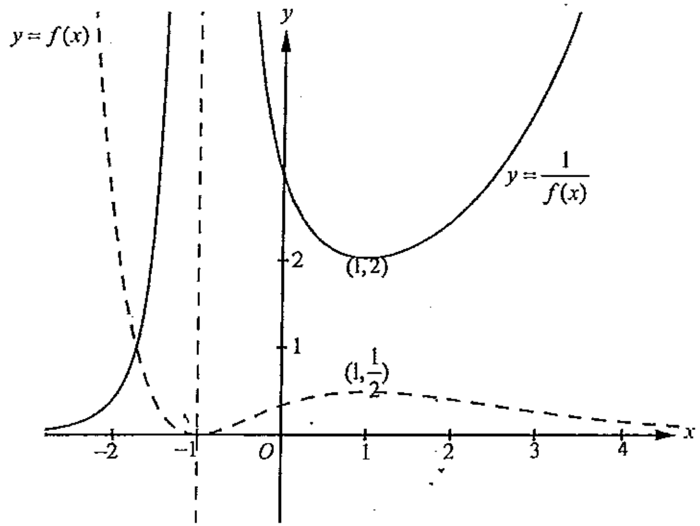
12 (a)(i) (2 marks)

Outcomes Assessed: E6

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct graph	2
* Demonstrates some correct working	1

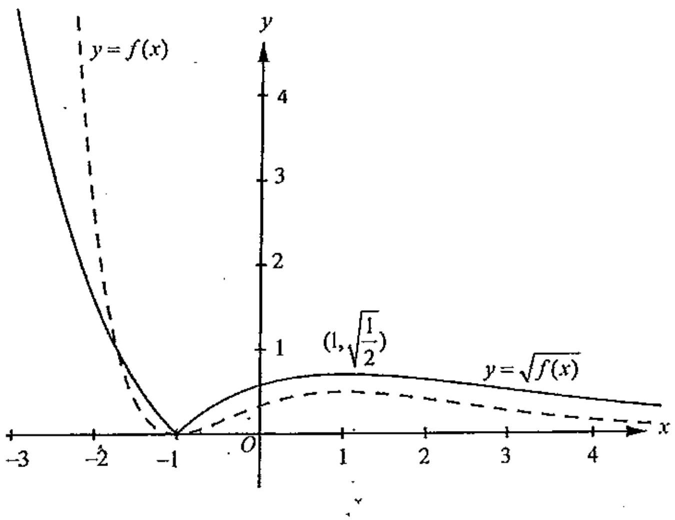
Sample Answer:



Question 12 (continued)
12 (a) (ii) (2 marks)
Outcomes Assessed: E6
Targeted Performance Bands: E2/3

Criteria	Marks
* Correct graph	2
* Demonstrates some correct working	1

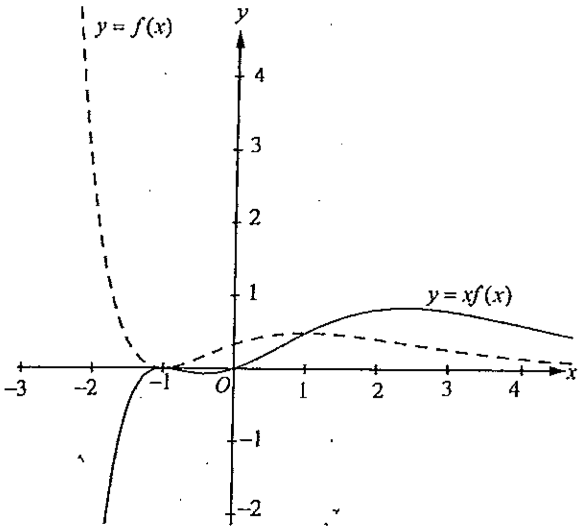
Sample Answer:



Question 12 (continued)
12 (a) (iii) (2 marks)
Outcomes Assessed: E6
Targeted Performance Bands: E2/3

Criteria	Marks
* Correct graph	2
* Demonstrates some correct working	1

Sample Answer:



Question 12 (continued)

12 (b) (i) (3 marks)

Outcomes Assessed: E3

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution and graph	3
* Finds all five fifth roots of -1	2
* Demonstrates some correct working	1

Sample Answer:

Roots of $z^5+1=0$ are given by $z_n=\cos(\frac{2k\pi+\theta}{5})+i\sin(\frac{2k\pi+\theta}{5})$ where $k=0,\pm1,\pm2,\theta=\pi$.

Hence roots are

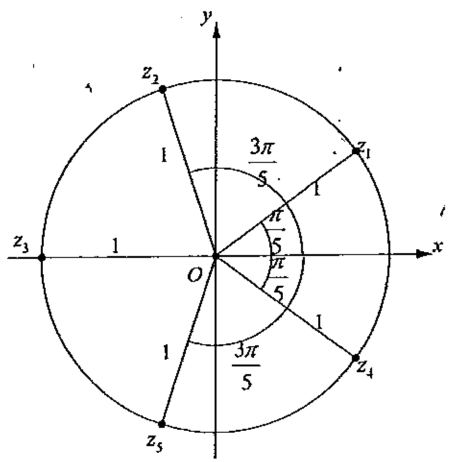
when $k=0,\ z_1=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$

when $k=1,\ z_2=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$

when $k=2,\ z_3=\cos\pi+i\sin\pi=-1$

when $k=-1,\ z_4=\cos(-\frac{\pi}{5})+i\sin(-\frac{\pi}{5})$

when $k=-2,\ z_5=\cos(-\frac{3\pi}{5})+i\sin(-\frac{3\pi}{5})$



Question 12 (continued)

12 (b) (ii) (2 marks)

Outcomes Assessed: E3

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

$z^5+1=(z-z_3)(z-z_1)(z-z_4)(z-z_2)(z-z_5)$

$=(z+1)(z^2-2\cos\frac{\pi}{5}z+1)(z^2-2\cos\frac{3\pi}{5}z+1)$

12(c) (i) (2 marks)

Outcomes Assessed: E6

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Shows $x^2+y^2\geq 2xy$	1

Sample Answer:

$(x-y)^2\geq 0$

$x^2+y^2\geq 2xy$

Similarly

$x^2+z^2\geq 2xz$

$y^2+z^2\geq 2yz$

Adding these results,

$2x^2+2y^2+2z^2\geq 2xy+2xz+2yz$

Hence $x^2+y^2+z^2\geq xy+xz+yz$

Question 12 (continued)
12(c) (ii) (2 marks)
Outcomes Assessed: E6
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

From (i), $x^2+y^2+z^2-xy-xz-yz\ge 0$

Multiply by $x+y+z$

$x^3+xy^2+xz^2-x^2y-x^2z-xyz+x^2y+y^3+yz^2-xy^2-xyz-y^2z+x^2z+y^2z+z^3-xyz-xz^2-yz^2\ge 0$

$x^3+y^3+z^3-3xyz\ge 0$

$\therefore x^3+y^3+z^3\ge 3xyz$

Question 13 (15 marks)
13 (a) (3 marks)
Outcomes Assessed: E7
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Finds integral of solid correctly	2
* Finds correctly height of strip	1

Sample Answer:

Consider a typical strip of the region

Height of strip $=2\sqrt{2}-(x+6x^3)$

$=2\sqrt{2}-x-6x^3$

Let thickness of strip be δx

Volume of shell $\approx 2\pi x(2\sqrt{2}-x-6x^3)\delta x$

Volume of solid $=2\pi\int_0^{\frac{\sqrt{2}}{2}} x(2\sqrt{2}-x-6x^3)dx$

$=2\pi\int_0^{\frac{\sqrt{2}}{2}}(2\sqrt{2}x-x^2-6x^4)dx$

$=2\pi\left[\frac{2\sqrt{2}x^2}{2}-\frac{x^3}{3}-\frac{6x^5}{5}\right]_0^{\frac{\sqrt{2}}{2}}$

$=2\pi\left[\sqrt{2}(\frac{\sqrt{2}}{2})^2-\frac{1}{3}(\frac{\sqrt{2}}{2})^3-\frac{6}{5}(\frac{\sqrt{2}}{2})^5\right]$

$=2\pi\left[\sqrt{2}(\frac{2}{4})-\frac{1}{3}(\frac{2\sqrt{2}}{8})-\frac{6}{5}(\frac{4\sqrt{2}}{32})\right]$

$=2\pi\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{12}-\frac{3\sqrt{2}}{20}\right]$

$=2.3695$

≈ 2.37 to 3 significant figures.

Question 13 (continued)

13 (b) (i) (1 mark)

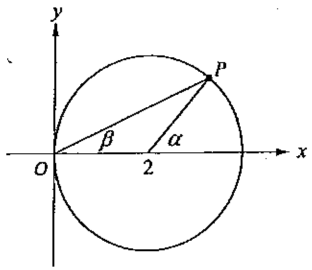
Outcomes Assessed: E3

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	1

Sample Answer:

The point representing z lies on a circle with centre at $(2,0)$ and radius 2.



13 (b) (ii) (2 marks)

Outcomes Assessed: E3

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Shows correctly $\arg(z-2)=k(\arg(z)+\arg(z-2))$	1

From (i), $\arg(z-2)=\alpha$ and $\arg(z)=\beta$.

$\therefore \arg(z-2)=2\arg(z)$ (angle at centre equals twice angle at circumference)

Given $\arg(z-2)=k(\arg(z^2-2z))$

$\arg(z-2)=k(\arg(z(z-2)))$

$\arg(z-2)=k(\arg(z)+\arg(z-2))$

$\arg(z-2)=k\arg(z)+k\arg(z-2)$

$\arg(z-2)-k\arg(z-2)=k\arg(z)$

$(1-k)\arg(z-2)=k\arg(z)$

$$\arg(z-2)=\frac{k}{1-k}\arg(z)$$

From line 2, $\frac{k}{1-k}=2$

$k=2-2k$

$\therefore k=\frac{2}{3}$

Question 13 (continued)

13 (c) (i) (3 marks)

Outcomes Assessed: E4

Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards answer	2
* Obtains $(\sqrt{x-1})^3-7(\sqrt{x-1})^2+18(\sqrt{x-1})-7=0$	1

Sample Answer:

Let $x=1+\alpha^2$, $\alpha=\sqrt{x-1}$

Since α is a root of the polynomial replace x by $\sqrt{x-1}$.

$(\sqrt{x-1})^3-7(\sqrt{x-1})^2+18(\sqrt{x-1})-7=0$

$\sqrt{x-1}((\sqrt{x-1})^2+18)=7(x-1)+7$

$\sqrt{x-1}(x+17)=7x$

Square both sides:

$(\sqrt{x-1})^2(x+17)^2=(7x)^2$

$(x-1)(x^2+34x+289)=49x^2$

$x^3+34x^2+289x-x^2-34x-289-49x^2=0$

$\therefore$ Required equation is $x^3-16x^2+235x-289=0$

13 (c) (ii) (1 mark)

Outcomes Assessed: E4

Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	1

Sample Answer:

$(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$ represents the product of the roots of $x^3-16x^2+235x-289=0$.

$$\therefore (1+\alpha^2)(1+\beta^2)(1+\gamma^2)=-\frac{d}{a}$$

$$=289$$

Question 13 (continued)

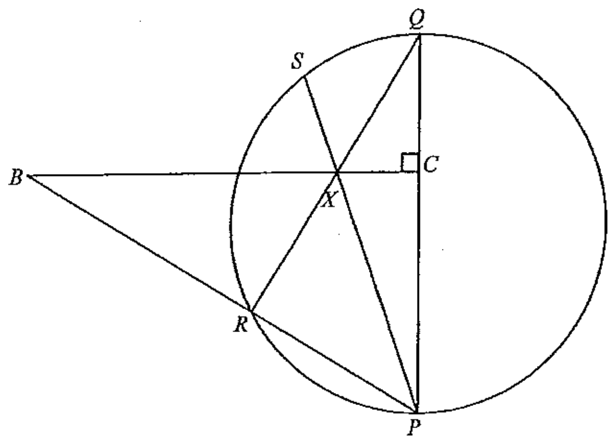
13 (d)(i) (1 mark)

Outcomes Assessed: E9

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	1

Sample Answer:



In triangles BCP and PRQ

$\angle PRQ = 90°$ (angle in a semicircle is a rightangle)

$\angle$BPC is a common angle

$\therefore \Delta BCP ||| \Delta PRQ$ (equiangular)

$\therefore \angle$PBC=$\angle$PQR (matching angles in similar triangles)

13 (d)(ii) (2 marks)

Outcomes Assessed: E9

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Demonstrates some correct work	1

Sample Answer:

Join SR.

$\angle RSP = \angle RQP$ (angles at the circumference standing on same arc RP)

$\angle RBC = \angle RSP$ (both equal to $\angle RQP$)

$\therefore SBRX$ is a cyclic quadrilateral (angles at circumference are equal)

Question 13 (continued)

13 (d)(iii) (2 marks)

Outcomes Assessed: E9

Targeted Performance Bands: E2/3

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

Join BS and SQ

$\angle PSQ = 90°$ (angle in a semicircle is a rightangle)

$\angle PRX = \angle BSX$ (exterior angle of cyclic quadrilateral equals interior opposite angle)

Since $\angle PRX = 90°$ then $\angle BSX = 90°$

$\angle BSQ = \angle BSX + \angle XSQ$

$= 90° + 90°$

$= 180°$

As $\angle BSQ$ is a straight angle then B, S and Q are collinear.

Question 14 (15 marks)
14 (a) (i) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Applies quadratic formula	1

Sample Answer:

$x^2 = (2\cos\theta)x - 1$

$x^2 - (2\cos\theta)x + 1 = 0$

$x = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \dfrac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2}$

$= \dfrac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$

$= \dfrac{2\cos\theta \pm 2i\sin\theta}{2}$

$\therefore \alpha = \cos\theta + i\sin\theta, \beta = \cos\theta - i\sin\theta$

14 (a) (ii) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Applies De Moirve's theorem to both roots	1

Sample Answer:

$\alpha^6 = (\cos\theta + i\sin\theta)^6$

$= \cos 6\theta + i\sin 6\theta$

$\beta^6 = (\cos\theta - i\sin\theta)^6$

$= \cos 6\theta - i\sin 6\theta$

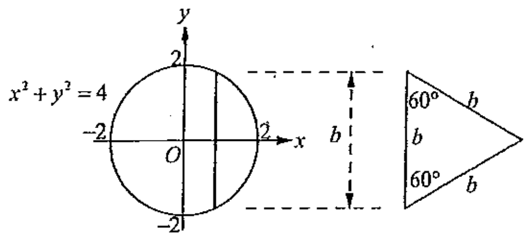
$\therefore \alpha^6 + \beta^6 = \cos 6\theta + i\sin 6\theta + \cos 6\theta - i\sin 6\theta$

$= 2\cos 6\theta$, as required.

Question 14 (continued)
14 (b) (3 marks)
Outcomes Assessed: E7
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Finds correct expression for volume	2
* Finds area of cross section	1

Sample Answer:



$x^2 + y^2 = 4$

$y = \pm\sqrt{4 - x^2}$

$\therefore$ Base of cross-section (b) $= 2\sqrt{4 - x^2}$

Area of cross-section $= \dfrac{1}{2} b^2 \sin 60°$

$= \dfrac{1}{2}(2\sqrt{4 - x^2})^2 \dfrac{\sqrt{3}}{2}$

$= \sqrt{3}(4 - x^2)$

$\therefore$ Volume $= \sqrt{3}\int_{-2}^{2}(4 - x^2)\,dx$

$= 2\sqrt{3}\int_{0}^{2}(4 - x^2)\,dx$

$= 2\sqrt{3}\left[4x - \dfrac{x^3}{3}\right]_0^2$

$= 2\sqrt{3}(8 - \dfrac{8}{3})$

$= 2\sqrt{3}(\dfrac{16}{3})$

$= \dfrac{32\sqrt{3}}{3}$

Question 14 (continued)

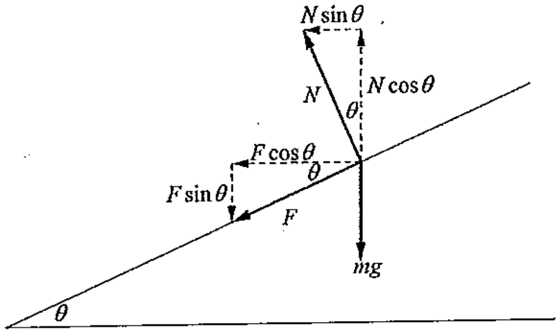
14 (c) (i) (3 marks)

Outcomes Assessed: E5

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Finds either sum of forces horizontally or vertically	1

Sample Answer:



Horizontally: Sum of forces $= \dfrac{mv^2}{R}$

$N\sin\theta + F\cos\theta = \dfrac{mv^2}{R}$ ....(1)

Vertically: Sum of forces = 0

$N\cos\theta - F\sin\theta - mg = 0$

$N\cos\theta - F\sin\theta = mg$ ....(2)

Eliminate N to find F

(1) $\times\cos\theta$ $\quad N\sin\theta\cos\theta + F\cos^2\theta = \dfrac{mv^2}{R}\cos\theta$

(2) $\times\sin\theta$ $\quad N\cos\theta\sin\theta - F\sin^2\theta = mg\sin\theta$

$F(\cos^2\theta + \sin^2\theta) = \dfrac{mv^2}{R}\cos\theta - mg\sin\theta$

$F = m(\dfrac{v^2}{R}\cos\theta - g\sin\theta)$, as required.

Question 14 (continued)

14 (c) (ii) (1 mark)

Outcomes Assessed: E5

Targeted Performance Bands: E3

Criteria	Marks
* Correct solution in either m/s or km/h	1

Sample Answer:

Using $F = m(\dfrac{v^2}{R}\cos\theta - g\sin\theta)$ with $F = 0,\ R = 130,\ \theta = 9°,\ g = 10$.

$0 = m(\dfrac{v^2}{130}\cos 9° - 10\sin 9°)$

$\dfrac{v^2}{130}\cos 9° = 10\sin 9°$

$v = \sqrt{\dfrac{1300\sin 9°}{\cos 9°}}$

$= 14.35$ m/s

$= 52$ km/h

Question 14 (continued)
14 (d) (i) (3 marks)
Outcomes Assessed: E8
Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Applies integration by parts correctly once	1

Sample Answer:

$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$, $n \geq 0$

$= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x\,dx$

$= (\frac{\pi}{2})^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x\,dx$

$\int v\,du = uv - \int u\,dv$

$v = x^n \qquad du = \cos x$

$dv = nx^{n-1} \qquad u = \sin x$

....

$v = x^{n-1} \qquad du = \sin x$

$dv = (n-1)x^{n-2} \qquad u = -\cos x$

Using integration by parts again

$I_n = (\frac{\pi}{2})^n - n\left(\left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x\,dx\right)$

$= (\frac{\pi}{2})^n - n\left((n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x\,dx\right)$

$= (\frac{\pi}{2})^n - n(n-1)I_{n-2}$, $n \geq 2$ as required.

Question 14 (continued)
14 (d) (ii) (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	1

Sample Answer:

$\int_0^{\frac{\pi}{2}} x^n \cos x\,dx = I_4$

$I_4 = \left(\frac{\pi}{2}\right)^4 - 12I_2$

$I_2 = \left(\frac{\pi}{2}\right)^2 - 2I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \cos x\,dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= 1$

$\therefore I_2 = \left(\frac{\pi}{2}\right)^2 - 2$

$I_4 = \left(\frac{\pi}{2}\right)^4 - 12(\left(\frac{\pi}{2}\right)^2 - 2)$

$= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$

Question 15 (15 marks)

15 (a) (3 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Shows it is true for $n = 4$	1

Sample Answer:

Let $P(n)$ be the proposition that $n! > 2^n$ for $n \geq 4$.

When $n = 4$, $n! = 24$ and $2^n = 16$.

$\therefore P(n)$ is true for $n = 4$.

Assume that $P(n)$ is true for $n = k$.

ie assume that $k! > 2^k$

Need to show that $P(n)$ is true for $n = k+1$.

ie Need to show that $(k+1)! > 2^{k+1}$

From assumption, $k! > 2^k$.

Multiplying both sides by $(k+1)$ does not change the inequality

$(k+1) \times k! > (k+1) \times 2^k$

$(k+1)! > (2+k-1) \times 2^k$

$(k+1)! > 2 \times 2^k + (k-1)2^k$

$(k+1)! > 2^{k+1} + (k-1)2^k$

Hence $(k+1)! > 2^{k+1}$ as required.

$\therefore$ If $P(n)$ is true for $n = k$ with $k \geq 4$, then $P(n)$ is true for $n = k+1$.

Hence by mathematical induction $P(n)$ is true for all integers $n \geq 4$.

Question 15 (continued)
15 (b) (3 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Obtains $4\cos x\cos\frac{x}{2}\sin\frac{5x}{2} = 0$	1

Sample Answer:

$\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

Re-arranging and using the given result

$$\begin{aligned} \text{LHS} &= \sin 3x + \sin x + \sin 4x + \sin 2x \\ &= 2\sin 2x\cos x + 2\sin 3x\cos x \\ &= 2\cos x(\sin 3x + \sin 2x) \\ &= 2\cos x \times 2\sin\frac{5x}{2}\cos\frac{x}{2} \\ &= 4\cos x\cos\frac{x}{2}\sin\frac{5x}{2} \end{aligned}$$

$\therefore$ Equation becomes $4\cos x\cos\frac{x}{2}\sin\frac{5x}{2} = 0$

Hence $\cos x = 0, \cos\frac{x}{2} = 0, \sin\frac{5x}{2} = 0$ for $0 \leq x \leq 2\pi$.

When $\cos x = 0$ and $0 \leq x \leq 2\pi$, $x = \frac{\pi}{2}, \frac{3\pi}{2}$

When $\cos\frac{x}{2} = 0$ and $0 \leq x \leq \pi$, $\frac{x}{2} = \frac{\pi}{2} \rightarrow x = \pi$

When $\sin\frac{5x}{2} = 0$ and $0 \leq \frac{5x}{2} \leq 5\pi$, $\frac{5x}{2} = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi \rightarrow x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$

$\therefore$ Solutions are $0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{3\pi}{2}, \frac{8\pi}{5}, 2\pi$

Question 15 (continued)

15 (c) (i) (2 marks)

Outcomes Assessed: E6

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Finds correctly $\dfrac{dy}{dx}$	1

Sample Answer:

$$x^{\frac{1}{2}}+(by)^{\frac{1}{2}}=b^{\frac{1}{2}}$$

$$\frac{1}{2}x^{-\frac{1}{2}}+b^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}.\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\left(\frac{y}{bx}\right)^{\frac{1}{2}}$$

As $\dfrac{dy}{dx}<0$, $x^{\frac{1}{2}}+(by)^{\frac{1}{2}}=b^{\frac{1}{2}}$ is a monotonic decreasing function

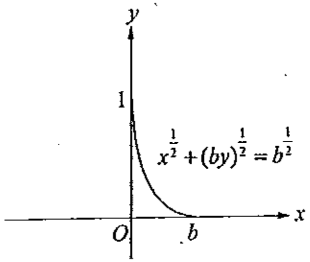
15 (c) (ii) (1 mark)

Outcomes Assessed: E6

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	1

Sample Answer:



Question 15 (continued)

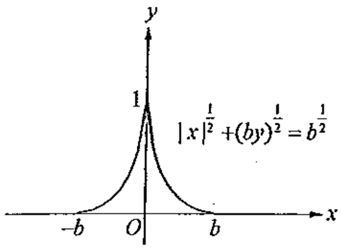
15 (c) (iii) (1 mark)

Outcomes Assessed: E6

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	1

Sample Answer:



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15 (d) (i) (2 marks)

Outcomes Assessed: E3

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Finds correctly $\dfrac{dy}{dx}$	1

Sample Answer:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{a^2}\frac{b^2}{2y}$$

$$=-\frac{b^2x}{a^2y}$$

$$\text{At } P,\ \frac{dy}{dx}=-\frac{b^2a\cos\theta}{a^2b\sin\theta}$$

$$=-\frac{b\cos\theta}{a\sin\theta}$$

Equation of tangent at P is

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$bx\cos\theta+ay\sin\theta-ab(\sin^2\theta+\cos^2\theta)=0$$

$\therefore bx\cos\theta+ay\sin\theta-ab=0$, as required.

Question 15 (continued)

15 (d) (ii) (1 mark)

Outcomes Assessed: E3

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	1

Sample Answer:

$$b^2\cos^2\theta+a^2\sin^2\theta=b^2\cos^2\theta+a^2(1-\cos^2\theta)$$

$$=b^2\cos^2\theta+a^2-a^2\cos^2\theta$$

$$=a^2+(b^2-a^2)\cos^2\theta$$

From $b^2=a^2(1-e^2)$, $b^2-a^2=-a^2e^2$

$$\therefore b^2\cos^2\theta+a^2\sin^2\theta=a^2-a^2e^2\cos^2\theta$$

$$=a^2(1-e^2\cos^2\theta)$$

15 (d) (iii) (2 marks)

Outcomes Assessed: E3

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Finds perpendicular distance ST	1

Sample Answer:

$S=(ae,0)$, $S'=(-ae,0)$

Using formula for distance from a point to a line,

$$ST\times S'T'=\frac{|abe\cos\theta-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\times\frac{|-abe\cos\theta-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Using the result from (ii),

$$ST\times S'T'=\frac{|abe\cos\theta-ab|\times}{\sqrt{a^2(1-e^2\cos^2\theta)}}\times\frac{|-abe\cos\theta-ab|}{\sqrt{a^2(1-e^2\cos^2\theta)}}$$

$$=\frac{|-ab(1-e\cos\theta)|\times|-ab(1+e\cos\theta)|}{a^2(1-e^2\cos^2\theta)}$$

$$=\frac{|[-ab(1-e\cos\theta)]\times[-ab(1+e\cos\theta])|}{a^2(1-e^2\cos^2\theta)}$$

$$=\frac{a^2b^2(1-e\cos\theta)\times(1+e\cos\theta)}{a^2(1-e^2\cos^2\theta)}$$

$$=\frac{a^2b^2(1-e^2\cos^2\theta)}{a^2(1-e^2\cos^2\theta)}$$

$=b^2$, as required.

Question 16 (15 marks)

16 (a)(i) (2 marks)

Outcomes Assessed: E4

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Finds number of ways of placing in a group $(x-2)$, x and $(x+2)$ ways	1

Sample Answer:

There are ${}^{3x}C_{x-2}$ ways of placing students in a group of $(x-2)$ students.

Then there are ${}^{2x+2}C_x$ ways of placing students in a group of x students.

The remaining $(x+2)$ students are placed only one way in a group of $(x+2)$ students.

Hence, total number of ways $= {}^{3x}C_{x-2} \times {}^{2x+2}C_x \times 1$

$$= \frac{(3x)!}{(x-2)!(2x+2)!} \times \frac{(2x+2)!}{x!(x+2)!}$$

$$= \frac{(3x)!}{x!(x-2)!(x+2)!}$$

16 (a)(ii) (2 marks)

Outcomes Assessed: E4

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

There are $(3-1)! = 2$ ways of arranging the three groups in a circle.

There are $(x-2)!$, $x!$ and $(x+2)!$ ways of arranging the students in each group.

Hence the total number of ways is $2x!(x+2)!(x-2)!$

Question 16 (continued)

16 (b)(i) (1 mark)

Outcomes Assessed: E5

Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	1

Sample Answer:

$m\ddot{x} = -mg - R$

In this case, $m = 1, g = 10, R = \frac{v}{5}$

$$\therefore \ddot{x} = -10 - \frac{v}{5}$$

$$= \frac{-50 - v}{5}$$

$$= -(\frac{50+v}{5})$$

16 (b)(ii) (3 marks)
Outcomes Assessed: E5
Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Finds correct expression for x	1

Sample Answer:

$v\frac{dv}{dx}=-(\frac{50+v}{5})$

$\frac{dv}{dx}=-(\frac{50+v}{5v})$

$\frac{dx}{dv}=-5\frac{v}{50+v}$

$=-5\frac{(50+v-50)}{50+v}$

$=-5(1-\frac{50}{50+v})$

$x=-5(v-50\log_e(50+v))+c$

When $x=0, v=50$.

$c=5(50-50\log_e 100)$

$x=5(50-50\log_e 100)-5(v-50\log_e(50+v))$

At maximum height, $v=0$.

$x=250-250\log_e 100+250\log_e 50$

$=250-250(\log_e 100-\log_e 50)$

$=250(1-\log_e 2)$

Question 16 (continued)
16 (b)(iii) (2 marks)
Outcomes Assessed: E5
Targeted Performance Bands: E3/4

Criteria	Marks
* Correct solution	2
* Obtains correct expression for time	1

Sample Answer:

$\frac{dv}{dt}=-\frac{(50+v)}{5}$

$\frac{dt}{dv}=-\frac{5}{50+v}$

$t=-5\log_e(50+v)+c$

When $t=0, v=50$

$c=5\log_e 100$

$\therefore t=5\log_e 100-5\log_e(50+v)$

At maximum height, $v=0$.

$\therefore t=5\log_e 100-5\log_e 50$

$=5(\log_e 100-\log_e 50)$

$=5\log_e 2$.

16 (c) (i) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

Since $\frac{d^2y}{dx^2}=-\frac{1}{x^2}$ and is always negative, the curve is concave down for all x.

Hence, the sum of the areas of the trapeziums is less than the area under the curve $y=\log_e x$ from $x=1$ to $x=n$.

Area of first trapezium is $\frac{1}{2}(\log_e 1+\log_e 2)$ from $x=1$ to $x=2$

Area of second trapezium is $\frac{1}{2}(\log_e 2+\log_e 3)$ from $x=2$ to $x=3$

and so on.

Hence $\frac{\log_e 1+\log_e 2}{2}+\frac{\log_e 2+\log_e 3}{2}+\ldots\ldots+\frac{\log_e(n-1)+\log_e n}{2}<\int_1^n \log_e x\,dx$

Question 16 (continued)
16 (c) (ii) (3 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Uses integration by parts correctly to find integral	1

Sample Answer:

From (i), $\frac{\log_e 1+\log_e 2}{2}+\frac{\log_e 2+\log_e 3}{2}+\ldots\ldots+\frac{\log_e(n-1)+\log_e n}{2}<\int_1^n \log_e x\,dx$

$$\text{LHS}=\log_e 2+\log_e 3+\ldots\ldots+\log_e(n-1)+\frac{1}{2}\log_e n$$

$$=\log_e 2+\log_e 3+\ldots\ldots+\log_e(n-1)+\log_e n-\frac{1}{2}\log_e n$$

$$=\log_e(2\times3\times4\times\ldots\ldots\times n)-\frac{1}{2}\log_e n$$

$$=\log_e(n!)-\frac{1}{2}\log_e n$$

$$\text{RHS}=\int_1^n \log_e x\,dx$$

$$=\left[x\log_e x\right]_1^n-\int_1^n x\times\frac{1}{x}\,dx \quad \text{using integration by parts}$$

$$=n\log_e n-\left[x\right]_1^n$$

$$=n\log_e n-n+1$$

Hence $\log_e(n!)-\frac{1}{2}\log_e n<n\log_e n-n+1$

$$\log_e(n!)<n\log_e n+\frac{1}{2}\log_e n-n+1$$

$$\log_e(n!)<(n+\frac{1}{2})\log_e n-n+1$$

$$\log_e(n!)<\log_e n^{n+\frac{1}{2}}-n+1$$

$$\log_e(n!)<\log_e n^{n+\frac{1}{2}}-n\log_e e+\log_e e$$

$$\log_e(n!)<\log_e n^{n+\frac{1}{2}}-\log_e e^n+\log_e e$$

$$\log_e(n!)<\log_e\left[\frac{en^{n+\frac{1}{2}}}{e^n}\right]$$

Hence $n!<\frac{en^{n+\frac{1}{2}}}{e^n}$