A SELECTION OF HSC MATHEMATICS QUESTIONS AND SOLUTIONS – 2001

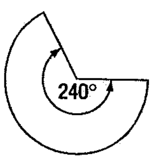
QUESTION 1:
Find the equation of the tangent to the curve $y = 2x^2 - 5/x + 3$ at the point where $x = 1$.

SOLUTION:
When $x = 1, y = 2 - 5 + 3$
$= 0.$
Now $y = 2x^2 - 5x^{-1} + 3$
$\therefore \, {}^{dy}/_{dx} = 4x + 5x^{-2}$
$= 4x + 5/x^2$
At (1, 0), ${}^{dy}/_{dx} = 4 + 5$
$= 9.$
Thus the gradient of the tangent at (1, 0) is 9.
The equation of the tangent is now obtained using the formula $y - y_1 = m(x - x_1)$:
$y - 0 = 9(x - 1)$
$\therefore y = 9x - 9.$

QUESTION TWO:
(a) Differentiate the following:
 (i) $x^3 ln x$
 (ii) $\cos(x^3)$.
(b) Find the exact value of:
 (i) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 3x dx$
 (ii) $\displaystyle\int_0^1 (e^{-2x} + 3) dx.$

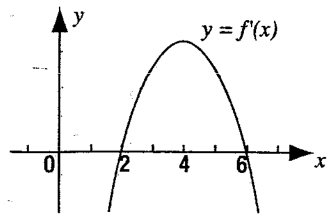
SOLUTION:
(a) (i) $\frac{d}{dx} x^3 ln x = ln x \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(ln x)$
$= 3x^2 ln x + x^3 . \frac{1}{x}$
$= 3x^2 ln x + x^2$
$= x^2(3 ln x + 1).$
(ii) $\frac{d}{dx} \cos(x^3) = -\sin(x^3) . 3x^2$
$= -3x^2 \sin(x^3).$
(b) (i) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 3x dx = \left[-\tfrac{1}{3}\cos 3x\right]_0^{\frac{\pi}{4}}$
$= -\frac{1}{3}\cos\frac{3\pi}{4} + \frac{1}{3}\cos 0$
$= -\frac{1}{3} \times -\frac{\sqrt{2}}{2} + \frac{1}{3}$
$= \frac{\sqrt{2}}{6} + \frac{1}{3}$
(ii) $\displaystyle\int_0^1 (e^{-2x} + 3) dx = \left[-\tfrac{1}{2} e^{-2x} + 3x\right]_0^1$

QUESTION THREE:
If the area of the sector of the circle shown is 20.00 cm², determine the radius of the sector.



SOLUTION:
The formula for the area A of a sector is given by $A = \frac{1}{2} r^2 \theta$.
In this question,
$A = 20$ and $\theta = 240° = \frac{4\pi}{3}$ radians.
Substituting these values into the formula above, we get:
$20 = \frac{1}{2} r^2 \times \frac{4\pi}{3}$
so that $r^2 = \frac{30}{\pi}$
$\therefore r = \sqrt{\frac{30}{\pi}}, r \neq -\sqrt{\frac{30}{\pi}}$
$\approx 9.55.$
∴ The radius of the sector is 9.55cm (to two decimal places).

QUESTION FOUR:



The diagram shows the graph of the gradient function of the curve $y = f(x)$. For what value of x does $f(x)$ have a local maximum? Justify your answer.

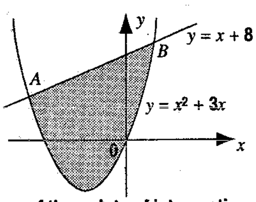
SOLUTION:
Stationary points occur when $f'(x) = 0$, that is, at $x = 2$ and $x = 6$.
To find the maximum, use the first derivative test or the second derivative test.

First derivative test:

	$x < 2$	$x = 2$	$2<x<6$	$x = 6$	$x > 6$
$f'(x)$	< 0	0	> 0	0	< 0
$f(x)$	\	—	/	—	\

It can thus be seen that there is a local maximum at $x = 6$.
Second derivative test:
At $x = 2, f'(x)$ is increasing so that the curve is concave up, i.e. $f''(x) > 0$. This is a local minimum.
At $x = 6, f'(x)$ is decreasing so that the curve is concave down, i.e. $f''(x) < 0$. This is a local maximum.

QUESTION FIVE:
The diagram shows the graphs of $y = x^2 + 3x$ and $y = x + 8$.



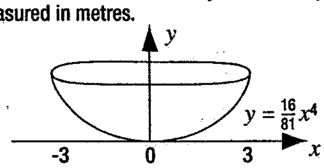
(i) Find the x values of the points of intersection, A and B.
(ii) Calculate the area of the shaded region.

SOLUTION:
(i) We find the x values of the points of intersection, A and B, by solving simultaneously, for x, $y = x^2 + 3x$ and $y = x + 8$.
$y = x^2 + 3x$ ________ (1)
$y = x + 8$ ________ (2)
Substitute for $y = x + 8$ from (2) into (1):
(or substitute for $y = x^2 + 3x$ from (1) into (2)):
$x + 8 = x^2 + 3x$
Rearrange, and collect like terms:
$x^2 + 2x - 8 = 0$
$(x + 4)(x - 2) = 0$
$\therefore x = -4, 2.$
(ii) The area of the shaded region can be found by evaluating:
$\left|\displaystyle\int_{-4}^{2} (x + 8) - (x^2 + 3x) dx\right|$
$\left|\displaystyle\int_{-4}^{2} (x + 8) - (x^2 + 3x) dx\right| = \left|\displaystyle\int_{-4}^{2} (x + 8 - x^2 - 3x) dx\right|$
$= \left|\displaystyle\int_{-4}^{2} (8 - 2x - x^2) dx\right|$

(ii) The area of the shaded region can be found by evaluating:
$\left|\displaystyle\int_{-4}^{2} (x + 8) - (x^2 + 3x) dx\right|$
$\left|\displaystyle\int_{-4}^{2} (x + 8) - (x^2 + 3x) dx\right| = \left|\displaystyle\int_{-4}^{2} (x + 8 - x^2 - 3x) dx\right|$
$= \left|\displaystyle\int_{-4}^{2} (8 - 2x - x^2) dx\right|$

$= \left|\left[8x - x^2 - \frac{1}{3}x^3\right]_{-4}^{2}\right|$
$= \left|(16 - 4 - \frac{8}{3}) - (-32 - 16 + \frac{64}{3})\right|$
$= \left|12 - \frac{8}{3} + 48 - \frac{64}{3}\right|$
$= |60 - 24|$
$= 36$
Thus the shaded area is 36 units².

QUESTION SIX:
An oil storage tank has the shape obtained by revolving the curve $y = \frac{16}{81} x^4$ from $x = 0$ to $x = 3$ about the y-axis, x and y being measured in metres.



Find the volume of the tank.

SOLUTION:
The volume can be found by evaluating $\pi\displaystyle\int_a^b x^2 dy$,
so we must find a, b and an expression for x^2.
When $x = 0, y = 0 \therefore a = 0$
When $x = 3, y = \frac{16}{81} \times 3^4 = 16 \therefore b = 16$
$y = \frac{16}{81} x^4 \Rightarrow x^4 = \frac{81}{16} y$
$\therefore x^2 = \frac{9}{4}\sqrt{y}$
(We ignore the solution $x^2 = -\frac{9}{4}\sqrt{y}$ as $x^2 \geq 0$ always).
$\therefore \pi\displaystyle\int_a^b x^2 dy = \pi\int_0^{16} \frac{9}{4}\sqrt{y}\, dy$
$= \frac{9\pi}{4}\displaystyle\int_0^{16} y^{1/2} dy$
$= \frac{9\pi}{4}\left[\frac{2}{3} y^{3/2}\right]_0^{16}$
$= \frac{3\pi}{2}(16^{3/2} - 0)$
$= \frac{3\pi}{2} \times 64$
$= 96\pi$
∴ The volume of the tank is 96π m³

QUESTION SEVEN:

Brittany signs a contract for a job with a salary in the first year of \$60,000 and an annual increase of 10% of the previous year's salary.

(a) How much will her salary be at the beginning of:
  (i) the second year?
  (ii) the third year?
  (iii) the twelfth year?

(b) Assuming the salary is paid monthly, find the length of time, in years and months, that Brittany would need to work in order for the total salary to exceed two million dollars.

SOLUTION:

(a) (i) At the beginning of the second year, Brittany's salary will be \$60,000 + 10% of \$60,000 = 110% of \$60,000, i.e. \$66,000.

(ii) At the beginning of the third year, Brittany's salary will be 110% of \$66,000, i.e. \$72,600.

(iii) The amounts of annual salary form a geometric progression where the first term a is 60,000, the common ratio r is 110% or 1.1, and the nth term $T_n = ar^{n-1}$.
At the beginning of the twelfth year, we will have $n = 12$, so we need to find T_{12}.
$T_{12} = 60{,}000 \times (1.1)^{11}$
$\doteq 171187.0024$
So, at the beginning of the twelfth year, her salary will be \$171,187.00.

(b) What is being asked, in mathematical terms, is "What would n have to be for the sum of the geometric progression to be at least 2,000,000?"
The formula for the sum to n terms of a geometric progression is given by:
$S_n = \frac{a(r^n - 1)}{r-1}, r \neq 1.$
I have chose this form because $r > 1$, with $r = 1.1$.
So, we have $S_n = 2{,}000{,}000$, $a = 60{,}000$ and $r = 1.1$ and we need to find n.
$\therefore \frac{60000((1.1)^n - 1)}{1.1-1} = 2000000$
$60000((1.1)^n - 1) = 2000000 \times 0.1$
$(1.1)^n - 1 = 3\frac{1}{3} + 1$
$\therefore (1.1)^n = 4\frac{1}{3}$
Now make n the subject by taking the logarithm, to the base e, of both sides, so that:
$\log_e(1.1)^n = \log_e 4\frac{1}{3}$
$n\log_e 1.1 = \log_e 4\frac{1}{3}$
$n = \frac{\log_e 4\frac{1}{3}}{\log_e 1.1}$
≈ 15.3849
So Brittany needs to stay in this job for 15 years and five months.
Note: You convert 0.3849 years to months by multiplying 0.3849 by 12.

QUESTION EIGHT:

(a) Show that $x = \frac{4\pi}{3}$ is a solution of $\tan x = -2\sin x$.

(b) On the same set of axes, sketch the graphs of the functions:
$y = \tan x$ and $y = -2\sin x$ for $-\pi \geq x \geq \pi$.

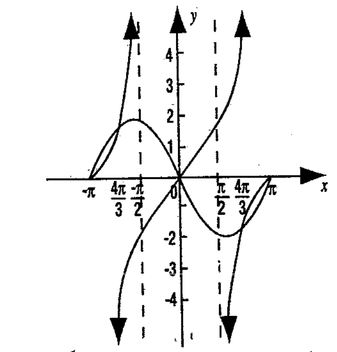
(c) Hence find the solutions of $\tan x = -2\sin x$ for $-\pi \leq x \leq \pi$.

(d) Use your graphs to solve $\tan x \geq -2\sin x$ for $-\pi \leq x \leq \pi$.

SOLUTION:

(a) LHS $= \tan\frac{4\pi}{3}$
$= \sqrt{3}$
RHS $= -2\sin\frac{4\pi}{3}$
$= -2 \times -\frac{\sqrt{3}}{2}$
$= \sqrt{3}$
= LHS
i.e. $\tan x = 2\sin x$ when $x = \frac{4\pi}{3}$

(b)



(c) The solutions are $x = -\pi, -\frac{4\pi}{3}, 0, \frac{4\pi}{3}, \pi$

(d) To solve $\tan x \geq -2\sin x$ graphically, we need to consider between what values of x the graph of $y = \tan x$ is ON or ABOVE the graphs of $y = -2\sin x$. This happens for $-\frac{4\pi}{3} \leq x < -\frac{\pi}{2}$ or $0 \leq x < \frac{\pi}{2}$ or $\frac{4\pi}{3} \leq x \leq \pi$.

QUESTION NINE:

The concentration $C(t)$ in milligrams per cubic centimetre of a particular drug in a patient's bloodstream is given by:
$C(t) = \frac{0.09t}{t^2 + 6t + 9}$
where t hours is the number of hours after the drug is taken.

(a) How many hours after the drug is given will the concentration be a maximum?

(b) What is the maximum concentration?

SOLUTION:

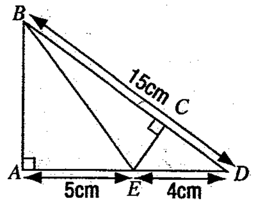
(a) $C(t) = \frac{0.09t}{t^2 + 6t + 9}$
Using the quotient rule:
$C'(t) = \frac{0.09(t^2 + 6t + 9) - 0.09t(2t + 6)}{(t^2 + 6t + 9)^2}$
$= \frac{0.09t^2 + 0.54t + 0.81 - 0.18t^2 - 0.54t}{(t + 3)^4}$
$= \frac{0.81 - 0.09t^2}{(t + 3)^4}$
Now, $C'(t) = 0 \Rightarrow 0.81 - 0.09t^2 = 0$
$\therefore 0.09t^2 = 0.81$
$t^2 = 9$
$t = 3, t \neq -3$
We now need to check to see whether a maximum does occur at $t = 3$ hours by applying the first derivative test and checking a point on either side of $C'(3)$.
$C'(2) = \frac{0.81 - 0.09 \times 4}{5^4} > 0$
$C'(3) = 0$
$C'(4) = \frac{0.81 - 0.09 \times 16}{7^4} < 0$
i.e.

t	3-	3	3+
$C(t)$	/	—	\

This means the gradient goes from + to 0 to - as t goes from 2 to 3 to 4, which maps out a maximum at time 3 hours.
You could have also checked for a maximum by first finding the second derivative $C''(t)$, then finding $C''(3)$, and showing that $C''(3) > 0$. This would mean using the quotient rule again. If you make an error using the quotient rule, you could be heavily penalised here. There is less chance of making an error if you use the first derivative test.

(b) $C(3) = \frac{0.09 \times 3}{3^2 + 6 \times 3 + 9}$
$= 0.0075$
So the maximum concentration is 0.0075 milligrams per cubic centimetre at 3 hours.

QUESTION TEN:



In the diagram, $AE = 5$cm, $ED = 4$cm and $BD = 15$cm. $\angle BAD = \angle BCE = 90°$

(a) Find the length of AB.

(b) Find the area of ΔEBD.

(c) Hence show that the length of EC is 3.2cm

(d) Find the size of $\angle BEC$.

SOLUTION:

(a) In ΔABD, using Pythagoras' Theorem, $AB^2 + 9^2 = 15^2$ so that $AB^2 = 144 \Rightarrow AB = 12$cm.

(b) The base $ED = 4$cm and the height $AB = 12$cm from (a).
$\therefore$ area $\Delta EBD = 1/2 \times 4 \times 12\text{cm}^2 = 24\text{cm}^2$.

(c) ΔEBD can also be looked at in a diferent way so that BD is the base and EC is the height. Now, we know that $BD = 15$cm and, from (b) the area of ΔEBD is 24cm². With this information, we can now find the height EC:
$\therefore 24 = 1/2 \times 15 \times EC \Rightarrow EC = 3.2$cm.

(d) In order to find the size of $\angle BEC$, we must first find the size of one of the other sides in ΔBEC – either BE or BC.
In ΔABE, BE can easily be found using Pythagoras' Theorem:
$AB^2 + AE^2 = BE^2$
$\therefore BE^2 = 12^2 + 5^2 = 169 \Rightarrow BE = 13$cm
So, in ΔBEC,
$\cos\angle BEC = EC/BE = 3.2/13$
$\therefore \angle BEC = 75°45'$.