Waverley College
Year 12 EXT 1Mathematics Examination
Term 4 2010

TIME ALLOWED: 50 MINUTES

NAME:

TEACHER:

INSTRUCTIONS:

Attempt all questions
Start each question on a new page
Calculators may be used
Write in blue or black pen only
Show all necessary working
Marks may be deducted for careless or badly arranged work

Question 1	5 /5
Question 2	5 /5
Question 3	14 /14
Question 4	14 /14
Total	38 /38
%	100%

Outcomes:

A student:
H1 – seeks to apply mathematical techniques to problems in a wide range of practical contexts
H2 – constructs arguments to prove and justify results
H3- manipulates algebraic expressions involving logarithmic and exponential functions
H4 – expresses practical problems in mathematical terms based on simple given models

Question 1 (begin new page) Mathematical Induction (5 marks)

a Prove by Mathematical Induction that if n is a positive integer then 5

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\text{---------}\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Question 2 (begin new page) Financial Mathematics (5 marks)

a A student borrows \$10 000 at $2\frac{1}{2}\%$ per month interest for his University fees and pays it off in equal monthly instalments. 5

i Let \$$A_3$ be the amount owing after 3 months and \$M be the monthly instalment. Write down an expression for A_3.

ii What should his instalments (\$M) be in order to pay off the loan at the end of 5 years ?

Question 3 (begin new page) Logarithmic Functions (14 marks)

a

i Evaluate $\log_9\left(\frac{1}{3}\right)$ 2

ii $\int\frac{4x-6}{x^2-3x-5}dx$ 1

b Differentiate $y=5^x$ 3

c Find the area bounded by the curve $y=\ln(x-1)$ the $x-axis$, the $y-axis$ and the line $y=4$. (Leave your answer as an exact value). 4

d Differentiate $y=(\ln x)^2$ and hence evaluate $\int_1^2\frac{\ln x}{x}dx$. 4

(Give answer to two decimal places)

Question 4 (begin new page) Exponentials (14 marks)

a Show that $\int_{-1}^{1}\left(e^{2x}+e^{-2x}\right)dx=\left(e-\frac{1}{e}\right)\left(e+\frac{1}{e}\right)$ 3

b Find $\frac{d}{dx}\left(e^{x^4}\right)$, hence evaluate $\int_0^1 x^3e^{x^4}dx$ 3

c Prove that the graph of the function $y=\frac{1}{x}e^{-x}$ has a stationary point at $(-1,-e)$ and determine the nature of this stationary point. 4

d Find the volume of the solid of revolution formed by rotating the curve $y=1+e^{-2x}$ about the $x-axis$ between $x=0$ and $x=\log_e 2$ 4

YEAR 12 EXT 1 TERM 4 SOLUTIONS 2010

Q 1

(a)

Step 1

Let $n=1$

$LHS = \frac{1}{1\times 2} = \frac{1}{2}$ $RHS = \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ the formula is true for $n=1$ ✓

Step 2

Assume that the formula is true for $n=k$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Step 3

Prove the formula is true for $n=k+1$

$$\left[\frac{1}{1\times 2} + \frac{1}{2\times 3} + \dots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

ie $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$ ✓

Now $LHS = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ ✓

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$ ✓

$= \frac{(k+1)^2}{(k+1)(k+2)}$ ✓

$= \frac{k+1}{k+2} = RHS$

Step 4

Since true for $n=k$ and true for $n=k+1$, when assumed true for $n=k+1$ when assumed true for $n=k$ therefore true for 2, 3, 4 .... etc

By principle of Mathematical induction true for all intergral $n \geq 1$

Q 2

(a)

i Let $P = \$10\,000$ and $R = 2.025$

Let A_n be the amount owing after the n^{th} instalment has been paid.

$A_1 = PR - m$

$A_2 = A_1R - M$

$A_2 = (PR - M)R - M$

$A_2 = PR^2 - MR - M$ ✓

$A_3 = A_2R - M$

$A_3 = (PR^2 - MR - M)R - M$

$A_3 = PR^3 - MR^2 - MR - M$

$= PR^3 - M(1 + R + R^2)$ ✓

$= \$10,000(1.025)^3 - M(1 + 1.025 + 1.025^2)$ ✓

ii At the end of 5 years $A_{60} = 0$ (5 years = 60 months)

$A_{60} = 10\,000(1.025)^{60} - M(1 + 1.025 + 1.025^2 + \dots 1.025^{59})$

$0 = 10\,000(1.025)^{60} - M\left(\frac{1.025^{60} - 1}{1.025 - 1}\right)$ ✓ (Sum of GP with 60 terms)

$M = 10\,000(1.025)^{60} \times \left(\frac{1.025 - 1}{1.025^{60} - 1}\right)$ ✓

$= 323.53$ ✓

The instalments would be \$ 323.53

Q3

a)

i Let $\log_9\left(\frac{1}{3}\right) = x$

$\frac{1}{3} = 9^x$ ✓

$3^{-1} = 3^{2x}$

$2x = -1$

$x = -\frac{1}{2}$ ✓

ii $\displaystyle\int \frac{4x-6}{x^2-3x-5}\,dx$

$= 2\displaystyle\int \frac{2x-3}{x^2-3x-5}\,dx$

$= 2\ln(x^2-3x-5) + c$ ✓

b) Differentiate $y = 5^x$

The only exponential we can differentiate ie $e^{f(x)}$

$\Rightarrow y = e^{\log_e 5^x}$

$y = e^{x\log_e 5}$ ✓

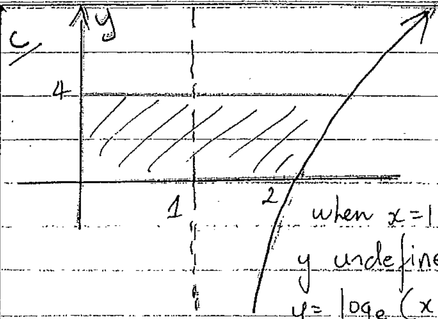
$\frac{dy}{dx} = \log_e 5\, e^{x\log_e 5}$

$\frac{dy}{dx} = \log_e 5 \times e^{\log_e 5^x}$ ✓

$= \log_e 5 \times 5^x$

$= 5^x \log_e 5$ ✓

c)



when $x=1$

y undefined

$y = \log_e(x-1)$

$(x-1) = e^y$

$x = e^y + 1$ ✓

$\text{Area} = \displaystyle\int_0^4 (e^y+1)\,dy$ ✓

$= \left[e^y + y\right]_0^4$ ✓

$= e^4 + 4 - (e^0 + 0)$

$= e^4 + 4 - 1$

$= e^4 + 3 \text{ unit}^2$ ✓

d)

$y = (\ln x)^2$

$\frac{dy}{dx} = 2 \times \frac{1}{x} \times (\ln x)' = \frac{2\ln x}{x}$ ✓

$\text{Now} \displaystyle\int_1^2 \frac{\ln x}{x}\,dx = \frac{1}{2}\int_1^2 \frac{2\ln x}{x}\,dx$ ✓

$= \frac{1}{2}\left[(\ln x)^2\right]_1^2$

$= \frac{1}{2}\left[(\ln 2)^2 - (\ln 1)^2\right]$ ✓

$= \frac{1}{2} \times 0.48$

$= 0.24$ ✓

Q4

a) $\displaystyle\int_{-1}^{1} (e^{2x} + e^{-2x})\,dx$

$= \left[\frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}\right]_{-1}^{1}$ ✓

$= \left(\frac{1}{2}e^2 - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^{-2} - \frac{1}{2}e^2\right)$ ✓

$= e^2 - e^{-2}$

$= e^2 - \frac{1}{e^2}$

$= \left(e - \frac{1}{e}\right)\left(e + \frac{1}{e}\right)$ ✓

b) $\frac{d}{dx}\left(e^{x^4}\right) = 4x^3 e^{x^4}$ ✓

Integrating both sides gives

$\displaystyle\int_0^1 \frac{d}{dx}\left(e^{x^4}\right) = \int_0^1 4x^3 e^{x^4}\,dx$

$\displaystyle\int 4x^3 e^{x^4} = \left[e^{x^4}\right]_0^1$

$\therefore \displaystyle\int_0^1 x^3 e^{x^4} = \frac{1}{4}\left[e^{x^4}\right]_0^1$ ✓

$= \frac{1}{4}(e-1)$ ✓

c) $y = \frac{e^{-x}}{x}$ $\quad u = e^{-x}$ $\quad v = x$

$u' = -e^{-x}$ $\quad v' = 1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{x(-e^{-x}) - e^{-x}(1)}{x^2}$ ✓

$\frac{dy}{dx} = \frac{-xe^{-x} - e^{-x}}{x^2}$ ---- ①

Sub $x=-1$ into ①, $\frac{dy}{dx} = 0$ ✓

$\therefore x = -1 \quad y = -e$ ✓

$\therefore$ Stationary point at $(-1, -e)$

$\frac{dy}{dx} = \frac{-e^{-x}(x+1)}{x^2}$

Nature

x	-1^-	-1	-1^+
$\frac{dy}{dx}$	$+$	0	$-$

✓

$\therefore$ Max at $(-1, -e)$

d

$$\text{Volume} = \pi \int_a^b y^2\,dx \qquad y = 1 + e^{-2x}$$
$$y^2 = (1+e^{-2x})^2$$

$$= \pi \int_0^{\ln 2} (1+e^{-2x})^2\,dx \checkmark$$

$$= \pi \int_0^{\ln 2} (1 + 2e^{-2x} + e^{-4x})\,dx \checkmark$$

$$= \pi \left[x - e^{-2x} - \frac{1}{4}e^{-4x} \right]_0^{\ln 2} \checkmark$$

$$= \pi \left[\left(\ln 2 - \frac{1}{4} - \frac{1}{4} \times \frac{1}{16} \right) - \left(0 - 1 - \frac{1}{4} \right) \right]$$

$$= \pi \left[\ln 2 + \frac{63}{64} \right] \text{units}^3 \checkmark$$

$= // =$