Waverley College
Year 12 2 Unit Mathematics Examination
Term 4 2010

TIME ALLOWED: 45 MINUTES.

<u>STUDENT NUMBER:</u>

<u>TEACHER:</u>

INSTRUCTIONS:

Attempt all questions
Calculators may be used
Write in blue or black pen only
Show all necessary working
Marks may be deducted for careless or badly arranged work

QUESTION 1	/26
QUESTION 2	/20
TOTAL	/46
%	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION ONE (26 Marks)

a) Sketch $y = x^2 - 5x + 6$ and hence or otherwise find the values of x for which $x^2 - 5x + 6 \leq 0$. (3)

b) Find all values of k for which $x^2 + kx + 1 = 0$ has equal roots. (2)

c) Show that $y = -x^2 + x - 2$ is a negative definite quadratic function. (2)

d) Find the values of A, B and C if $x^2 + x - 2 \equiv A(x-2)^2 + Bx + C$. (4)

e) Find the quadratic equation whose roots are $3 - \sqrt{2}$ and $3 + \sqrt{2}$. (3)

f) If α and β are roots of the quadratic equation $x^2 - 3x - 6 = 0$ find

i) $\alpha + \beta$ (1)

ii) $\alpha\beta$ (1)

iii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (1)

iv) $\alpha^2 + \beta^2$ (1)

g) Find the values of k in the equation $(k-2)x^2 - (6+k)x + 2k + 3 = 0$ if the roots are reciprocals of each other. (2)

h) Solve for x, $4^x - 10.2^x + 16 = 0$ giving exact values. (3)

i) Show that the line $y = 5x - 2$ is a tangent to the parabola $y = x^2 + 3x - 1$ (3)

END OF QUESTION ONE

QUESTION TWO (20 Marks)

a) Find the coordinates of the vertex, focus and the equation of the directix of the following parabolas

i) $y^2 = -8x$. (3)

ii) $(x-2)^2 = 16(y+1)$ (3)

b) Find the equation of the locus of a point P(x, y) that moves so that distance PA to PB is in the ratio 1:2 where A is (-6, 5) and B is (3,-1). (3)

c) i) Find the equation of the normal to the curve $x^2 = 16y$ at the point (4,1). (3)

ii) This normal meets the parabola again at the point Q. Find the coordinates of Q. (2)

d) For the parabola $x^2 - 2x + 4y + 5 = 0$ find

i) the coordinates of the vertex (2)

ii) the coordinates of the focus (1)

iii) the equation of the directrix (1)

iv) hence sketch the curve showing these important features (2)

END OF EXAMINATION

11 Question 1

GC

26

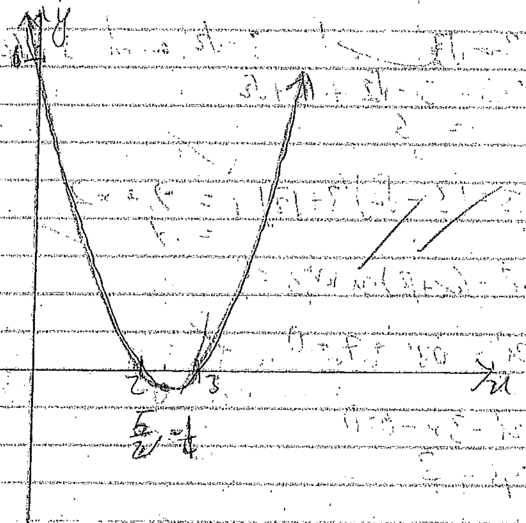
a) $y = x^2 - 5x + 6$

$x^2 - 5x + 6 = 0$ for x-intercepts: $x^2 - 5x + 6 \le 0$ for

$(x-3)(x-2) = 0$ $\qquad$ $(x-3)(x-2) \le 0$

$x = 3$ or $x = 2$ $\qquad$ $2 \le x \le 3$

vertex at $\frac{5}{2}, \; 5\frac{1}{4}$



b) $x^2 + kx + 1 = 0$

$\Delta = 0$ for equal roots

$k^2 - 4(1)(1) = 0$

$k^2 - 4 = 0$

$(k-2)(k+2) = 0$

$k = 2$ or $k = -2$

c) $-x^2 + x - 2 = 0$

$a < 0$ $\quad$ $\Delta < 0$ for negative definite

$-1 < 0$

$\Delta = 1^2 - 4(-1)(-2)$

$= -7 \;\therefore < 0$ ✓ function is negative definite

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d) $x^2 + x - 2 = A(x-2)^2 + Bx + C$

$= A(x^2 - 4x + 4) + Bx + C$

$= Ax^2 - 4Ax + 4A + Bx + C$

$= Ax^2 + (B - 4A)x + 4A + C$

$A = 1$ $\quad$ $B - 4A = 1$ $\quad$ $4A + C = -2$

$B - 4 = 1$ $\quad$ $4 + C = -2$

$B = 5$ $\quad$ $C = -6$

e) ~~$3 - \sqrt{2}$ and~~ $3 - \sqrt{2}$ and $3 + \sqrt{2}$

$\alpha + \beta = 3 - \sqrt{2} + 3 + \sqrt{2}$

$= 6$

$\alpha\beta = (3 - \sqrt{2})(3 + \sqrt{2}) = 9 - 2$

$= 7$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - 6x + 7 = 0$

f) $x^2 - 3x - 6 = 0$

i) $\alpha + \beta = 3$

ii) $\alpha\beta = -6$

iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{3}{-6} = -\frac{1}{2}$

iv) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 3^2 - 2(-6) = 21$

g) $(k-2)x^2 - (6+k)x + 2k + 3 = 0$

Since roots are reciprocals let them be α and $\frac{1}{\alpha}$

$\alpha\left(\frac{1}{\alpha}\right) = \frac{2k+3}{k-2} = 1$

$2k + 3 = k - 2$

$k = -5$

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Question 1 GC

h) $4^x - 10.2^x + 16 = 0$

$(2^2)^x - 10.2^x + 16 = 0$

$(2^x)^2 - 10.2^x + 16 = 0$

let $2^x = a$

$a^2 - 10a + 16 = 0$

$(a-8)(a-2) = 0$

$a = 8$ or $a = 2$

$2^x = 8$ or $2^x = 2$

$2^x = 2^3$ or $2^x = 2^1$

$x = 3$ or $x = 1$

i) $y = x^2 + 3x - 1$ $y = 5x - 2$

$x^2 + 3x - 1 = 5x - 2$

$x^2 - 2x + 1 = 0$

If line is tangent it will touch at 1 point

∴ equation of intersection must have $\Delta = 0$

$\Delta = 2^2 - 4(1)(1)$

$= 4 - 4 = 0$

∴ line is tangent to parabola

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Question 2 20 GC

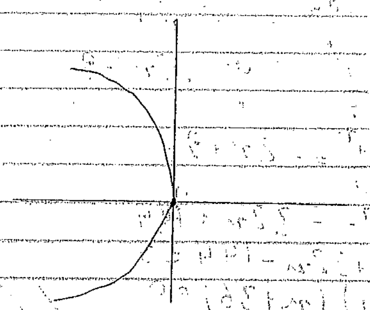
i) $y^2 = -8x$

$V(0,0)$

$a = 2$

$F(-2,0)$

d: $x = 2$



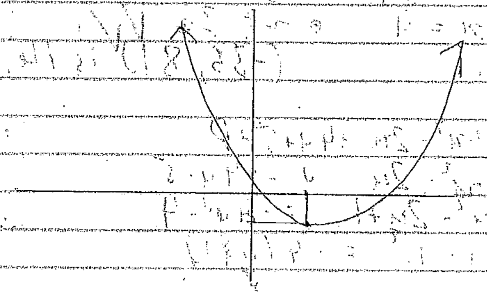
ii) $(x-2)^2 = 16(y+1)$

$V(2,-1)$

$a = 4$

$F(2,3)$

d: $y = -5$



b) $PA : PB = 1 : 2$ $A(-6,5)$ $B(3,-1)$

$2PA = PB$ $PA = \sqrt{(x+6)^2 + (y-5)^2}$ $PB = \sqrt{(x-3)^2 + (y+1)^2}$

$4PA^2 = PB^2$

$4((x+6)^2 + (y-5)^2) = (x-3)^2 + (y+1)^2$

$4(x^2 + 12x + 36 + y^2 - 10y + 25) = x^2 - 6x + 9 + y^2 + 2y + 1$

$4x^2 + 48x + 4y^2 - 40y + 244 = x^2 - 6x + y^2 + 2y + 10$

$3x^2 + 54x + 3y^2 - 42y + 234 = 0$

$x^2 + 18x + y^2 - 14y + 78 = 0$

$x^2 + y^2 + 18x - 14y + 78 = 0$

c) i) $x^2 = 16y$ $(4,1)$

$y = \frac{x^2}{16}$ $\frac{dy}{dx} = \frac{x}{8}$

$x = 4$

$\frac{dy}{dx} = \frac{4}{8} = \frac{1}{2}$ $m_n = -2$

$y - 1 = -2(x - 4)$

$y - 1 = -2x + 8$

$2x + y - 9 = 0$

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ii) $x^2 = 16y \qquad 2x + y - 9 = 0$

$y = \frac{x^2}{16} \qquad y = -2x + 9$

$\frac{x^2}{16} = -2x + 9$

$x^2 = -32x + 144$

$x^2 + 32x - 144 = 0$

$(x-4)(x+36) = 0$

$x = 4 \quad \text{or} \quad x = -36$

$(-36, 81)$ is the other point of intersection

d) $x^2 - 2x + 4y + 5 = 0$

$x^2 - 2x = -4y - 5$

$x^2 - 2x + 1 = -4y - 4$

$(x-1)^2 = -4(y+1)$

$V(1, -1) \quad (a = 1)$

$F(1, -2)$

$d: \; y = 0$