Student Number

WAVERLEY COLLEGE 2012

HALF YEARLY EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 8-11
- Write your Student Number at the top of this page and the Multiple Choice Answer Sheet.

Total marks – 76

Section I Pages 3-5
7 marks

- Attempt Questions 1-7
- Allow 10 minutes for this section

Section II Pages 6-10
69 marks

- Attempt Questions 8-11
- Allow 1 hour and 50 minutes for this section
- Answer Each question in a new booklet

Section I

7 marks
Attempt all questions
Allow about 10 minutes for this section

Use the multiple choice answer sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2+4=$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think that you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

Section 1 (7 marks) **Marks**

(1) Which of the following is a simplification of the expression $\frac{n^2-25}{5n-25}$. **1**

(A) $\frac{n}{5}$

(B) $\frac{n+5}{5}$

(C) $\frac{n-5}{n+5}$

(D) $(n+5)^2$

(2) If $\frac{4}{2\sqrt{3}-3}=p+q\sqrt{3}$, which statement is true.

(A) $q=2$

(B) $p=\frac{4}{3}$

(C) $q=-2$

(D) $p=4$

(3) The function $y=3+(x-2)^2$ is:

(A) An EVEN function

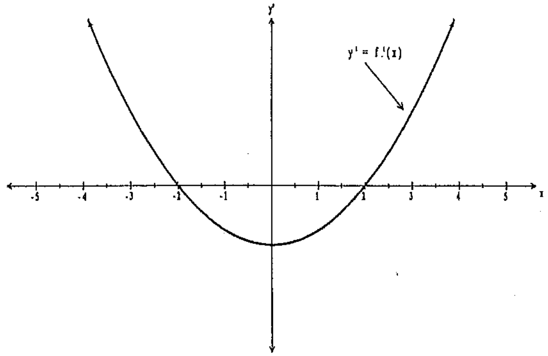
(B) An ODD function

(C) Symmetric about the line $x=2$

(D) Symmetric about the line $y=3$

Marks

(4) **1**



The diagram above represents a sketch of the **gradient function** of the curve $y=f(x)$.
Which of the following is a true statement: The curve $y=f(x)$ is

(A) A cubic function with a stationary point at $x=0$

(B) A cubic function with turning points at $x=\pm 2$

(C) A quadratic function with a point of inflection at $x=0$

(D) A straight line with a positive gradient

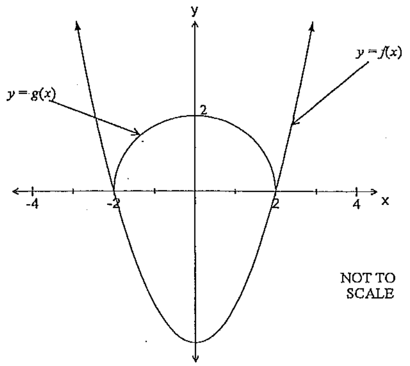
(5) Which of the following is equal to $e^x(e^x-\frac{1}{e})$? **1**

(A) $e^{x^2}-e$

(B) $e^{2x}-e$

(C) $e^{x^2}-e^{x+1}$

(D) $e^{2x}-e^{x-1}$

(6) 

Marks 1

Noting that $y = g(x)$ is a semi-circle;

If $\int_0^2 f(x)+g(x)\,dx = \pi - 10$ $units^2$, Which of the following equates to $\int_{-2}^{2} f(x)\,dx$.

(A) $2\pi - 20$

(B) 20

(C) $20 - 2\pi$

(D) -20

(7) The values of n for which $nx^2 + 2\sqrt{2}x - 1 + n > 0$; for all real values of x are: 1

(A) $n > 2$

(B) $-1 < n < 2$

(C) $0 < n < 2$

(D) $n < -1$

Section II

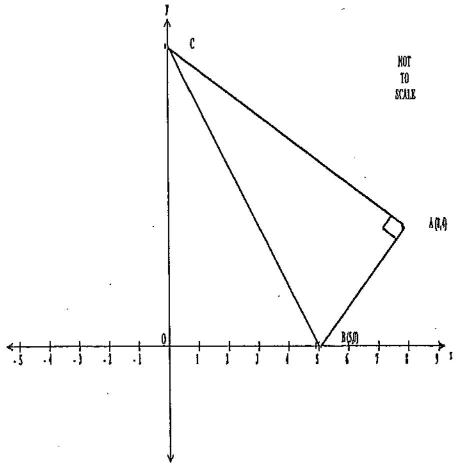
Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\sqrt[3]{\dfrac{\pi + 90}{2}}$ correct to three significant figures. 2

(b) Factorise fully $a^3 - 8a^6$. 2

(c) Evaluate $\lim_{x \to -1}\left(\dfrac{3x^2 + x - 2}{x + 1}\right)$. 2

(d) Solve $|1 - 3x| \le 5$. 2

(e) 

In the diagram above $A = (8,4)$, $B = (5,0)$, C lies on the $y-axis$ and AB is perpendicular to AC.

(i) Find the gradient of AB. 1

(ii) Hence show that the equation of AC is $3x + 4y - 40 = 0$. 2

(iii) Find the coordinates of point C. 1

(iv) Hence prove that ΔOBC is congruent to ΔABC. 2

(v) Find the area of quadrilateral $OBAC$. 1

Question 9 (18 marks) Use a SEPARATE writing booklet. **Marks**

(a) Graph the following on the same number plane and shade in the region, which satisfies both equations. 3

$$y \geq |x+2| \quad \text{and} \quad y \leq \sqrt{4-x^2}$$

(b) Solve for x: $e^{2x} + e^{x} - 2 = 0$ 3

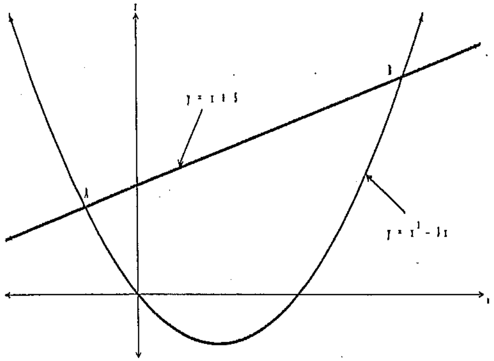
(c) Two ships A and B leave Port C at the same time. Ship A travels on a bearing of $150°$T for 4 nautical miles, while ship B travels on a bearing of $210°$T for 8 nautical miles.

(i) Draw a diagram representing this information and explain why $\angle ACB = 60°$. 2

(ii) Find the distance between A and B leaving your answer in exact form. 2

(iii) Find the bearing of ship A from ship B. 2

(d)



In the above diagram, the line $y = x + 5$ meets the parabola $y = x^2 - 3x$ in A and B.

(i) Find the x-values of A and B. 1

(ii) Hence, find the area bounded by $y = x + 5$ and $y = x^2 - 3x$. *(leave your answer in exact form)* 3

(e) If $2x + 3 = A(x-1)^2 + B(x-1) + C$, find the value of C. 2

Question 10 (18 marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate the following with respect to x.

(i) $y = \dfrac{4x^3 - 2x}{x^5}$. 2

(ii) $y = \ln\left(e^{x}\right)$. 1

(iii) $y = (\ln x)^4$. 2

(b) A parabola has the equation $2y = x^2 - 8x + 6$. Find the coordinates of:

(i) The vertex. 1

(ii) The focus. 1

(c) The depth of water (*d metres*) in a dam over a six year period is given the equation $d = t^3 - 9t^2 + 15t + 30$, $0 \leq t \leq 6$ where $t = \text{time (years)}$.

(i) Find the initial and final depths of the water during the six year period. 2

(ii) Find the greatest depth of water in the dam during this period. 3

(iii) Find the time when the depth of the water level was decreasing and the greatest rate. 2

(iv) Sketch the graph of $d = t^3 - 9t^2 + 15t + 30$, $0 \leq t \leq 6$ 2

(v) Find the time when the depth of water in the dam is greater than 30 metres. *(leave your answer to 2 decimal places).* 2

Question 11 (18 marks) Use a SEPARATE writing booklet. **Marks**

(a) State the domain of $y=\dfrac{\sqrt{16-x^2}}{x}$. **2**

(b) Prove that $\dfrac{1+\tan^2(2x)}{1+\cot^2(2x)}=\sec^2(2x)-1$ **2**

(c) Evaluate $\displaystyle\int_1^e \frac{2x+e}{x^2+ex}\,dx$. *(give your answer correct to 1 decimal place).* **2**

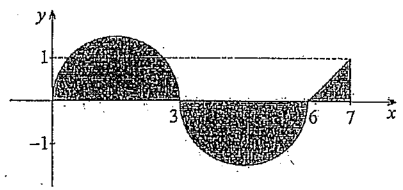
(d) At a particular location, a river 30 metres wide is measured for depth every 5 metres across its width. The measurements from bank to bank are given in the following table:

Distance Across river (m)	0	5	10	15	20	25	30
Depth (m)	0	10	12	18	7	3	0

(i) Use Simpson's rule to find the cross-sectional area of the river at this point. **2**

(ii) Use your answer in part (i) to find the volume of water passing this point in two hours, if the water is passing this spot at $\frac{1}{8}\ m/\sec$. **2**

(e)



The function $y=f(x)$, for $0\le x\le 7$ is shown above. The curves are semicircular arcs.

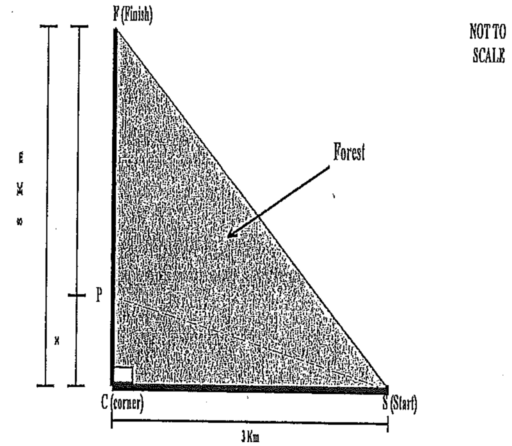
(i) Find $\displaystyle\int_0^7 f(x)\,dx$ **1**

(ii) Find the exact total area of the shaded parts. **2**

Question 11 continues over the page

Question 11 (continued) **Marks**

(f)



At the annual school fun run, students race each other. Students can take any course they choose, however, must start at S and finish at F. Mia can run along the flat straight path SC and then CF at 10km/hr. If she chooses, she can run through the forest at 4km/hr. Let P be the optimal position for Mia to come out of the forest along the path CF so that she finishes the race in the **shortest** time and let $CP=x\,km$. (Note the path SF is in the forest)

(i) Show that the time (T) taken for Mia to start at S and reach F following the path $S\longrightarrow P\longrightarrow F$ is given by **2**

$$T=\frac{\sqrt{x^2+9}}{4}+\frac{8-x}{10}.$$

(ii) Hence, find the value of x to enable Mia to complete the race in the shortest time. **3**

END OF PAPER

2012 2U Half Yearly Solutions

Multiple choice

1) $\dfrac{n^2-25}{5n-25} = \dfrac{\cancel{(n-5)}(n+5)}{5\cancel{(n-5)}}$

$= \dfrac{n+5}{5}$

B

2) $\dfrac{4}{2\sqrt{3}-3} = p + q\sqrt{3}$

$\dfrac{4}{2\sqrt{3}-3} = \dfrac{4}{2\sqrt{3}-3} \times \dfrac{2\sqrt{3}+3}{2\sqrt{3}+3}$

$= \dfrac{4(2\sqrt{3}+3)}{3}$

$= \dfrac{8\sqrt{3}+12}{3}$

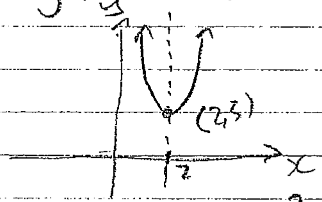
$= \dfrac{8}{3}\sqrt{3} + 4$

$p = 4$

D

3) $y = 3 + (x-2)^2$

* Concave up
* turning pt $(2,3)$



AOS: $x = 2$

C

4) $f'(x) = 0$ at $x = \pm 2$
$\therefore f(x)$ has stat pts at $x = \pm 2$

B

5) $e^x\left(e^x - \dfrac{1}{e}\right) = e^x(e^x - e^{-1})$

$= e^{2x} - e^{x-1}$

D

M/C continued

6) $g(x)$ semi circle
$f(x)$ parabola

$\int_0^2 g(x)\,dx = \dfrac{\pi r^2}{4}$ $(r=2)$

$= \dfrac{4\pi}{4}$

$= \pi$

$\therefore \int_0^2 f(x)\,dx = -10$

Symmetry

$\int_{-2}^{2} f(x)\,dx = 2\int_0^2 f(x)\,dx$

$= 2(-10)$

$= -20$

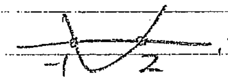
D

7) $nx^2 + 2\sqrt{2}x - 1 + n > 0$
for all real x

positive definite
$\Rightarrow a > 0 \Rightarrow n > 0$
$\Delta < 0$

$\Delta = b^2 - 4ac$
$= 8 - (4n(-1+n))$
$= 8 + 4n - 4n^2$

let $\Delta = 0 \Rightarrow -4n^2 + 4n + 8 = 0$
$n^2 - n - 2 = 0$
$(n-2)(n+1) = 0$



$\therefore -1 < n < 2$ but be careful!

$\therefore 0 < n < 2$

C

Q8 Solutions 2012 2u Half Yearly

a) $\sqrt[3]{\frac{\pi+90}{2}} = 3.5978\ldots$ (1)

$= 3.60$ (3 sig fig) (1)

b) $a^3 - 8a^6 = a^3(1-8a^3)$ (1)

$= a^3(1-2a)(1+2a+4a^2)$ (1)

c) $\lim_{x\to -1}\left(\frac{3x^2+x-2}{x+1}\right) = \lim_{x\to -1}\left(\frac{(3x-2)\cancel{(x+1)}}{\cancel{x+1}}\right)$

$= \lim_{x\to -1}(3x-2)$ (1)

$= 5$ (1)

d) $|1-3x| \le 5$

Distance from zero is ≤ 5

$\therefore -5 \le 1-3x \le 5$

$-6 \le -3x \le 4$

$2 \ge x \ge -1\frac{1}{3}$

$\therefore -1\frac{1}{3} \le x \le 2$

(1) (1)

Q8

e) i) $m_{AB} = \frac{4-0}{8-5}$

$= \frac{4}{3}$ (1)

ii)

$m_{AB}\, m_{AC} = -1$

$m_{AC} = -\frac{3}{4}$ (1)

equation of AC

$y-4 = -\frac{3}{4}(x-8)$

$4y-16 = -3x+24$

$3x+4y-40=0$ (1)

iii) when $x=0$

$3(0)+4y-40=0$

$4y=40$

$y=10$

$\therefore C(0,10)$ (1)

iv) In $\triangle OBC$ & $\triangle ABC$

BC is common H

$\angle BOC = \angle CAB = 90°$ (axis meet at right angles & $\angle CAB$ is given) (1)

$AB^2 = 3^2+5^2$ (1) for any 2 correct lines

$AB = 5$

$= OB$ (S)

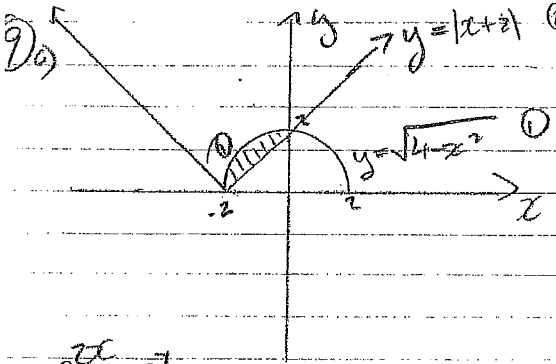
$\therefore \triangle OBC \equiv \triangle ABC$ (RHS) (1)

v) $A = 2(\frac{1}{2}\times 5\times 10)$

$= 50$ units^2 (1)

olutions 2012 2U Half Yearly

9a)



b) $e^{2x}+e^{x}-2=0$

let $m=e^{x}$

$m^2+m-2=0$ ①

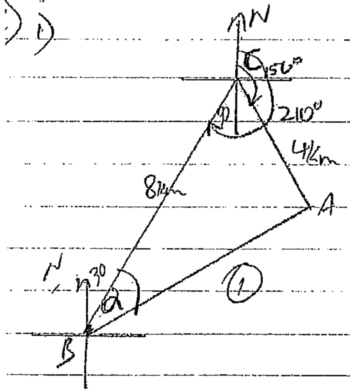
$(m+2)(m-1)=0$

$\therefore m=-2$ or 1 but $m=e^{x}$

$\therefore e^{x}=-2$ or $e^{x}=1$

no solution ① $x=0$ ①

c) i)



$\angle ACB = 210-150$
$= 60°$ ①

ii) $AB^2 = 8^2+4^2-2\times8\times4\times\cos60°$ ①
$= 48$

$AB = 4\sqrt{3}$ ① or $\sqrt{48}$

iii) $\frac{\sin\alpha}{4} = \frac{\sin 60}{4\sqrt{3}}$ ①

$\sin\alpha = \frac{1}{2}$

$\therefore \alpha = 30$

$\therefore$ Bearing of A from B is

$30+30° = 060°T$ ①

Solutions 2U Half yearly

Q9 d) i)

$x^2-3x=x+5$

$x^2-4x-5=0$

$(x+1)(x-5)=0$

$\therefore x=-1, 5$ ①

ii) $A = \int_{-1}^{5}\left((x+5)-(x^2-3x)\right)dx$

$= \int_{-1}^{5}(-x^2+4x+5)\,dx$ ①

$= \left[\frac{-x^3}{3}+\frac{4x^2}{2}+5x\right]_{-1}^{5}$

$= \left[\frac{-x^3}{3}+2x^2+5x\right]_{-1}^{5}$

$= \left(\frac{-125}{3}+50+25\right)-\left(\frac{1}{3}+2-5\right)$

$= 36$ units^2

9e) let $x=1$

$2(1)+3 = A(1-1)^2+B(1-1)+C$ ①

$\therefore C=5$ ①

Q10

a) i) $y = \dfrac{4x^3 - 2x}{x^5}$

$= \dfrac{4x^3}{x^5} - \dfrac{2x}{x^5}$

$= 4x^{-2} - 2x^{-4}$

$\therefore y' = -8x^{-3} + 8x^{-5}$

$= \dfrac{-8}{x^3} + \dfrac{8}{x^5}$

ii) $y = \ln(e^x)$

$y' = \dfrac{e^x}{e^x}$

$= 1$

iii) $y = (\ln x)^4$

$y' = 4(\ln x)^3 \times \frac{1}{x}$

$= \dfrac{4(\ln x)^3}{x}$

b) $2y = x^2 - 8x + 6$

$2y - 6 + 16 = x^2 - 8x + 16$

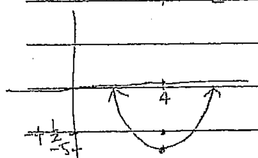
$2y + 10 = (x-4)^2$

$2(y+5) = (x-4)^2$

i) $\therefore$ Vertex $(4, -5)$

$4a = 2$

$a = \frac{2}{4} = \frac{1}{2}$



ii) Focus $(4, -4\frac{1}{2})$

c) i) $d = t^3 - 9t^2 + 15t + 30$

when $t = 0$, $d = 30\text{m}$

when $t = 6$, $d = 6^3 - 9(6)^2 + 15\times6 + 30$

$= 12\text{m}$

ii) $d' = 3t^2 - 18t + 15 = 0$

$t^2 - 6t + 5 = 0$

$(t-5)(t-1) = 0$

$t = 1, 5$

when $t = 1$, $d = 1 - 9 + 15 + 30 = 37\text{m}$

when $t = 5$, $d = 5^3 - 9\times5^2 + 15\times5 + 30 = 5\text{m}$

$\therefore$ Greatest depth is 37m

iii) Water level is decreasing when $1<t<5$

Greatest rate occurs when $d'' = 0$

$d'' = 6t - 18 = 0$

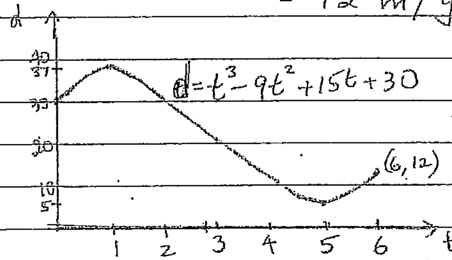
$6t = 18$

$t = 3$

Greatest rate is $d' = 3\times3^2 - 18\times3 + 15$

$= -12\ \text{m/yr}$

iv)



v) $t^3 - 9t^2 + 15t + 30 > 30$

$t^3 - 9t^2 + 15t > 0$

$t(t^2 - 9t + 15) > 0$

$t = \dfrac{9 \pm \sqrt{81 - 4\times15}}{2}$

$= 6.79$ or 2.21

$\therefore$ Depth of water is greater than 30m when $0 < t < 2.21$

Q11

a) $16 - x^2 \geq 0$

$16 \geq x^2$

$x^2 \leq 16$

$-4 \leq x \leq 4$

and $x \neq 0$

b) LHS $= \dfrac{1 + \tan^2(2x)}{1 + \cot^2(2x)}$

$= \dfrac{\sec^2(2x)}{\operatorname{cosec}^2(2x)}$

$= \dfrac{\frac{1}{\cos^2(2x)}}{\frac{1}{\sin^2(2x)}}$

$= \dfrac{\sin^2(2x)}{\cos^2(2x)}$

$= \tan^2(2x)$

$= \sec^2(2x) - 1$

$=$ RHS

c) $\displaystyle\int_1^e \frac{2x+e}{x^2+ex} = \Big[\ln(x^2+ex)\Big]_1^e$

$= \ln(e^2+e^2) - \ln(1+e)$

$= \ln(2e^2) - \ln(1+e)$

$= 1.4$

d) i) $A \approx \frac{5}{3}\left[0 + 4(10+18+3) + 2(12+7) + 0\right]$

$= 270\text{m}^2$

ii) $V = 270\text{m}^2 \times \frac{1}{8}\text{m/s} \times (60\times60\times2)\text{s}$

$= 243\,000\text{m}^3$

e) i) $\displaystyle\int_0^7 f(x)\,dx = \frac{1}{2}\times1\times1$

$= \frac{1}{2}$

ii) $A = \pi\times\left(\frac{3}{2}\right)^2 + \frac{1}{2}$

$= \dfrac{9\pi}{4} + \dfrac{1}{2}$

$= \dfrac{9\pi + 2}{4}$ units^2

f) i) $SP = \sqrt{3^2 + x^2}$

$= \sqrt{x^2 + 9}$ km

speed (SP) $= 4$ km/h

$\therefore$ Time (SP) $= \frac{D}{S} = \dfrac{\sqrt{x^2+9}}{4}$

D
ST

$PF = (8 - x)$ km

Speed PF $= 10$ km/h

$\therefore$ Time (PF) $= \dfrac{8-x}{10}$

$\therefore$ Time $(S \to P \to F) =$ Time (SP) + Time (

$= \dfrac{\sqrt{x^2+9}}{4} + \dfrac{8-}{1}$

ii) $T' = \dfrac{\frac{1}{2}(x^2+9)^{-\frac{1}{2}}\cdot 2x}{4} - \dfrac{1}{10}$

$0 = \dfrac{x}{4\sqrt{x^2+9}} - \dfrac{1}{10}$

$\dfrac{1}{10} = \dfrac{x}{4\sqrt{x^2+9}}$

$4\sqrt{x^2+9} = 10x$

$16(x^2+9) = 100x^2$

$16x^2 + 144 = 100x^2$

$84x^2 = 144$

$x^2 = \frac{12}{7}$

$x = \sqrt{\frac{12}{7}}$

x	1	$\sqrt{\frac{12}{7}}$	2
T'	$-$	0	$+$

$\therefore$ MIN

$\therefore$ when $x = \sqrt{\frac{12}{7}}$ Mia will complete th