# WESTERN REGION

## TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# 1998

# MATHEMATICS

**3 Unit (Additional)**
**and**
**3/4 Unit (Common)**

*Time allowed - TWO hours*
*(Plus 5 minutes reading time)*

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on the last page. These may be removed for your convenience.
- Board-approved calculators may be used.
- Each question should be started on a new page.

**Question One.** *Start a new page* **Marks**

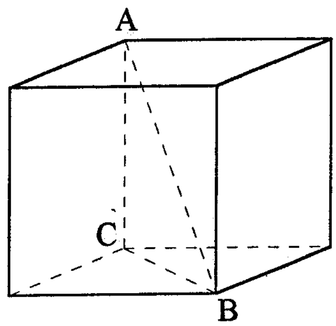
(a) Use the substitution $u = x^2$ to find $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\, dx$ . **3**

(b) Given that $1-\sqrt{2}$ is a root of the polynomial $x^2 + 2ax + a = 0$, where $a$ is an integer, determine the value of $a$. **2**

(c) Differentiate $\log_e(\sin^3 x)$ writing your answer in the simplest form. **2**

(d) AB is the diagonal of a cube and the point C is a vertex on the base of the cube as shown in the diagram. **3**

Determine the size of $\angle$ ABC, correct to the nearest minute.

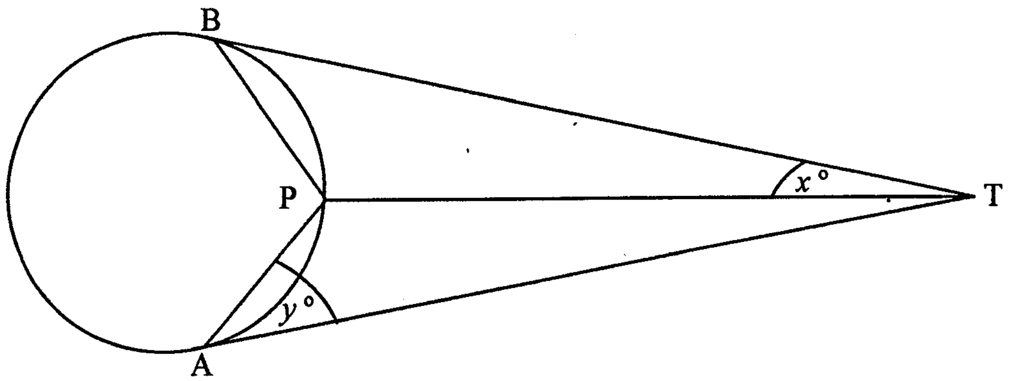



(e) A is the point ( − 4 , 2 ) and B is the point ( 3 , − 1 ). Find the coordinates of the point P which divides the interval AB externally in the ratio 2 : 1. **2**

**Question Two.** *Start a new page* **Marks**

(a) The quadratic polynomial P($x$) has a stationary point at ( 2 , − 1 ) and passes through the point ( 1 , 2 ). Show that $P(x) = 3x^2 - 12x + 11$. **3**

(b) **6**




In the diagram above, TB and TA are tangents to the circle. P is a point on the circumference of the circle such that $\angle$ BTA is bisected by the line PT.
Also, $\angle BTP = x°$ and $\angle PAT = y°$.

(i) Prove that the chords BP and AP are equal.

(ii) Hence, by considering Δ APB, or otherwise, prove that

$$x + 2y = 90°$$

(c) Determine the coefficient of $a^4$ in the expansion of $(a - 2a^{-1})^{10}$ **3**

**Question Three.** *Start a new page* **Marks**

(a) Solve $\frac{x^2 - 25}{2x} < 0$ **2**

(b) A function is defined as $f(x) = 1 + e^{2x}$. **10**

(i) Write down the domain and range of the function.

(ii) Show that the inverse function can be defined as $f^{-1}(x) = \frac{1}{2} \ln(x - 1)$.

(iii) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

(iv) Show that the equation of the normal to $y = f^{-1}(x)$ at the point where $f^{-1}(x) = 0$ is

$$2x + y - 4 = 0.$$

(v) Show that the point of intersection of this normal and $y = f(x)$ can be derived from the equation

$$e^{2x} + 2x = 3$$

(vi) By taking $x = 0{\cdot}4$ as the first approximation of the root to $e^{2x} + 2x = 3$, use one application of Newton's Method to find a better approximation of the root, correct to 3 significant figures.

**Question Four.** *Start a new page* **Marks**

(a) A teacher organising an examination timetable has to schedule seven examinations. Of these examinations, one is English and two are Mathematics. The Mathematics examinations are not to be scheduled consecutively. **3**

(i) In how many ways can the seven examinations be scheduled ?

(ii) If the English examination is scheduled first, find the probability that one Mathematics examination will be scheduled second and the other Mathematics examination scheduled last.

(b) The portion of the curve $y = \sin x + \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the $x$ axis. Show that the volume of the solid of revolution generated is $\frac{\pi}{2}(\pi + 2)$ cubic units. **4**

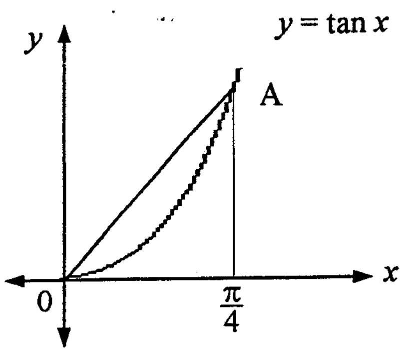
(c) Triangle PQR is isosceles, having PQ = PR = 6 cm and $\angle RPQ = \theta$. **5**

(i) Show that the area of $\Delta$ PQR = $18 \sin \theta$, where $\theta$ is expressed in radians.

(ii) If the area of $\Delta$ PQR is increasing at the rate of $3 \text{ cm}^2$ per second, determine the rate at which $\angle RPQ$ is changing at the instant the area of $\Delta$ PQR is $9 \text{ cm}^2$.

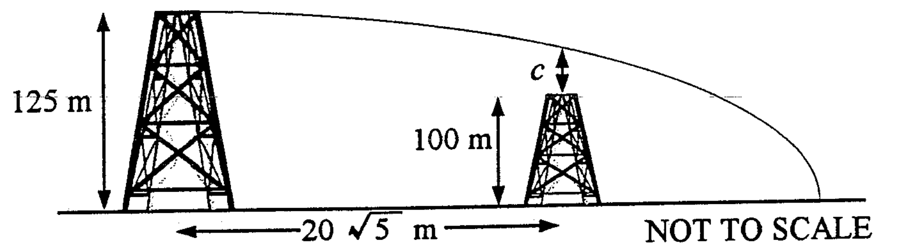
**Question Five.** *Start a new page* **Marks**

(a) **4**




In the diagram, A is a point on the curve $y = \tan x$ with an $x$ coordinate of $\frac{\pi}{4}$. The chord OA has been drawn from the origin to the point A. Show that the area enclosed by the chord OA and the curve $y = \tan x$ between $x = 0$ and $x = \frac{\pi}{4}$ has a magnitude of $\frac{1}{8}(\pi - 4\ln 2)$ units $^2$.

(b) **8**




A projectile is shot horizontally from the top of a 125 metre tower with a velocity of V metres per second. It clears a second tower of height 100 metres by a distance of $c$ metres, as shown in the diagram. The two towers are $20\sqrt{5}$ metres apart.

(i) The equations of position for this system are

$$x = \mathrm{V}t \text{ and } y = -5t^2 + 125$$

where V is velocity in metres per second and, $t$ is time in seconds. Explain where the origin of this system is being taken from.

(ii) Show that $\mathrm{V} = \dfrac{100}{\sqrt{25 - c}}$ .

(iii) Prove that the minimum speed of projection for the projectile to just clear the 100 metre tower is 20 metres per second.

(iv) Hence, find how far past the 100 metre tower the projectile will strike the ground.

(v) Determine the vertical velocity of the projectile when it strikes the ground.

**Question Six.** *Start a new page* **Marks**

(a) The acceleration of a particle P is given by the equation **6**

$$\ddot{x} = 8x(x^2 + 1)$$

where $x$ is the displacement of P from the origin in metres after $t$ seconds, with movement being in a straight line.

Initially, the particle is projected from the origin with a velocity of 2 metres per second in the positive direction.

(i) Show that the velocity of the particle can be expressed as

$$V = 2(x^2 + 1).$$

(ii) Hence, show that the equation describing the displacement of the particle at time $t$ is given by $x = \tan 2t$.

(iii) Determine the velocity of the particle after $\frac{\pi}{8}$ seconds.

(b) Let $2\cos\theta - 3\sin\theta = R\cos(\theta + \alpha)$, where $R > 0$. **6**

(i) Find the exact value of R and the size of $\alpha$ to the nearest minute.

(ii) Hence, or otherwise, find all values of $\theta$, to the nearest minute, over the domain $0° \le \theta \le 360°$, where

$$2\cos\theta - 3\sin\theta = 1.$$

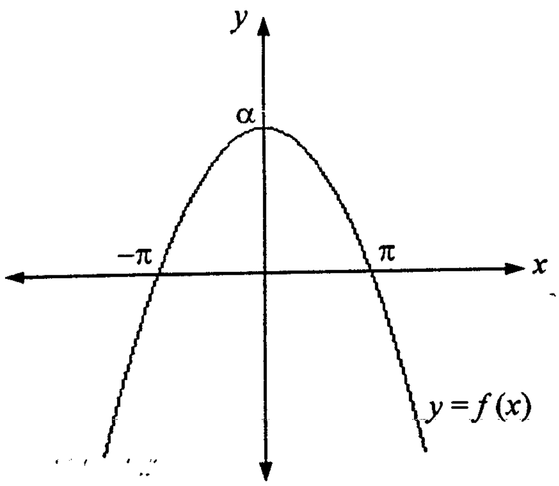
**Question Seven.** *Start a new page* **Marks**

(a) For k = 0 , 1 , 2 , 3 , 4 , ... , n, $P_k$ is defined as **3**

$$P_k = \binom{n}{k} a^k (1-a)^{n-k} \quad \text{, where } a \text{ is real and } n > 0.$$

Prove that $\sum_{k=0}^{n} P_k = 1$ .

(b) **6**




The diagram shows a parabola $y = f(x)$, with vertex $(0, \alpha)$ and passing through the points $(-\pi, 0)$ and $(\pi, 0)$.

If $a$ is the focal length of the parabola $y = f(x)$ ;

(i) show that $4a = \frac{\pi^2}{\alpha}$ .

(ii) show that $f(x)$ can be expressed as $f(x) = \alpha\left(1 - \frac{x^2}{\pi^2}\right)$ .

(iii) find the exact value of $\alpha$ given that the area between $y = f(x)$ and the $x$ axis is 4 square units.

(c) Assuming that $\sin(k\pi + \phi) = (-1)^k \sin\phi$ is true for some positive integer $k$, **3**

prove that

$$\sin[(k+1)\pi + \phi] = (-1)^{k+1} \sin\phi.$$

(1)(a) $\frac{\pi}{12}$

(b) $a = -1$

(c) $3\cot x$

(d) $35^0 16'$

(e) $P(10, -4)$

(2) (a) Proof

(b) (i) & (ii) Proofs

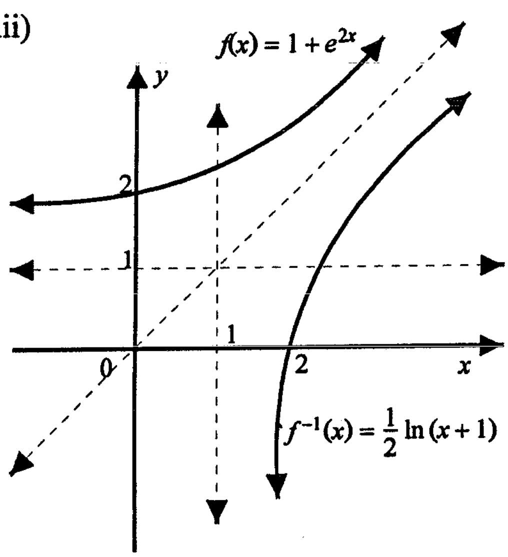
(c) $-960$

(3)(a) $x < -5$ and $0 < x < 5$

(b)(i) $D$ :All real $x$; $R : y > 1$

(ii) Proof

(iii)




(iv), (v) Proofs

(v) 0.396 (to 3 sig. fig.)

(4) (a) (i) 3600 ways

(ii) $\frac{1}{105}$

(b) (i) Proof

(c) (i) Proof

(ii) $\frac{\sqrt{3}}{9}$ rad/s

(5) (a) Proof

(b)(i) Origin is taken at the base of the tower

(ii), (iii) Proofs

(iv) $100 - 20\sqrt{5}$ m $\approx 55.3$ m

(v) $\dot{y} = -50$ m/s

(6)(a)(i), (ii) Proofs

(iii) $v = 4$ m/s

(b)(i) $R = \sqrt{13}$, $\alpha = 56^0 18'$

(ii) $\theta = 21^0 36'$ or $287^0 36'$

(7)(a) Proof

(b)(i), (ii) Proofs

(iii) $\alpha = \frac{3}{\pi}$

(c) Proof

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$