WESTERN REGION

2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (120)

- Attempt Questions 1-8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your name and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) Use the technique of integration by parts to find **2**

$$\int x^2 e^x dx$$

b) i) Use partial fractions to evaluate **3**

$$\int_0^1 \frac{5\,dt}{(2t+1)(2-t)}$$

ii) Hence, and by using the substitution $t = \tan\frac{\theta}{2}$, evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta}$$

c) By using the table of standard integrals and manipulation, find **2**

$$\int_0^1 \frac{dx}{\sqrt{4x^2+36}}$$

d) If $I = \int e^x \sin x\, dx$, find I. **3**

e) Without evaluating, explain why $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x\, dx = 0$. **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

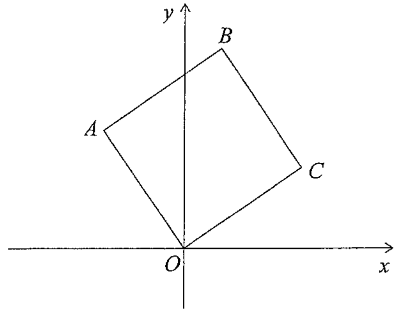
a) Given that A and B represent the complex numbers $\sqrt{3}+i$ and $1-\sqrt{2}i$ respectively, find:

i) $\frac{A}{B}$ in the form $x+iy$ **2**

ii) the modulus and the argument of A. **2**

iii) the complex number C in $x+iy$ form, given that $\arg C = 2\arg A$ and $|C| = 3|A|$. **3**

b)



On the Argand diagram above, $OABC$ is a square, with OC representing the complex number $4+3i$.

Write down the complex numbers represented by:

i) OA **1**

ii) OB **2**

iii) AC **2**

c) On an Argand diagram sketch the regions defined by the following, where Z is a complex number:

i) $1 \le |Z| \le 3$ **1**

ii) $\frac{\pi}{4} \le \arg Z \le \frac{\pi}{2}$ **1**

iii) $-1 \le \operatorname{Re}(Z) \le 2$ **1**

End of Question 2

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) i) On the same set of axes draw reasonably accurate graphs of $y = \ln x$ and $y = \sin x$ for $0 \le x \le 2\pi$. **2**

ii) By using your graphs from part (i), find an approximate solution for the equation $\ln x - \sin x = 0$. **1**

iii) Using one application of Newton's method, find a more accurate approximation for your answer to part (ii). **2**

iv) On the same set of axes as you used for part (i), sketch $y = \ln x + \sin x$ over the same domain. **2**

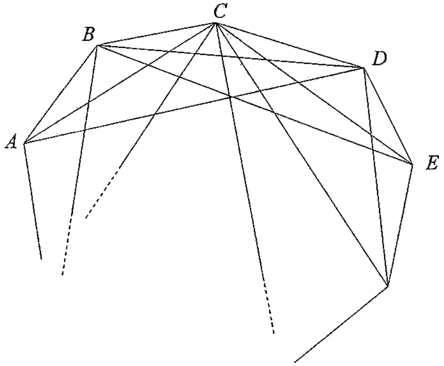
b) Given $f(x) = \ln x$, and considering your graph of $y = \ln x$ in part (a), draw separate sketches of:

i) $y = |f(x)|$ **1**

ii) $y = [f(x)]^2$ **1**

iii) $y = \frac{1}{f(x)}$ **2**

c) Prove, using Mathematical Induction, that the total number of diagonals in an n-sided polygon is given by $\frac{n(n-3)}{2}$. **4**



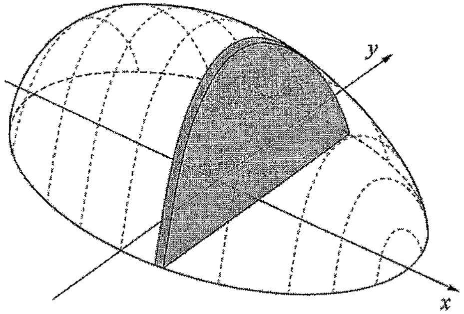
End of Question 3

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) Show that the area enclosed between the parabola $x^2 = 4ay$ and its latus rectum is $\frac{8a^2}{3}$ units^2. **3**

b) A solid figure has as its base, in the xy plane, the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Cross-sections perpendicular to the x-axis are parabolas with latus rectums in the xy plane.



i) Show that the area of the cross-section at $x = h$ is $\frac{16-h^2}{6}$ units^2. **3**

[Use your answer to part (a)]

ii) Hence, find the volume of this solid. **2**

c) i) Show that a reduction formula for $I_n = \int (\ln x)^n dx$ is **2**

$$I_n = x(\ln x)^n - I_{n-1}.$$

ii) Hence, or otherwise, evaluate $\int_1^e (\ln x)^3 \, dx$. **2**

d) Find an equation which has roots which are the squares of the roots of $3x^3 + 2x^2 + x - 6 = 0$. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $ax\sin\theta + by = \left(a^2+b^2\right)\tan\theta$. **3**

ii) If the normal in part (i) intersects the x-axis at A and the y-axis at B, find the coordinates of A and B. **2**

iii) Show that the coordinates of M, the midpoint of AB are given by **2**

$$x = \frac{1}{2a^2}\left(a^2+b^2\right)\sec\theta \quad \text{and} \quad y = \frac{1}{2b}\left(a^2+b^2\right)\tan\theta.$$

iv) Hence, find the equation of the locus of M in Cartesian form. **2**

v) If $a = b$, what can you say about the locus in part (iv)? **1**

b) Find the gradient of the tangent to the curve $x^2 - 3xy + 2y^2 = 3$ at the point $(5, 2)$. **2**

c) i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **1**

ii) Hence, or otherwise, solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) If $z = \cos\theta + i\sin\theta$,

i) Find the complex roots of $z^5 + 1 = 0$. **2**

ii) Hence express $z^5 + 1 = 0$ as the product of linear and quadratic factors. **2**

iii) Using your result for part (i), show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$. **2**

b) i) Given $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2\cos n\theta$. **1**

ii) Hence, express $\cos^6\theta$ in terms of $n\theta$. **3**

iii) Using your answer to part (ii), evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$. **2**

c) Over the complex field, the polynomial $P(x) = 2x^3 - 15x^2 + 18x - 13$ has a zero at $x = 3 - 2i$. **3**

Determine the other two zeros of $P(x)$.

End of Question 6

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) A mass of m kg is allowed to fall under gravity from a stationary position h metres above the ground. It experiences resistance proportional to the square of its velocity, $v\,\text{ms}^{-1}$.

i) Explain why the equation for this motion is $\ddot{x} = g - kv^2$, (where k is a constant). **1**

ii) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$ **1**

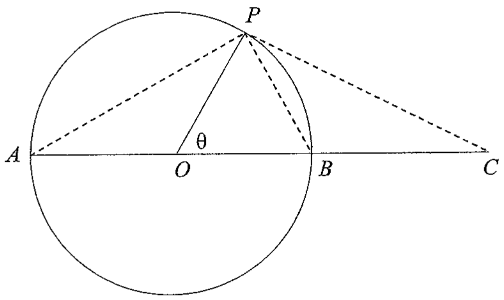
iii) Hence show that $v^2 = \frac{g}{k}\left(1 - e^{-kx}\right)$ **4**

iv) Find the velocity at which the mass hits the ground in terms of g, h and k. **1**

v) What would be the terminal velocity of this mass if it were allowed to fall without resistance? **1**

b) The diagram below shows a point P rotating in a circle of radius 1 metre, whose centre is at O.

AB is a diameter produced to C such that $OC = 2$ metres.



NOT TO SCALE

The angular velocity of P is given by $\dot{\theta} = \pi$ rad/sec. $(\theta = \angle POB)$

i) Find the angular velocity of P about A and about B. **2**

ii) If $\angle PCO = \alpha$, show that $\sin(\alpha + \theta) = 2\sin\alpha$. **1**

iii) Hence find the angular velocity of P about C at the instant when $\theta = \frac{\pi}{2}$. **4**

End of Question 7

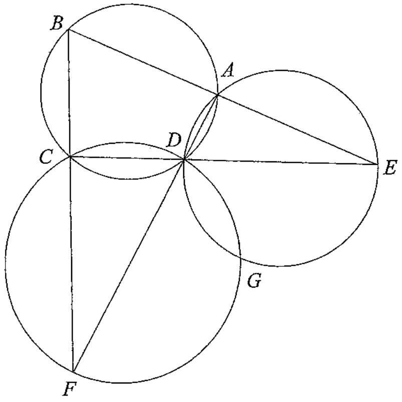
Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) A motorcyclist is riding inside a sphere of radius 10 metres. What must be his minimum speed if he is to successfully ride in a vertical circle passing through the top of the sphere? $(g = 10\,\text{ms}^{-2})$ **2**

b) If pqr represents a three digit number (i.e. $pqr = 100p + 10q + r$) and $p + q + r = 3A$, where A is a positive integer; show that the number pqr is also divisible by 3. **2**

c) $ABCD$ is a cyclic quadrilateral. BA and CD produced intersect at E. The circles EAD and FCD intersect at both G and D. BC and AD produced intersect at F. **3**

Prove that the points E, G and F are collinear. (Hint: join FG, EG and DG)



d) A triangle is formed by joining three points on the number plane. Verify that the coordinates of the centroid (the point of intersection of the medians of the triangle) is given by averaging the x-coordinates and averaging the y-coordinates of the vertices. **4**

e) Use the method of Mathematical Induction to prove De Moivre's theorem, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. **4**

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Mathematics
Extension 2

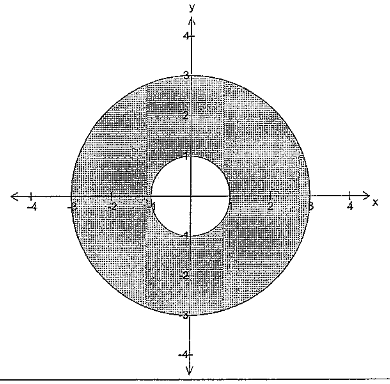
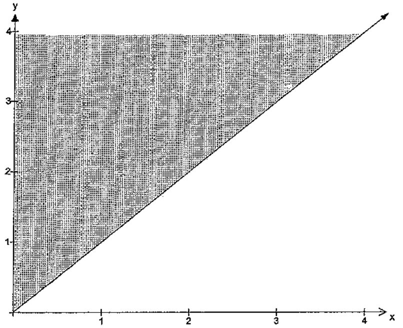
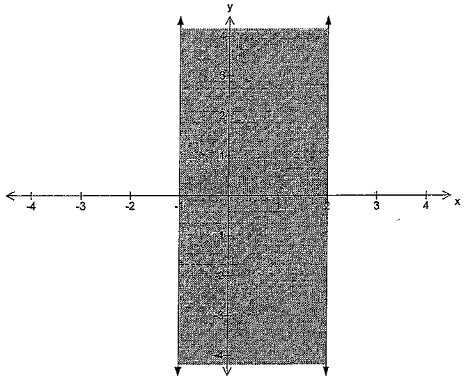
SOLUTIONS

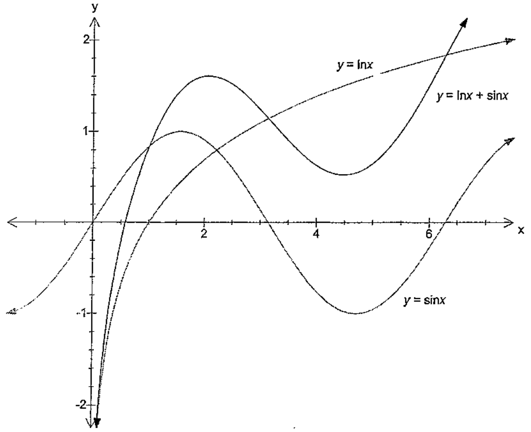
Question 1	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(a)	$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$ $= x^2e^x - \left[2xe^x - \int 2xe^x\,dx\right]$ $= x^2e^x - 2xe^x + 2e^x + c$ $= e^x\left(x^2 - 2x + 2\right) + c$	2	1 1
(b)	i) Let $\frac{A}{2t+1} + \frac{B}{2-t} = \frac{5}{(2t+1)(2-t)}$ $\therefore A(2-t) + B(2t+1) = 5$ Put $t = 2$, then $5B = 5 \to B = 1$ put $t = -\frac{1}{2}$, then $\frac{5}{2}A = 5 \to A = 2$ $\therefore \int_0^1 \frac{5dt}{(2t+1)(2-t)} = \int_0^1 \left(\frac{2}{2t+1} + \frac{1}{2-t}\right)dt$ $= \left[\ln(2t+1) - \ln(2-t)\right]_0^1$ $= \left[\ln\left(\frac{2t+1}{2-t}\right)\right]_0^1$ $= \ln 5 - \ln\left(\frac{1}{2}\right)$ $= \ln 10$	3	1 1 1

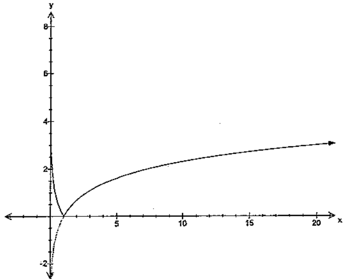
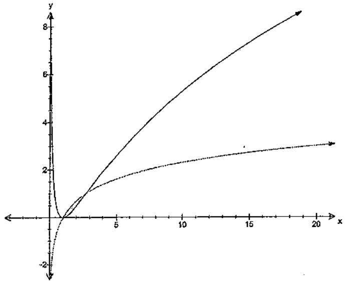
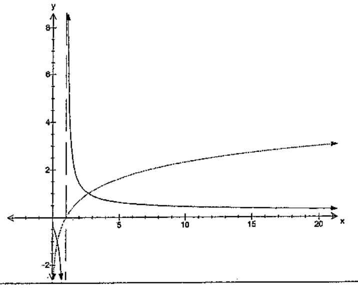
Question 1	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(b)			
ii)	$t=\tan\frac{\theta}{2} \quad \to \quad \frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$	3	
	$\frac{d\theta}{dt}=\frac{1+t^2}{2} \quad \to \quad \frac{dt}{d\theta}=\frac{2}{1+t^2}$		
	When $\theta=\frac{\pi}{2}$, $t=1$ and When $\theta=0$, $t=0$		1
	$\int_0^{\frac{\pi}{2}}\frac{d\theta}{3\sin\theta+4\cos\theta}=\int_0^1\frac{\frac{2}{1+t^2}}{\frac{6t}{1+t^2}+\frac{4(1-t^2)}{1+t^2}}\,dt$		
	$=\int_0^1\frac{dt}{3t+2-2t^2}$		1
	$=\frac{1}{5}\int_0^1\frac{5dt}{(2t+1)(2-t)}$		1
	$=\frac{1}{5}\ln 10$ [from (i)]		
(c)	$\int\frac{dx}{\sqrt{4x^2+36}}=\frac{1}{2}\int\frac{dx}{\sqrt{x^2+9}}$	2	1
	$=\frac{1}{2}\int\frac{dx}{\sqrt{x^2+9}}$		1
	$=\frac{1}{2}\ln\left(x+\sqrt{x^2+9}\right)+c$		

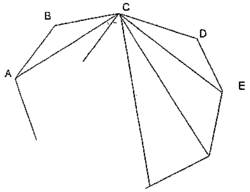
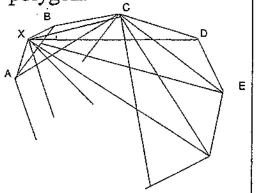
Question 1	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(d)	$I=\int e^x\sin x\,dx$	3	
	$=e^x\sin x-\int e^x\cos x\,dx$		1
	$=e^x\sin x-\left[e^x\cos x--\int e^x\sin x\,dx\right]$		
	$=e^x\sin x-e^x\cos x-\int e^x\sin x\,dx$		1
	$=e^x\left(\sin x-\cos x\right)-I$		1
	$2I=e^x\left(\sin x-\cos x\right)$		
	$I=\frac{1}{2}e^x\left(\sin x-\cos x\right)+c$		
(e)	As $y=\sin^5 x$ is an odd function, then	2	
	$\int_{-\frac{\pi}{2}}^{0}\sin^5 x\,dx=-\int_0^{\frac{\pi}{2}}\sin^5 x\,dx$		1 Other ways possible
	$\therefore\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^5 x\,dx=\int_{-\frac{\pi}{2}}^{0}\sin^5 x\,dx+\int_0^{\frac{\pi}{2}}\sin^5 x\,dx$		1
	$=0$		

Question 2	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(a) i)	$\frac{A}{B}=\frac{\sqrt{3}+i}{1-\sqrt{2}i}\times\frac{1+\sqrt{2}i}{1+\sqrt{2}i}$ $=\frac{\sqrt{3}+\sqrt{6}i+i-\sqrt{2}}{3}$ $=\frac{\sqrt{3}-\sqrt{2}}{3}+\frac{\sqrt{6}+1}{3}i$	2	1 1
ii)	$A=\sqrt{3}+i$ $\lvert A\rvert=\sqrt{3+1}=2 \ \therefore \text{mod}\,A=2$ $\therefore A=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ If $\arg A=\theta$ $\cos\theta=\frac{\sqrt{3}}{2}\sin\theta=\frac{1}{2}$ $\therefore\theta^2=\frac{\pi}{6}$ $\therefore \arg A=\frac{\pi}{6}$	2	1 1
iii)	$\arg C=2\arg A$ $\therefore \ \arg C=\frac{\pi}{3}$ $\text{mod}\,C=3\,\text{mod}\,A$ $=3\times 2$ $=6$ $C=6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ $=6\left(\frac{1}{2}+i\sqrt{\frac{3}{2}}\right)$ $=3+i3\sqrt{3}$	3	1 1 1
(b) i)	$OA=iOC$ $=i(4+3i)$ $=-3+4i$	1	
ii)	$OB+OA=(4+3i)+(-3+4i)$ $=1+7i$	1	
(b) iii)	$AC=OC-OA$ $=(-3+4i)-(4+3i)$ $=-7+i$	2	1 mark for wrong subtraction

Question 2	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(c) i)		1	
(c) ii)		1	
(c) iii)		1	

Question 3	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(a) i) & iv)		2	1 for sine graph 1 for log graph
(a) ii)	For $ln\,x - sin\,x = 0$ $x \approx 2.2$	1	
iii)	$f(2.2) = \ln(2.2) - \sin(2.2) = -0.020$ $f'(x) = \frac{1}{x} - \cos x$ $f'(2.2) = \frac{1}{2.2} - \cos(2.2) = 1.043$ $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$ $= 2.2 - \frac{-0.020}{1.043}$ $= 2.22$ (to 2 d.p.)	2	1 for use of Newtons Method 1 for evaluating
iv)	See graph above.	2	1 for general shape 1 for location of t.p. and intercept.

Question 3	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(b) i)		1	
ii)		1	
iii)		2	1 for curve x>1 1 for asymptote and curve 0< x <1

Question 3	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(c)	When $n=3$, a 3-sided polygon (triangle) has no diagonals. and $\frac{n(n-3)}{2}=\frac{3\times 0}{2}=0$		1
	Assume true for $n=k$ i.e. a k-sided polygon has $\frac{k(k-3)}{2}$ diagonals. k-sided polygon  $k+1$-sided polygon obtained by adding a vertex, X, between A and B in a k-sided polygon.  We can join X to every other existing vertex of the k-sided polygon, but two of these are the sides AX and BX, so there are $k-2$ new diagonals. There is also a new diagonal joining AB so there are a total of $k-1$ new diagonals. The $k+1$-sided polygon has all the diagonals of the k-sided polygon plus the extra $k-1$ diagonals.		2 marks for any reasonable case being put for the number of diagonals when $n=k+1$
	The number of diagonals $=\frac{k(k-3)}{2}+k-1$ $=\frac{k(k-3)+2k-2}{2}$ $=\frac{k^2-k-2}{2}$ $=\frac{(k+1)(k-2)}{2}$ $=\frac{(k+1)(k+1-3)}{2}$ $\therefore$ if true for $n=k$ it is also true for $n=k+1$ Since true for $n=3$, by induction it is true for all polygons with $n\geq 3$		1 for k+1 formula and conclusion

Question 4	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(a)	End points of latus rectum are $(2a,a)$ and $(-2a,a)$ Area $=4a^2-\int_{-2a}^{2a}\frac{x^2}{4a}\,dx$	3	1
	$=4a^2-\frac{1}{12a}\left[x^3\right]_{-2a}^{2a}$ $=4a^2-\frac{1}{6a}\left[x^3\right]_{0}^{2a}$		1
	$=4a^2-\frac{8a^2}{6}$ $=\frac{8a^2}{3}unit^2$		1
(b) i)	$a+x=h$ $\frac{y^2}{4}=1-\frac{h^2}{16}$ $y^2=4-\frac{h^2}{4}$	3	1
	$y=\pm 2\sqrt{1-\frac{h^2}{16}}$ For parabola $2a=2\sqrt{1-\frac{h^2}{16}}$		1
	$a=\sqrt{1-\frac{h^2}{16}}$ $\therefore Area=\frac{8}{3}\left(1-\frac{h^2}{16}\right)$ $=\frac{16-h^2}{6}units^2$		1

Question 4	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(b) ii)	$\partial V = \frac{16-h^2}{6}\partial h$ $V = \lim_{x\to\infty}\sum_{h=-4}^{4}\frac{16-h^2}{6}\ \partial h$ $= 2\int_0^4 \frac{16-h^2}{6}\,dh$ $= \frac{1}{3}\left[16h - \frac{h^3}{3}\right]_0^4$ $= \frac{1}{3}\left[\left(6-\frac{64}{3}\right)-0\right]$ Volume $= \frac{128}{9}$ units^3 $\left(14\frac{2}{9}\right)$	2	1 1
(c) i)	$(i) I_n = \int (\ln x)^n\,dx$ $\therefore I_n = x(\ln x)^n - \int x(\ln x)^{n-1}\,dx$ $= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$ $= x\ln^n x - nI_{n-1}$	2	1 1
(c) ii)	$I_3 = \int_1^e \ln^3 x\,dx$ $= \left[x\ln^3 x\right]_1^e - 3\int_1^e \ln^2 x\,dx$ $= \left[x\ln^3 x - 3\left(x\ln^2 x\right)\right]_1^e + 6\int_1^e \ln^1 x\,dx$ $= \left[x\ln^3 x - 3x\ln^2 x + 6x\ln x\right]_1^e - 6\int_1^e \ln^0 x\,dx$ $= \left[x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x\right]_1^e$ $= (e-3e+6e-6e)-(-6)$ $= 6-2e$	2	1 1

Question 4	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(d)	$3x^3+2x^2+5x-6=0$ Let A $= x^2$ $\sqrt{A} = x$ $\therefore 3A\sqrt{A}+2A+5\sqrt{A}-6=0$ $\sqrt{A}(3A+5) = 6-2A$ Squaring gives… $A\left(9A^2+30A+25\right) = 36-24A+4A^2$ $9A^3+30A^2+25A = 36-24A+4A^2$ $\therefore 9A^3+26A^2 49A-36=0$ $\therefore$ Required equation is… $9x^3+26x^2+49x-36=0$	3	1 1 1

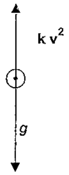
Question 5	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(a) i)	$\frac{x^2}{a^2} - \frac{y^2}{6^2} = 1$ $\therefore \frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ $\therefore$ Gradient of normal $= -\frac{a^2 y}{b^2 x}$ at P $= \frac{-a^2 b \tan\theta}{b^2 a \sec\theta}$ $= -\frac{a}{b}\sin\theta$ $\therefore$ Eq'n is $y - b\tan\theta = -\frac{a}{b}\sin\theta\left(x - a\sec\theta\right)$ $by - b^2\tan\theta = ax\sin\theta + a^2\sin\theta\sec\theta$ $= -ax\sin\theta + a^2\tan\theta$ $\therefore\ ax\sin\theta + by = \left(a^2 + b^2\right)\tan\theta$	3	1 1 1
ii)	At A, $y = 0$ $\therefore x = \frac{\left(a^2+b^2\right)}{a\sin\theta}\tan\theta$ At B, $x = 0$ $\therefore y = \frac{a^2+b^2}{b}\tan\theta$	2	1 1

Question 5	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(a) iii)	Coords of M $x = \frac{1}{2}\left[\left(\frac{a^2+b^2}{a\sin\theta}\right)\tan\theta + 0\right]$ $= \frac{\left(a^2+b^2\right)\tan\theta}{2a\sin\theta}$ $= \frac{\left(a^2+b^2\right)\frac{\sin\theta}{\cos\theta}}{2a\sin\theta}$ $x = \frac{\left(a^2+b^2\right)}{2a}\sec\theta$ $y = \frac{1}{2}\left[\frac{\left(a^2+b^2\right)}{b}\tan\theta + 0\right]$ $y = \left(\frac{a^2+b^2}{2b}\right)\tan\theta$	2	1 1
(a) iv)	From (iii) $\tan\theta = \frac{2by}{a^2+b^2} + \sec\theta = \frac{2ax}{a^2+b^2}$ as $\sec^2\theta - \tan^2\theta = 1$ $\therefore \frac{4a^2x^2}{\left(a^2+b^2\right)^2} - \frac{4b^2y^2}{\left(a^2+b^2\right)^2}$ $\therefore\ 4a^2x^2 - 4b^2y^2 = \left(a^2+b^2\right)^2$	2	1 1
(a) v)	If $a = b$, $4a^2x^2 - 4a^2y^2 = 4a^4$ $x^2 - y^2 = a^2$ $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ Which would be the original hyperbola when $a = b$ (OR by putting $a = b$ in parametric equations for x & y)	1	

Question 5	Trial HSC Examination- Mathematics Extension 2		2007
Part	Solution	Marks	Comment
(b)	$x^2 - 3xy + y^2 = 3$	2	
	$\therefore\ 2x - \left(3y + 3x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$		
	$(2y - 3x)\frac{dy}{dx} = 3y - 2x$		
	$\frac{dy}{dx} = \frac{3y - 2x}{2y - 3x}$		1
	At (5, 2) $\frac{dy}{dx} = \frac{15-4}{10-6} = \frac{11}{4}$		
	$\therefore$ Gradient of tangent $= \frac{11}{4}$		1
(c) i)	$\sin x + \sin 3x$	1	
	$= \sin(2x - x) + \sin(2x + x)$		
	$= \sin 2x \cos x - \cos 2x \sin x + \sin 2x \cos x + \cos 2x \sin x$		
	$= 2\sin 2x \cos x$		
ii)	$\sin x + \sin 2x + \sin 3x = 0$	2	
	$2\sin 2x \cos x + \sin 2x = 0$		1
	$\sin 2x(2\cos x + 1) = 0$		
	$\therefore \sin 2x = 0\ ,\quad \cos x = -\frac{1}{2}$		
	$2x = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi$		
	$x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$		1

Question 6	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(a) i)	$z^5 + 1 = 0$	2	
	$z^5 = -1$		
	By De Moivre's Theorem,		
	$\cos 5\theta + i\sin 5\theta = -1$		
	Equating real parts gives…		
	$\cos 5\theta = -1$		
	$5\theta = \pi,\ 3\pi,\ 5\pi,\ 7\pi,\ 9\pi$		1
	$\theta = \frac{\pi}{5},\ \frac{3\pi}{5},\ \pi,\ \frac{7\pi}{5},\ \frac{9\pi}{5}$		
	$\therefore$ The roots are: $z_1 = cis\frac{\pi}{5};\quad z_2 = cis\frac{3\pi}{5},\quad z_3 = cis\pi = -1,$		
	$z_4 = cis\frac{7\pi}{5} = \overline{z}_2,\quad z_5 = cis\frac{9\pi}{5} = \overline{z}_1$		
			1
ii)	$z^5 + 1 = (z - z_3)\left[(z - z_2)(z - z_4)\right]\left[(z - z_1)(z - z_5)\right]$	2	1
	$= (z+1)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)\left(z^2 \mp 2z\cos\frac{\pi}{5} + 1\right)$		1
iii)	Sum of the roots = 0	2	
	$\therefore\ -1 + cis\left(\frac{\pi}{5}\right) + cis\left(\frac{3\pi}{5}\right) + cis\left(\frac{-\pi}{5}\right) + cis\left(\frac{-3\pi}{5}\right) = 0$		1
	$-1 + 2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 0$		
	$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 1$		
	$\therefore\ \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$		1
(b) i)	$z^n = \cos n\theta + i\sin n\theta$	1	
	$z^{-n} = \cos n\theta - i\sin n\theta$		
	$z^n + z^{-n} = 2\cos n\theta$		

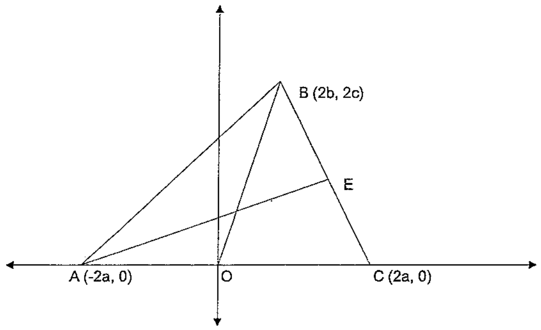
Question 6	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(b) ii)	$\left(z+\frac{1}{z}\right)^6=(2\cos n\theta)^6$ $=64\cos^6\theta$ Also, $\left(z+\frac{1}{z}\right)^6=z^6+6z^5\frac{1}{z}+15z^4\frac{1}{z^2}+20z^3\frac{1}{z^3}+15z^2\frac{1}{z^4}+6z\frac{1}{z^5}+\frac{1}{z^6}$ $64\cos^6\theta=\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$ $=2\cos6\theta+12\cos4\theta+30\cos2\theta+20$ $\cos^6\theta=\frac{1}{32}\cos6\theta+\frac{3}{16}\cos4\theta+\frac{15}{32}\cos2\theta+\frac{5}{16}$	3	1 1 1
(b) iii)	$\int_0^{\frac{\pi}{6}}\cos^6\theta d\theta=\frac{1}{32}\int_0^{\frac{\pi}{6}}(\cos6\theta+6\cos4\theta+15\cos2\theta+10)d\theta$ $=\frac{1}{32}\left[\frac{1}{6}\sin6\theta+\frac{3}{2}\sin4\theta+\frac{15}{2}\sin2\theta+10\theta\right]_0^{\frac{\pi}{6}}$ $=\frac{1}{32}\left[\left(0+\frac{3}{2}\times\frac{\sqrt{3}}{2}+\frac{15}{2}\times\frac{1}{2}+10\frac{\pi}{6}\right)-(0)\right]$ $=\frac{1}{32}\left(\frac{3\sqrt{3}+15}{4}+\frac{5\pi}{3}\right)$	2	1 1
(c)	$P(x)=2x^3-15x^2+18x-13$ If $x=3-2i$ is a root of $P(x)=0$, then $\bar{x}=3+2i$ is also a root. Let K be the third root. Product of the roots $=(3-2i)(3+2i)K=\frac{13}{2}$ $(9+4)K=\frac{13}{2}$ $13K=\frac{13}{2}$ $K=\frac{1}{2}$ The zeros are $(3-2i)$, $(3+2i)$ and $\frac{1}{2}$	2	1 Any other valid method okay to find 3rd root. 1

Question 7	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(a) ✓ i)	Resistance $=kv^2$ this is acting against the gravity g $\therefore\ x=g-kv^2$ 	1	
ii)	$\frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}=v\frac{dv}{dx}$ $=\frac{dx}{dt}\times\frac{dv}{dx}$ $=\frac{dv}{dt}$ $=\ddot{x}$	1	
iii)	$v\frac{dv}{dx}=g-kv^2$ $\frac{dv}{dx}=\frac{g-kv^2}{v}$ $\frac{dx}{dv}=\frac{v}{g-kv^2}$ $\therefore x=-\frac{1}{k}\ln\left(g-kv^2\right)+c$ When $x=0,\ v=0$ $\therefore c=\frac{1}{k}\ln g$ $\therefore x=\frac{1}{k}\ln g-\frac{1}{k}\ln\left(g-kv^2\right)$ $x=\frac{1}{k}\ln\left(\frac{g}{g-kv^2}\right)$ $e^{kx}=\frac{g}{g-kv^2}$ $ge^{kx}-kv^2e^{kx}=g$ $\therefore v^2=\frac{g\left(e^{kx}-1\right)}{ke^{kx}}$ or $\frac{g\left(1-e^{-kx}\right)}{k}$	4	1 1 1 1

Question 7	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(a) iv)	When $x=h$ $v^2=\frac{g\left(1-e^{-kh}\right)}{k}$ $\therefore v=\sqrt{\frac{g}{k}\left(1-e^{-kh}\right)}$	1	
v)	Terminal velocity when $\ddot{x}=g-kv^2=0$ $kv^2=g$ $v=\sqrt{\frac{g}{k}}$ Terminal Velocity $=\sqrt{\frac{g}{k}}$	1	
(b) i)	$Let\angle OAP=\phi$ and $\angle OPB=\gamma$ $\therefore\ \phi=\frac{1}{2}\theta$ ($\angle$ at centre twice angle at circumference) $\dot{\phi}\ =\frac{1}{2}\dot{\theta}\ =\frac{\pi}{2}$ Angular velocity about A is $\frac{\pi}{2}$rad/sec $\gamma=\frac{\pi}{2}-\theta\ \left(\angle APB=\frac{\pi}{2},\ \angle\text{ in semicircle, angle sum of }\Delta\right)$ $\therefore\dot{\gamma}\ =-\dot{\phi}\ =-\frac{\pi}{2}$ Angular velocity about B is $\frac{-\pi}{2}$ rad/sec	2	1 1
ii)	If $\angle PCO=\alpha$ $\angle OPC=\pi-(\alpha+0)$ By Sine Rule, $\frac{\sin\alpha}{1}=\frac{\sin\left(\pi-(\alpha+\theta)\right)}{2}$ (As $\sin(\pi-a)=\sin A$) $2\sin\alpha=\sin(\alpha+\theta)$	1	

Question 7	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(b) iii)	Differentiate with respect to t $2\cos\alpha\times\dot{\alpha}\ =\cos(\alpha+\theta)\times\left(\dot{\alpha}+\dot{\theta}\right)$ $\left[2\cos\alpha-\cos(\alpha+\theta)\right]\dot{\alpha}\ =\dot{\theta}\cos(\alpha+\theta)$ $\therefore\dot{\alpha}\ =\frac{\dot{\theta}\cos(\alpha+\theta)}{2\cos\alpha-\cos(\alpha+\theta)}$ When $\theta=\frac{\pi}{2}$ and $\dot{\theta}=\pi$ $\cos\alpha=\frac{2}{\sqrt{5}}$ $\sin\alpha=\frac{1}{\sqrt{5}}$ $\dot{\alpha}\ =\frac{\pi\cos\left(\alpha+\frac{\pi}{2}\right)}{2\cos\alpha-\cos\left(\alpha+\frac{\pi}{2}\right)}$ $=\frac{\pi.(-\sin\alpha)}{2\cos\alpha+\sin\alpha}$ $=\frac{-\pi\times\frac{1}{\sqrt{5}}}{\frac{4}{\sqrt{5}}+\frac{1}{\sqrt{5}}}=-\frac{\pi}{5}$ Angular Velocity about B is $-\frac{\pi}{5}$ rad/sec	4	1 1 1 1

Question 8	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(a)	At the top of the sphere $\frac{mv^2}{r} \geq mg$ $\therefore v^2 \geq Rg$ $\geq 10\times 10$ ≥ 100 $\therefore \ v \geq 10$ ∴ He must have a speed of at least 10 ms^{-1}	2	1 1
(b)	$\alpha\beta\gamma = 100\alpha + 10\beta + \gamma$ $= 99\alpha + 9\beta + \alpha + \beta + \gamma$ $= 3(11\alpha + 3\beta) + 3A$ $= 3(11\alpha + 3\beta + A)$ Which is divisible by 3	2	1 1
(c)	Let $\angle BCD = x°$ and $\angle BAD = y°$ $\angle DGF = \angle BCD$ (exterior $\angle$of cyclic quad CDGF) $= x°$ $\angle DGE = \angle BAD$ (exterior $\angle$of cyclic quad AEGD) $= y°$ $\angle FGE = \angle DGF + \angle DGE = x° + y°$ $=180°$ (Opposite angles of cyclic quad ABCD) $\therefore F, G$ and E are collinear.	3	Any correct reasoning receives 3 marks. 2 if simple error or if not quite finished, 1 if major error or start made but not followed through.

Question 8	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(d)	Let the triangle be as shown  Midpoint of AC = O (0, 0) Midpoint of BC = E $(a+b,\ c)$ Equation of OB $y=\frac{c}{b}x$ Equation of AE $\frac{y-0}{x--2a}=\frac{c-0}{3a+b}$ $y=\frac{c}{3a+b}(x+2a)$ Solving simultaneously… $\frac{c}{b}x=\frac{c}{3a+b}x+\frac{2ac}{3a+b}$ $\left(\frac{c}{b}-\frac{c}{3a+b}\right)x=\frac{2ac}{3a+b}$ $\frac{c(3a+b-b)}{b(3a+b)}x=\frac{2ac}{3a+b}$ $\frac{3a}{b}x=2a$ $x=\frac{2b}{3}$ $\therefore\ y=\frac{c}{b}.\frac{2b}{3}$ $=\frac{2c}{3}$ ∴ Average of x coordinates $=\frac{2b+2a-2a}{3}=\frac{2b}{3}$ Average of y coordinates $=\frac{2c+0+0}{3}=\frac{2c}{3}$	4	1 1 1 1

Question 8	Trial HSC Examination- Mathematics Extension 2	2007	
Part	Solution	Marks	Comment
(e)	When $n=1$ $\left(\cos\theta+i\sin\theta\right)^1=\cos\theta+i\sin\theta$ Assume true for $n=k$ $\therefore\left(\cos\theta+i\sin\theta\right)^k=\cos k\theta+i\sin k\theta$ When $n=k+1$ $\left(\cos\theta+i\sin\theta\right)^{k+1}=\left(\cos\theta+i\sin\theta\right)^k\left(\cos\theta+i\sin\theta\right)$ $=\left(\cos k\theta+i\sin k\theta\right)\left(\cos\theta+i\sin\theta\right)$ $=\cos k\theta\cos\theta+i\sin\theta\cos k\theta+i\sin k\theta\cos\theta-\sin k\theta\sin\theta$ $=\cos k\theta\cos\theta-\sin k\theta\sin\theta+i\left(\sin k\theta\cos\theta+\sin\theta\cos k\theta\right)$ $=\cos\left(k\theta+\theta\right)i\sin\left(k\theta+\theta\right)$ $=\cos\left(k+1\right)\theta+i\sin\left(k+1\right)\theta$ $\therefore$ if true for $n=k$, is also true for $n=k+1$ But since true for $n=1$, by induction is true for all positive integer $n\geq 1$	4	1 1 1 1