WESTERN REGION

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time- 5 minutes
- Working Time – 2 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**84**)

- Attempt Questions 1-7
- All questions are of equal value

Question 1 (12 Marks) Begin a new booklet. **Marks**

(a) Divide the interval from A (-3, 6) to B (12, -4) in the ratio 2:3 **2**

(b) Find the value of $\sin 105^\circ$ in simplest exact form **2**

(c) Solve the inequality $\frac{4}{1-x} \leq 3$ and graph your solution on the number line **3**

(d) Use the substitution $u = \cos x$, to find $\int \cos^2 x \sin x dx$ **2**

(e) Find $\lim_{x \to \infty} \frac{3x^3 - 2x}{x^2 + 4}$ **1**

(f) Find the acute angle between the lines: $x - \sqrt{3}y + 1 = 0$ **2**

$y = x - 4$

End of Question 1

Question 2 (12 Marks) Begin a new booklet. **Marks**

(a) i) Show that $x-2$ is a factor of x^3-4x^2+7x-6 **1**

ii) Show why $x^3-4x^2+7x-6=0$ has only 1 real root. **2**

(b) (i) Prove $\dfrac{1-\cos 2x}{\sin 2x}=\tan x$ **1**

(ii) Hence express $\tan 15^\circ$ in simplest exact form. **1**

(c) Find the value of $\displaystyle\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}}\frac{dx}{x^2+4}$ **3**

(d) How many distinct permutations of the letters of the word **ARRANGE** are possible

(i) In a straight line **1**

(ii) In a straight line when the "word" begins and ends with the letter R. **1**

(ii) In a circle **2**

End of Question 2

Question 3 (12 Marks) Begin a new booklet. **Marks**

(a) Determine the exact value of $\displaystyle\int_2^3\frac{x\,dx}{x^2-2}$ **3**

(b) State the domain and range of $y=\cos^{-1}\left(\dfrac{3x}{2}\right)$ **2**

(c) If we take $t=\tan\dfrac{\theta}{2}$ then $\tan\theta=\dfrac{2t}{1-t^2}$ **3**

Use the t results or otherwise to obtain θ correct to the nearest minute

when $\dfrac{7\sin\theta}{2}+2\cos\theta=4$

(d) A tower CX is observed at an angle of elevation 14° from a point A on level ground. The same tower is observed from B, 1 km from A, with an angle of elevation 17°. $A\hat{C}B=120^\circ$. C is the base of the tower.

(i) Draw a diagram showing this information. **1**

(ii) Calculate h, the height of the tower CX. (nearest m) **3**

End of Question 3

Question 4 (12 Marks) Begin a new booklet. **Marks**

(a) The polynomial equation $x^3 - 5x^2 + 7x + 5 = 0$ has 3 roots, α, β, γ

(i) Find $\alpha + \beta + \gamma$ **1**

(ii) Find $\alpha\beta + \beta\gamma + \gamma\alpha$ **1**

(iii) Find $\alpha^2 + \beta^2 + \gamma^2$ **2**

(b) (i) Express $\sqrt{3}\sin 2\theta - \cos 2\theta$ in the form $R\sin(2\theta - \alpha)$, α acute. **2**

(ii) Hence solve $\sqrt{3}\sin 2\theta - \cos 2\theta = 1$; $0 \le \theta \le \pi$. Answer in exact form. **2**

(c) Newton's Law of Cooling states that the rate of change of temperature of a body is proportional to the difference between the temperature of the body and its surrounds.

$$\frac{dT}{dt} = -k(T - D) \quad \text{where } D \text{ is the surrounding temperature}$$

(i) Show that $T = D + Ce^{-kt}$ satisfies Newton's Law. **1**

(ii) An ingot of Aluminium has an initial temperature of $1350^\circ C$ **3**

After 10 minutes in an environment at $25^\circ C$ its temperature has fallen to $720^\circ C$. What total time elapses for the ingot to cool to $50^\circ C$

End of Question 4

Question 5 (12 Marks) Begin a new booklet. **Marks**

(a) The function $f(x) = x^4 - 5x^3 + 11x^2 - 12x + 6$

This function has only 1 minimum near $x = 1.3$

(i) Use one application of Newton's Method to obtain a better approximation to the x value **of this minimum.** **3**

(ii) Justify why $f(x) = 0$ has no roots **1**

(b) How many times should a die be thrown so that the probability of throwing an even number is greater than 0.99? **2**

(c) A cube, side s, is growing at a rate of 200cm^3 per second. **3**

At what rate is the surface area growing at the moment when $s = 15$cm?

(d) Prove by Mathematical Induction that, **3**

$(n)^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all positive whole numbers n

End of Question 5

Question 6 (12 Marks) Begin a new booklet. **Marks**

(a) Consider the function $h(x)=\dfrac{3x}{1-x^2}$ for which $\lim\limits_{x\to\infty}\dfrac{3x}{1-x^2}=0$

(i) Describe the domain of $h(x)$ **1**

(ii) Find $h(-2)$ and $h(2)$ **1**

(iii) Show why $h(x)$ has no turning points **1**

(iv) Sketch $h(x)$ showing the important features **2**

(b) Find the volume generated when $y=\sec x$ **2**

between $x=0$ and $x=\dfrac{\pi}{3}$ is rotated around the x axis.

Express your answer in simplest exact form.

(c) (i) Prove that $\dfrac{d^2x}{dt^2}=\dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$ **1**

(ii) A particle is moving in a straight line with $v^2=36-4x^2$

α Prove the particle is undergoing SHM **1**

β What is the amplitude and period of the motion? **2**

γ If the particle is initially at the origin, write an expression for its displacement in terms of t **1**

End of Question 6

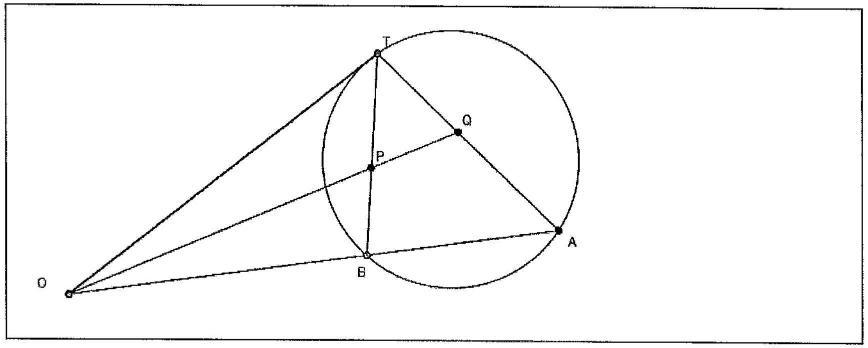
Question 7 (12 Marks) Begin a new booklet. **Marks**

(a) AB is a chord in a circle. AB is produced to O outside the circle. **3**

From O. the tangent OT is drawn to the circle.

The bisector of $T\hat{O}B$ Meets TB at P and TA at Q.

Prove ΔTPQ is isosceles



(b) A rock is hurled from the top of a 15m cliff with an initial velocity of 26ms^{-1}

at an angle of projection equal to $\tan^{-1}\left(\dfrac{5}{12}\right)$ above the horizontal.

The cliff overlooks a flat paddock.

The equations of motion of the stone are $\ddot{x}=0$ and $\ddot{y}=-10$

(i) Taking the origin as the base of the cliff, show the components of the rock's displacement are, $x=24t$ and $y=-5t^2+10t+15$ **2**

(ii) Calculate the time until impact with the paddock, and the distance of the impact from the base of the cliff. **2**

Question 7 continues on page 9

Question 7 (12 Marks) continued **Marks**

(c) The parabola $x^2 = 4y$ is shown in the diagram.

The point P has coordinates $(2t, t^2)$. S(0, 1) is the focus and

M is the foot of the perpendicular from P on the directrix. MP is produced to A.

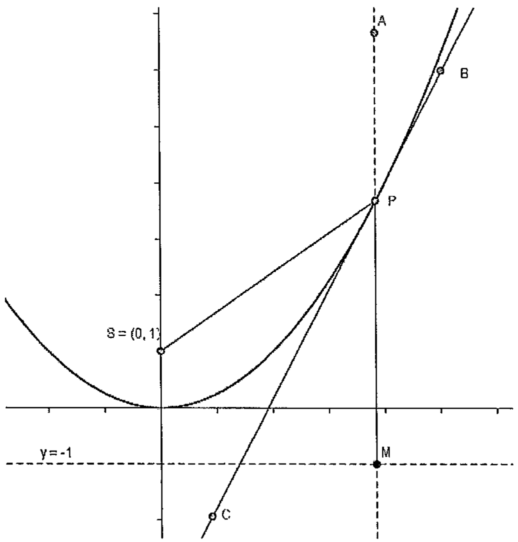
BPC is tangent to the parabola at P.

(i) Find the length PS and PM. Describe ΔPSM **2**

(ii) Find the gradient of the tangent BPC and of SM. **2**

What is true about the tangent and SM?

(iii) Prove $A\hat{P}B = S\hat{P}C$ **1**



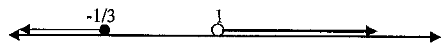
End of Examination

Western Region

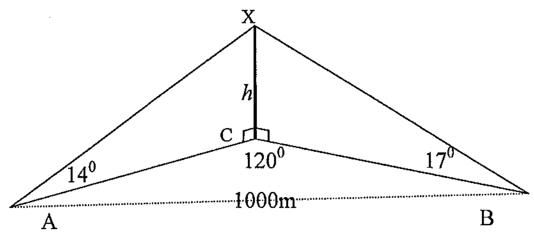
2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Solutions

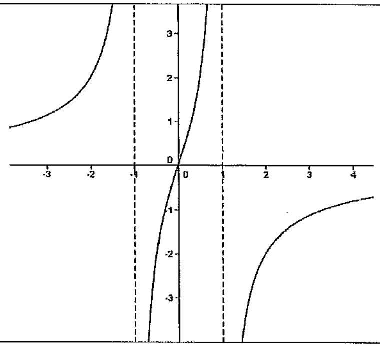
Solutions Question 1 2009	Marks/Comments
a. $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)=\left(\frac{2\times12+3\times-3}{5},\frac{2\times-4+3\times6}{5}\right)$ $=(3,2)$	1 1
b. $\sin(60+45)=\sin60\cos45+\cos60\sin45$ $=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}$	1 1
c. $x\neq1$, $4=3-3x$, $x=-\frac{1}{3}$ are critical points test $x=$ -1TRUE 4/2<3 test $x=0$ FALSE 4/1 > 3 test $x=2$ TRUE 4/-1<3 solution is $x\le-\frac{1}{3}$ *or* $x>1$ 	1 **Or by multiplying by $(1-x)^2$** **1 pay 2 for any legitimate** **1 must have open circle on x = 1**
d. $u=\cos x$ $\frac{du}{dx}=-\sin x$ $du=-\sin x dx$ $I=-\int u^2du=-\frac{1}{3}u^3=-\frac{1}{3}\cos^3x+c$	1 **1 ignore constant of integration**
e. $L=\lim_{x\to\infty}\frac{\frac{3x^3}{x^2}-\frac{2x}{x^2}}{1+\frac{4}{x^2}}=\lim_{x\to\infty}\frac{3x-\frac{2x}{x^2}}{1+\frac{4}{x^2}}=\lim_{x\to\infty}3x=\infty$	**Or divide by x^3** 1
f $m_2=\frac{1}{\sqrt{3}}$, $m_1=1$ $\tan\theta=\left\|\frac{m_2-m_1}{1+m_2m_1}\right\|$ $=\left\|\frac{\frac{1}{\sqrt{3}}-1}{1+\frac{1}{\sqrt{3}}}\right\|$ $=\left\|\frac{1-\sqrt{3}}{\sqrt{3}+1}\right\|$ $\theta=15^o$	1 1 **/12** **pay 1 for successful sub in formula 2 if correct conclusion**

Solutions Question 2 2009	Marks/Comments
a.(i) x^3-4x^2+7x-6 $P(2)=0 \quad 2^3-4\times 2^2+14-6=0$ $\therefore (x-2)$ is a factor (ii)Dividing $\dfrac{x^3-4x^2+7x-6}{x-2}=x^2-2x+3$ a quadratic with negative discriminant and no real roots	1 1 1
b (i) $\dfrac{1-\cos 2x}{\sin 2x}=\dfrac{1-(\cos^2 x-\sin^2 x)}{2\cos x\sin x}=\dfrac{1-(1-\sin^2 x-\sin^2 x)}{2\cos x\sin x}$ $=\dfrac{2\sin^2 x}{2\cos x\sin x}=\dfrac{\sin x}{\cos x}=\tan x$ (ii) $\dfrac{1-\cos 30^\circ}{\sin 30^\circ}=\dfrac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\dfrac{2-\sqrt{3}}{1}=2-\sqrt{3}$	 1 1
c. $\left[\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}\right]_{2/\sqrt{3}}^{2\sqrt{3}}=\dfrac{1}{2}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\dfrac{1}{\sqrt{3}}\right)=\dfrac{1}{2}\left(\dfrac{\pi}{3}-\dfrac{\pi}{6}\right)=\dfrac{\pi}{12}$	**3 1 per step**
d. (i) $\dfrac{7!}{2!2!}=1260$ (ii) $\dfrac{5!}{2!}=60$ (iii) Fix the E or any of the single letters $\dfrac{6!}{2!2!}=180$	1 1 2 **/12**

Solutions Question 3 2009	Marks/Comments
a. $\displaystyle\int_2^3 \frac{x\,dx}{x^2-2}=\left[\frac{1}{2}\ln(x^2-2)\right]_2^3=\left[\ln\sqrt{x^2-2}\right]_2^3=\ln\sqrt{7}-\ln\sqrt{2}$ $=\ln\left(\dfrac{\sqrt{7}}{\sqrt{2}}\right)$	**3 lose 1 per error** **Any equivalent exact value is okay.**
b. $-1\le\dfrac{3x}{2}\le 1 \qquad -\dfrac{2}{3}\le x\le\dfrac{2}{3}$ domain range $0\le y\le\pi$	1 1
c $\dfrac{7}{2}\sin\theta+2\cos\theta=4 \quad \dfrac{7t}{1+t^2}+2\dfrac{1-t^2}{1+t^2}=4$ $7t+2-2t^2=4+4t^2 \quad 6t^2-7t+2=0 \quad (3t-2)(2t-1)=0$ $\tan\dfrac{\theta}{2}=\dfrac{1}{2},\dfrac{2}{3} \quad \dfrac{\theta}{2}=\tan^{-1}\left(\dfrac{1}{2}\right)$ *and* $\tan^{-1}\left(\dfrac{2}{3}\right)$ $\theta=53°8'$ *or* $67°23'$	 1 1 1
d 	1
d (ii) $AC=b=h\cot 14 \quad BC=a=h\cot 17$ $c^2=a^2+b^2-2ab\cos C$ $1000^2=h^2\cot^2 14+h^2\cot^2 17-2\times h\cot 14 h\cot 17\cos 120$ $=h^2(\cot^2 14+\cot^2 17+\cot 14\cot 17)$ $=h^2\times 39.9035$ $h=\sqrt{1000000\div 39.9035}=158.3\approx 158m$	1 1 1 **/12**

Solutions Question 4 2009	Marks/Comments
a $x^3 - 5x^2 + 7x + 5 = 0$ (i) $\alpha + \beta + \gamma = -\frac{b}{a} = 5$ (ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 7$ (iii) $(\alpha+\beta+\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$ $\therefore \alpha^2 + \beta^2 + \gamma^2 = 5^2 - 2\times 7 = 11$	1 1 1 1
b (i) $\sqrt{3}\sin 2\theta - \cos 2\theta = 2\left(\frac{\sqrt{3}}{2}\sin(2\theta) - \frac{1}{2}\cos(2\theta)\right)$ $= 2\sin\left(2\theta - \frac{\pi}{6}\right)$ noting $\sin(A-B) = \sin A\cos B - \cos A \sin B$ (ii) $2\sin\left(2\theta - \frac{\pi}{6}\right) = 1$ $\left(2\theta - \frac{\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}$ $2\theta = \frac{\pi}{3}, \pi$ $\theta = \frac{\pi}{6}, \frac{\pi}{2}$	**1 for r and α** **1 for correct form** **1 for solving for 2θ** **1 for θ**
c) i) $T = D + Ce^{-kt}$ $\frac{dT}{dt} = -k(Ce^{-kt} + D - D) = -k(T-D)$ ii) $D = 25, C = 1325$ $T(10) = 720 = 25 + 1325e^{-10k}$ $\frac{695}{1325} = e^{-10k}$ $\ln\frac{695}{1325} \div -10 = k = 0.0645255...$ $50 = 25 + 1325e^{-0.0645255t}$ $e^{-0.0645255t} = \frac{25}{1325}$ $t = \ln\frac{25}{1325} \div 0.0645255 = 61.53\text{min}$	1 **1 for equation** **1 for *k*** **1 for answer.** **/12**

Solutions Question 5 2009	Marks/Comments
a. $f'(x) = 4x^3 - 15x^2 + 22x - 12$ $f'(1.3) = 0.038$ $f''(x) = 3.28$ $x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = 1.3 - \frac{0.038}{3.28} = 1.288...$ $f'(1.288) = -0.00128$ a ii $f(1.3) = 0.8611$ x=1.3 is so close to the minimum that the curve cannot get much lower.	1 1 1 1
b) 0.5 chance not even $0.5^7 = .0078125$ chance of not throwing < 1% Therefore chance of at least 1 even > 99% if 7 rolls	1 1
c $\frac{ds}{dt} = \frac{ds}{dV}\times\frac{dV}{dt} = \frac{1}{3s^2}\times 200 = \frac{200}{675} = \frac{8}{27}$ $\frac{dSA}{dt} = \frac{dSA}{ds}\times\frac{ds}{dt} = 12s\times\frac{8}{27} = 53\frac{1}{3}cm^2s^{-1}$	1 1 1
d) If $n = 1$ $1^3 + 2^3 + 3^3 = 36 = 9\times 4$ so the result holds when $n = 1$ Assume that when $n = k$ $(k)^3 + (k+1)^3 + (k+2)^3 = 9m,\ m\in C$ RTP $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9p,\ p \in C$ $LHS = 9m - k^3 + (k+3)^3 = 9m - k^3 + k^3 + 9k^2 + 27k + 27$ $= 9(m + k^2 + 3k + 3) = 9p$ as req[d] Hence since true for $n = 1$, and since if true for $n = k$. Also true for $n = k+1$, by induction the result holds for all positive integers n	1 1 1 **/12** **Penalise 1 if conclusion not stated**

Solutions Question 6 2009	Marks/Comments
a) i) $x \neq \pm 1$	1
a) ii) $h(2) = -2,\ h(-2) = 2$	1
a) iii) $h(x) = \frac{3x}{1-x^2}$ $\quad \frac{vu'-uv'}{v^2} = \frac{3(1-x^2) - 3x(-2x)}{(1-x^2)^2} = \frac{3+3x^2}{(1-x^2)^2}$ positive numerator and denominator ∴never zero	 1
a) iv) 	**2 award 1 for progress**
b) $V = \pi \int_0^{\frac{\pi}{3}} y^2 dx = \pi \int_0^{\frac{\pi}{3}} \sec^2 dx = \pi[\tan x]_0^{\frac{\pi}{3}} = \sqrt{3}\pi\ units^3$	2
c) i) $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v.\frac{dv}{dx} = \frac{d(\frac{1}{2}v^2)}{dv}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$	1
c) ii) α $\quad v^2 = 36 - 4x^2 \quad \ddot{x} = \frac{d}{dx}(18 - 2x^2) = -4x$ $\ddot{x} = -n^2 x$ with $n = 2$ signifying SHM	1
c) ii) β At the endpoints $v = 0,\ x = \pm 3$ amplitude = 3 $T = \frac{2\pi}{n} = \pi$	1 1
c) ii) γ $x = 3\sin(2t)$ OR $x = 3\cos(2t - \frac{\pi}{2})$	1 **/12**

Solutions Question 7 2009	Marks/Comments
a) $T\hat{O}P = B\hat{O}P$ (given bisector) $O\hat{T}P = T\hat{A}B$ (angle between chord and tangent equals the angle in the alternate segment) $O\hat{Q}A = O\hat{P}T$ (angle sum triangle) But $O\hat{P}T = P\hat{T}Q + P\hat{Q}T$ and $O\hat{Q}A = P\hat{T}Q + T\hat{P}Q$ (exterior angle triangle) $\therefore P\hat{Q}T = P\hat{T}Q$ (equals - $P\hat{T}Q$) ΔTPQ is isosceles as req[d]	 1 1 1
If $\tan\theta = \frac{5}{12}$ $\cos\theta = \frac{12}{13}$ $\sin\theta = \frac{5}{13}$ $\ddot{x} = 0$ $\dot{x} = \int \ddot{x}dt = c$ $\qquad \ddot{y} = -10$ $\dot{y} = \int \ddot{y}dt = -10t + c$ $= -10t + 10$ b) $\dot{x} = 26 \times \frac{12}{13} = 24$ $\qquad y = \int \dot{y}dt = -5t^2 + 10t + c$ $x = \int \dot{x}dt = 24t + c = 24t$ $\qquad = -5t^2 + 10 + 15$ since components given in first line and origin is 15m below point of projection	 1 1 **explanation of evaluation of Cs of I must be given**
b)ii) Impact when $t = 0$ $0 = -5t^2 + 10t1 = 15$ $0 = t^2 - 2t - 3 = (t-3)(t+1)$ $whence t = 3$ $x(3) = 24 \times 3 = 72m$	 1 1
c) i) $PS = \sqrt{(2t-0)^2 + (t^2-1)^2} = \sqrt{t^4 + 2t^2 + 1} = t^2 + 1$ $PM = t^2 + 1$ which is the distance of P above the x axis plus the distance of the directrix below the x axis.. The lengths are equal *Δ PSM is isosceles*	 1 1
c)ii) $y = \frac{x^2}{4}$ $\quad y' = \frac{x}{2} = t\ atP$ $m_{SM} = \frac{-2}{2t} = -\frac{1}{t}$ ∴ The two lines are perpendicular	1 1
c)iii) The altitude of an isosceles triangle bisects the angle at the apex AND $A\hat{P}B = C\hat{P}M$ vertically opposite ∴ $A\hat{P}B = S\hat{P}C$ as req[d]	 1 **/12**