WESTERN REGION

2011
Preliminary Course
FINAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time - 5 minutes.
- Working Time - $1\frac{1}{2}$ hours.
- Write using a blue or black pen.
- Board Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**60**)

- Attempt Questions 1-5.
- All questions are of equal value.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Question 1 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) A tangent to a curve makes an angle of $60°$ to the positive x axis. What is the exact gradient of this tangent. **2**

b) Factorise $(x+3)^3+(y-4)^3$ **2**

c) Find a and b if $\dfrac{\sqrt{7}-5}{-2+3\sqrt{7}}=a+b\sqrt{7}$ **2**

d) i) Show that $\dfrac{x^3+2x^2-3}{x^3}=1+\dfrac{2}{x}-\dfrac{3}{x^3}$ **1**

ii) Hence find $\lim\limits_{x\to\infty}\dfrac{x^3+2x^2-3}{x^3}$ **1**

e) Find to the nearest degree the acute angle between the lines $y=6x-4$ and $4y-3x+2=0$ **2**

f) Divide the interval P(-4, 3) and Q(8,7) in the ratio 3:4 externally. **2**

End of Question 1

2011 Preliminary Final Examination Mathematics Extension 1

Question 2 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Georgina skis along a straight snow trail. At one point along the trail she notices a ski lift station on a bearing of $047°$ with an angle of elevation of $29°$. After skiing 350m along the trail the station is now on a bearing of $320°$ with an angle of elevation of $19°$. Find the height of the ski lift station correct to the nearest metre. (Hint: Use the cosine rule) **3**

b) Find the exact value of $\cos 15°\sin 15°$ **2**

c) Find the general solution of: **2**

$$3\tan^2\theta-1=0$$

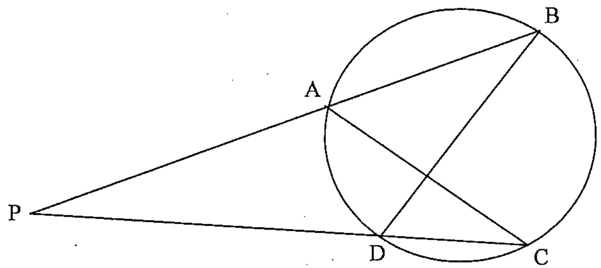
d) Solve $6\sin\theta+8\cos\theta=4$ for $0°\le\theta\le 360°$ **3**

e) Write $\sec\theta$ in terms of t where $t=\tan\dfrac{\theta}{2}$ **2**

End of Question 2

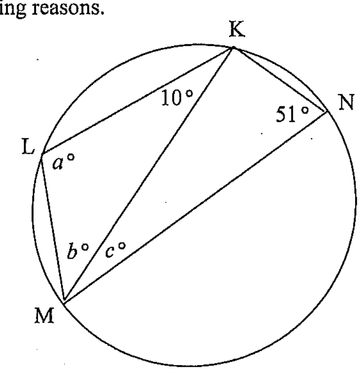
Question 3 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Two chords AB and CD intersect at point P outside the circle.
Prove that $\Delta APC \parallel\!| \Delta DPB$ **3**



b) MN is the diameter of the circle below:

Find a, b, c giving reasons. **3**



c) Differentiate and simplify the following:

i) $\left(3x^2+2x\right)^{10}$ **2**

ii) $\dfrac{2x^2}{\sqrt{\left(x^2-4\right)}}$ **4**

End of Question 3

Question 4 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Given the parabola $x = 4t^2$ and $y = 8t$, find

i) the Cartesian equation of this parabola **1**
ii) the focus **1**
iii) the directrix **1**
iv) sketch the parabola on the Cartesian axes, showing the directrix, vertex and focus **2**

b) Find the equation of the normal to the parabola $x^2 = 12y$ at the point $x = -2$. **2**

c) Find the remainder $R(x)$ when $P(x) = x^5 - 3x^3 + 2x$ is divided by $Q(x) = x^2 - 4$ **2**

d) If α, β, γ are the roots of $3x^3 - 4x^2 + 7x - 11 = 0$
Find:

i) $\alpha + \beta + \gamma$ **1**

ii) $\alpha\beta\gamma$ **1**

iii) $(\alpha+1)(\beta+1)(\gamma+1)$ **1**

End of Question 4

Question 5 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) How many arrangements of the letters of the word Woolloomooloo are possible? **1**

b) 9 students are to sit around a circular table.

i) How many ways can they be seated? **1**

ii) John and Casey refuse to sit together. How many ways can they be seated now? **1**

c) Mrs Smith wants to find out how many ways 4 boys and 3 girls can be arranged in a row, if there are 17 boys and 13 girls in her class. **1**

d) Norton surveyed 30 Year 11 students at Westburg High School to see what type of mobile phone they owned. The results are listed below:

Type of Mobile Phone	Number of Students
i-phone	3
Motorola	8
Nokia	10
Samsung	15

i) How many students had more than one phone? **1**

ii) The students who had two phones each had a Motorola and a Nokia. What is the probability that a person just had one Nokia phone only? **1**

e) Solve $\frac{x-7}{x+4}<3$ **3**

f) Prove by Mathematical Induction that: $\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$ **3**

End of Examination

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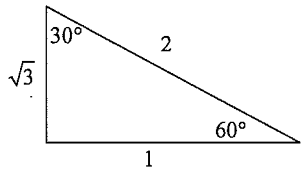
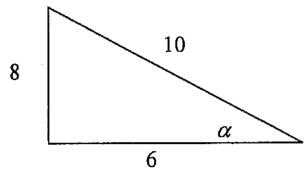
SOLUTIONS

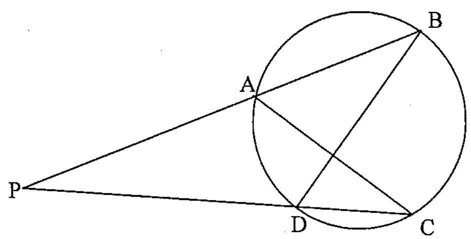
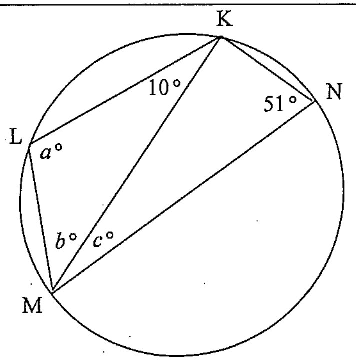
Question 1	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
a)	$\tan\theta = m$ $\tan 60° = m$  $m = \sqrt{3}$	2	1 1
b)	$(x+3)^3 + (y-4)^3$ Using $(a+b)^3 = (a+b)(a^2 - ab + b^2)$ *Let* $a = (x+3)$ $b = (y-4)$ $(x+3)^3 + (y-4)^3$ $= [x+3+(y-4)][(x+3)^2 - (x+3)(y-4) + (y-4)^2]$ $= (x+y-1)(x^2+6x+9-xy+4x-3y+12+y^2-8y+16)$ $= (x+y-1)(x^2+10x+y^2-11y-xy+37)$	2	1 1
c)	$\dfrac{\sqrt{7}-5}{-2+3\sqrt{7}} \times \dfrac{-2-3\sqrt{7}}{-2-3\sqrt{7}}$ $= \dfrac{-2\sqrt{7}-21+10+15\sqrt{7}}{4-63}$ $= \dfrac{13\sqrt{7}-11}{-59}$ $= \dfrac{11-13\sqrt{7}}{59}$ $a = \dfrac{11}{59}, b = \dfrac{-13}{59}$	2	1 1

Question 1	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
d)	i) $\frac{x^3+2x^2-3}{x^3}=\frac{x^3}{x^3}+\frac{2x^2}{x^3}-\frac{3}{x^3}$ $=1+\frac{2}{x}-\frac{3}{x^3}$ ii) $\lim_{x\to\infty}\frac{x^3+2x^2-3}{x^3}$ $=\lim_{x\to\infty}1+\frac{2}{x}-\frac{3}{x^3}$ (since $\lim_{x\to\infty}\frac{1}{x}=0$) $=1$	1 1	1 1
e)	$y=6x-4$ $m_1=6$ $4y-3x+2=0$ $4y=3x-2$ $y=\frac{3x}{4}-\frac{1}{2}$ $m_2=\frac{3}{4}$ $\tan\alpha=\left\|\frac{m_1-m_2}{1+m_1m_2}\right\|$ $\tan\alpha=\left\|\frac{6-\frac{3}{4}}{1+6\times\frac{3}{4}}\right\|$ $\alpha=44°$	2	1 for gradients 1
f)	$P(-4,3),\ Q(8,7),\ ratio\ 3:-4\ externally$ $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$ $=\left(\frac{3\times8+-4\times-4}{3-4},\frac{3\times7+-4\times3}{3-4}\right)$ $=(-40,-9)$	2	1 1
		/12	

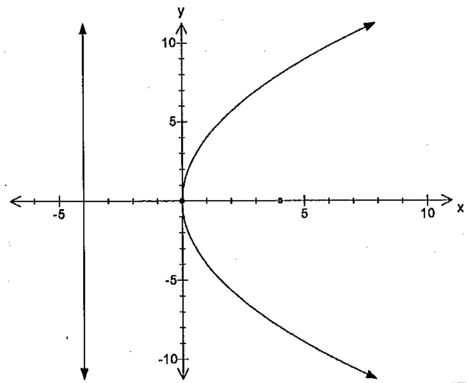
Question 2	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
a)	Diagram at the end of the solutions $\tan19°=\frac{h}{x}\quad\tan29°=\frac{h}{y}$ $x=\frac{h}{\tan19°}\quad y=\frac{h}{\tan29°}$ $a^2=b^2+c^2-2bc\cos A$ $350^2=\left(\frac{h}{\tan19°}\right)^2+\left(\frac{h}{\tan29°}\right)^2-2\times\frac{h}{\tan19°}\times\frac{h}{\tan29°}\cos87°$ $350^2=\frac{(\tan29°)^2h^2+(\tan19°)^2h^2-2h^2(\tan19°)(\tan29°)\cos87°}{(\tan19°)^2(\tan29°)^2}$ $350^2(\tan19°)^2(\tan29°)^2=h^2\left[(\tan29°)^2+(\tan19°)^2-2(\tan19°)(\tan29°)\cos87°\right]$ $h^2=\frac{350^2(\tan19°)^2(\tan29°)^2}{\left[(\tan29°)^2+(\tan19°)^2-2(\tan19°)(\tan29°)\cos87°\right]}$ $h^2=10995.80$ $h=105m$	3	1 1 1
b)	$\sin2x=2\sin x\cos x$ $\cos15°\sin15°$ $=\frac{1}{2}2\sin15°\cos15°$ $=\frac{1}{2}\sin(2\times15)°$ $=\frac{1}{2}\sin30°$ $=\frac{1}{2}\times\frac{1}{2}$ $=\frac{1}{4}$	2	1 1

Question 2	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
c)	$3\tan^2\theta - 1 = 0$ $3\tan^2\theta = 1$ $\tan^2\theta = \frac{1}{3}$ $\tan\theta = \pm\sqrt{\frac{1}{3}}$ $\tan\theta = \pm\frac{1}{\sqrt{3}}$ $\theta = 30°$ $\therefore$ *general solution* $\theta = 180n + 30°$	2	1 1
d)	$6\sin\theta + 8\cos\theta = 4$ for $0° \le \theta \le 360°$ $a\sin\theta + b\cos\theta = r\sin(\theta + \alpha)$  $r = \sqrt{6^2 + 8^2}$ $r = 10$ $\tan\alpha = \frac{8}{6}$ $\alpha = 53°08'$ $10\sin(\theta + 53°08') = 4$ $\sin(\theta + 53°08') = \frac{4}{10}$ *Let* $x = \sin(\theta + 53°08')$ $\sin x = \frac{4}{10}$ $x = 23°35'$ $x = 23°35', 180° - 23°35', 360° + 23°35', 540° - 23°35'$ $x = 23°35', 156°25', 383°35', 516°25'$ $\theta + 53°8' = 23°35', 156°25', 383°35', 516°25'$ $\theta = 103°17', 330°27'$ *as* $0 \le \theta \le 360°$	3	1 1 1

Question 2	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
e)	$\sec\theta = \frac{1}{\cos\theta}$ $\cos\theta = \frac{1-t^2}{1+t^2}$ $\sec\theta = \frac{1+t^2}{1-t^2}$	2	1 1
		/12	

Question 3	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
a)		3	
	$\angle BPD = \angle APC$ (common)	1	
	$\angle PBD = \angle PCA$ (*angles subtended at the circumference by the same arc are equal*)	1	
	$\angle PDB = \angle PAC$ (*angle sum* $\Delta = 180°$)	1	
	$\therefore \Delta APC \parallel\!\mid \Delta DPB$ (AAA) equal angular		
b)		3	
	$\angle MKN = 90°$($\angle$ *in semi-circle* $= 90°$) $c° = 180 - 90 - 51$	1	
	$= 39°$(*angle sum* $\Delta = 180°$) $a° = 180 - 51$		
	$= 129°$(*opp* $\angle$'s in *cyclic quad* $= 180°$) $b° = 180 - 100 - 39$	1	
	$= 41°$(*opp* $\angle$'s in *cyclic quad* $= 180°$) *or any other valid reason*	1	

Question 3	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
c)	i) $\frac{d}{dx}\left(3x^2+2x\right)^{10}$ $=10(6x+2)\left(3x^2+2x\right)^9$ $=(60x+20)\left(3x^2+2x\right)^9$	2	2 either of these lines acceptable
c)	ii) $u = 2x^2 \qquad v = \sqrt{(x^2-4)} = \left(x^2-4\right)^{\frac{1}{2}}$	2	1 for quotient rule
	$u' = 4x \qquad v' = \frac{1}{2}\times 2x\left(x^2-4\right)^{-\frac{1}{2}} = \frac{x}{\sqrt{(x^2-4)}}$ $\frac{d}{dx}\frac{2x^2}{\sqrt{(x^2-4)}}$ $=\frac{vu'-uv'}{v^2}$ $=\frac{4x\left(x^2-4\right)^{\frac{1}{2}}-2x^2\times\frac{x}{\sqrt{(x^2-4)}}}{\left(\sqrt{(x^2-4)}\right)^2}$ $=\frac{4x\left(x^2-4\right)^{\frac{1}{2}}-\frac{2x^3}{\sqrt{(x^2-4)}}}{x^2-4}$ $=\frac{4x\left(x^2-4\right)-2x^3}{\sqrt{(x^2-4)}}\times\frac{1}{x^2-4}$ $=\frac{4x^3-16x-2x^3}{\left(x^2-4\right)^{\frac{3}{2}}}$ $=\frac{2x^3-16x}{\sqrt{\left(x^2-4\right)^3}}$		1
		/12	

Question 4	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
a)	$x=4t^2$......(1) $y=8t$......(2) i) $y=8t$ $t=\frac{y}{8}$.......(3) *sub* (3) into (1) $x=4\times\left(\frac{y}{8}\right)^2$ $x=\frac{4y^2}{64}$ $x=\frac{y^2}{16}$ *or* $y^2=16x$	5	1
	ii) $y^2=4ax$ $\therefore a=4$ $focus=(a,0)$ $=(4,0)$		1
	iii) Directrix $x=-a$ $x=-4$		1
	iv) 		2 if all parts shown 1 only if not labelled 1 if correct but in wrong direction

Question 4	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
b)	$x^2=12y$ $y=\frac{x^2}{12}$ $y'=\frac{2x}{12}$ $y'=\frac{x}{6}$ *when* $x=-2$ $y'=\frac{-2}{6}$ $y'=-\frac{1}{3}$ $m_1m_2=-1$ *for normal* $-\frac{1}{3}\times m_2=-1$ $m_2=3$ *when* $x=-2$ $y=\frac{1}{3}$ $y-\frac{1}{3}=3(x--2)$ $3y-1=9(x+2)$ $3y-1=9x+18$ $9x-3y+19=0$	2	1 1
c)	$x^2-4\overline{)x^5-3x^3+2x}$ quotient x^3+x x^5-4x^3 x^3+2x x^3-4x $6x$ $R(x)=6x$	2	1 for division 1 for answer

Question 4	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
d)	$3x^3 - 4x^2 + 7x - 11 = 0$ i) $\alpha + \beta + \gamma = -\frac{b}{a}$ $= -\frac{-4}{3}$ $= \frac{4}{3}$ ii) $\alpha\beta\gamma = -\frac{d}{a}$ $= -\frac{-11}{3}$ $= \frac{11}{3}$ iii) $(\alpha+1)(\beta+1)(\gamma+1)$ $= (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$ $= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \gamma + \alpha\beta + \alpha + \beta + 1$ $= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \alpha\beta + \alpha + \beta + \gamma + 1$ $= -\frac{d}{a} + \frac{c}{a} + -\frac{b}{a} + 1$ $= \frac{11}{3} + \frac{7}{3} + \frac{4}{3} + 1$ $= \frac{25}{3}$ or $8\frac{1}{3}$	3	 1 1 1
		/12	

Question 5	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
a)	Arrangements of Woolloomooloo $= \frac{13!}{8! \times 3!}$ $= 25740$	1	1
b)	i) 8! = 40320 ways ii) John & Casey sit together 7! × 2! = 10080 Don't sit together 40320-10080 = 30240 ways	2	 1 1
c)	$^{17}C_4 \times {}^{13}C_3 \times 7!$ $= 3430627200$	1	1
d)	i) Total Phones = 3+8+10+15=36 ∴ 6 people had 2 phones ii) P(1Nokia) $= 10 - 6 = 4$ $P(1Nokia) = \frac{4}{30}$ $= \frac{2}{15}$	2	 1 1

Question 5	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
e)	$\frac{x-7}{x+4}<3$ $(x+4)^2\times\frac{x-7}{x+4}<3(x+4)^2$ $(x+4)(x-7)<3(x+4)^2$ $3(x+4)^2-(x+4)(x-7)>0$ $(x+4)[3(x+4)-(x-7)]>0$ $(x+4)[3x+12-x+7]>0$ $(x+4)(2x+19)>0$ $x+4=0 \quad 2x+19=0$ $x=-42 \quad x=-19$ $x\neq-4 \quad x=-\frac{19}{2}=-9.5$ *Test* $x=0$ $x=-5\times$ $x=-10$ $\therefore x<-9.5 \ \ or \ \ x>-4$	3	2 - 1 for each correct solution 1 test and correct answer

Question 5	Preliminary Final Examination - Mathematics Extension 1		2011
Part	**Solution**	**Marks**	**Comment**
f)	Step 1 - Prove true for $n=1$ $1^3=\frac{1^2}{4}(1+1)^2$ $1=\frac{1}{4}\times4$ $1=1$ $\therefore$ *true for* $n=1$ Step 2 – Assume true for $n=k$ $1^3+2^3+3^3+.......+k^3=\frac{k^2}{4}(k+1)^2$ Step 3 – Prove true for $n=k+1$ $1^3+2^3+3^3+.......+k^3+(k+1)^3=\frac{(k+1)^2}{4}[(k+1)+1]^2$ $\frac{k^2}{4}(k+1)^2+(k+1)^3=\frac{(k+1)^2}{4}(k+2)^2$ *LHS* $=\frac{k^2}{4}(k+1)^2+(k+1)^3$ $=(k+1)^2\left[\frac{k^2}{4}+(k+1)\right]$ $=(k+1)^2\left[\frac{k^2+4(k+1)}{4}\right]$ $=(k+1)^2\left(\frac{k^2+4k+4}{4}\right)$ $=\frac{(k+1)^2}{4}(k+2)^2$ $=RHS$ $\therefore$ if the formula is true for $n=k$, then it is true for $n=k+1$ We know the formula is true for $n=1$, so it must be true for $n=2$. If it is true for $n=2$ then it must be true for $n=3$ and so on. $\therefore$ it is true for any positive integer n	3	1 1 1
		/12	

Question 2 (a) diagram