WESTERN REGION

2011
Preliminary Course
FINAL EXAMINATION

Mathematics

General Instructions

- Reading Time - 5 minutes.
- Working Time - 2 hours.
- Write using a blue or black pen.
- Board Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**84**)

- Attempt Questions 1-7.
- All questions are of equal value.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Find the value of $5\pi\sqrt{\frac{a}{g}}$ if $a = 2.75$ *and* $g = 9.8$ correct to 2 significant figures. **1**

b) If $a = 2.7 \times 10^5$ write $\frac{1}{a}$ in scientific notation. **1**

c) Evaluate $\frac{|x-6|}{|x|-6}$ when $x = -2$. **1**

d) Simplify as a single fraction with a rational denominator $\frac{1}{\sqrt{3}} + \frac{1}{3+\sqrt{3}}$. **2**

e) Factorise $6a^3 - 48$. **2**

f) Solve i. $x - \frac{2x-4}{3} = 0$ **1**

ii. $2x = 7 - \frac{5}{x}$ **2**

g) Solve simultaneously **2**

$$x - y = 2$$
$$3x + 2y = 1$$

End of Question 1

Question 2 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) State the domain of $y = \frac{1}{(x-3)(1-x)}$ **1**

b) Which of the following equations would **not** represent a function? **1**

i. $y = 3^x$

ii. $f(x) = 1 - x^2$

iii. $y = \frac{-2}{x}$

iv. $x^2 + y^2 = 1$

c) A function is defined by

$$f(x) = \begin{cases} -1 & x \le -2 \\ 2x & -2 < x < 0 \\ 3 - x & x \ge 0 \end{cases}$$

Find the value of $f(-3) + f\left(\frac{1}{2}\right) + 2f(4)$ **2**

d) Sketch and state the range of $y = |2x| - 1$. **2**

e) Evaluate $\lim_{x \to 2} \frac{4-x^2}{x-2}$ **2**

f) Is the function $f(x) = 2x^3 - x$ even, odd or neither? (give reasons) **2**

g) Sketch the region defined by the inequalities **2**

$y \ge -\sqrt{4-x^2}$ *and*

$y < 0$

End of Question 2

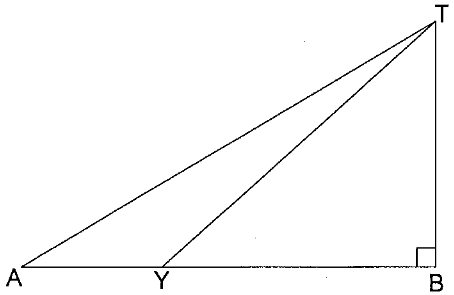
Question 3 (12 Marks) Use a Separate Sheet of paper **Marks**

a) If $\sin\theta = 0.251$ evaluate $\sin(180^\circ + \theta)$. **1**

b) Find the exact value of $\sec 225^\circ$. **1**

c) Prove that $\dfrac{\cot\theta\cos\theta}{\cot\theta + \cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$. **3**

d) From a point A, Peter finds that the angle of elevation of the top, T, of a cliff BT is 11°. After walking 275 metres directly towards the cliff to the point Y, he finds that the angle of elevation is 19°.



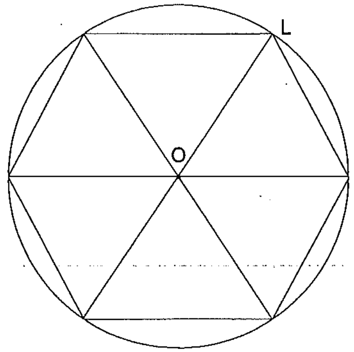
i. Calculate the length TY (nearest metre). **2**

ii Find the height of the cliff BT (nearest metre). **1**

Question 3 continued on page 5

Question 3 (continued) **Marks**

e) A regular hexagon is drawn inside a circle, with centre O so that its vertices lie on the circumference. The circle has radius 1cm.



i. Prove that ΔLMO is equilateral. **2**

ii. Find the area of ΔLMO and hence find the area of the hexagon (exact form). **2**

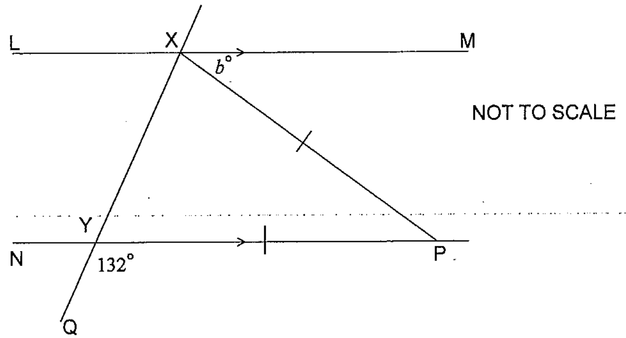
End of Question 3

Question 4 **(12 Marks)** Use a Separate Sheet of paper **Marks**

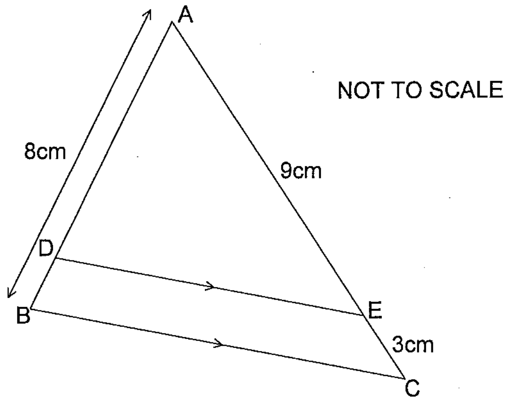
a) In the diagram LM is parallel NP, $XP=YP$, $\angle PYQ = 132^\circ$ and $\angle PXM = b^\circ$. **3**

Copy the diagram onto your answer sheet.

Find the value of b°, giving complete reasons.



b)



The diagram shows ΔABC. $BC \parallel DE$, $AB=8cm$, $AE=9cm$ and $EC=3cm$.

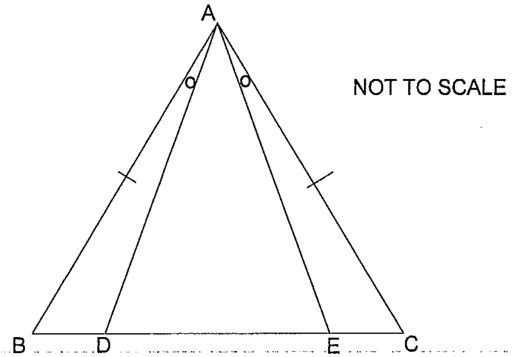
i. Prove that $\Delta ABC \,|||\, \Delta ADE$. **2**

ii. Find the length of DB. **2**

Question 4 continued on page 7

Question 4 (continued) **Marks**

c)



In the diagram $AB=AC$ and $\angle BAD = \angle CAE$.

i. Prove that $\Delta ABD \equiv \Delta ACE$. **2**

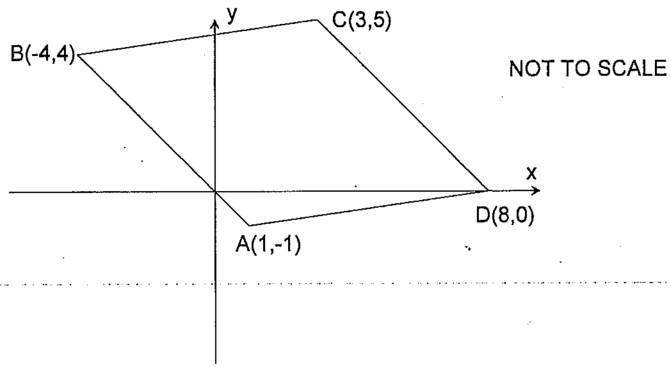
ii. Prove that $\Delta ABE \equiv \Delta ACD$. **3**

End of Question 4

Question 5 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) The line $2x + ky = 7$ passes through the point (2, -1). Find the value of k. **1**

b)



i. Find the midpoints of AC and BD. **2**

ii. Show that AC and BD are perpendicular. **2**

iii. What type of quadrilateral is $ABCD$? Justify your answer. **1**

iv. Find the length of AC and BD. **2**

v. Find the area of $ABCD$. **1**

vi. Find the equation of AD. **2**

vii. What angle does BD make with the positive direction of the x axis? (nearest degree) **1**

End of Question 5

Question 6 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Find the values of y for which $12 + 4y - y^2 > 0$ **2**

b) Find the value of k for which the quadratic equation $3x^2 + 2x + k = 0$ has real roots. **1**

c) One of the roots of $x^2 - (m + 1)x + 2m + 2 = 0$ is twice the other. Find the roots. **3**

d) A is the point (8, 0) and O is the origin. If the variable point $P(x, y)$ moves so that $PO = 3PA$,

i. prove that the locus of P is $x^2 + y^2 - 18x + 72 = 0$ **2**

ii. show that P moves in a circle and find its centre and radius. **2**

e) The focus of a parabola is (4, 1) and its directrix is $y = -3$. Find the equation of the parabola. **2**

End of Question 6

Question 7 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Differentiate

i. $\frac{2x^3}{\sqrt{x}}$ **1**

ii. $\frac{3}{\sqrt{2x-1}}$ **2**

b) If $y = \frac{2x}{(4x+3)^2}$, find $\frac{dy}{dx}$. **3**

c) If $f(x) = 5x^2(3x-1)^4$, find $f'(2)$. **3**

d) Find the point on the curve $y = x^2 + 5x + 4$ where the tangent is perpendicular to $y = \frac{x}{5}$. **3**

End of Examination

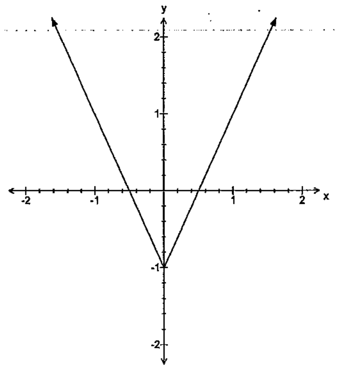
WESTERN REGION

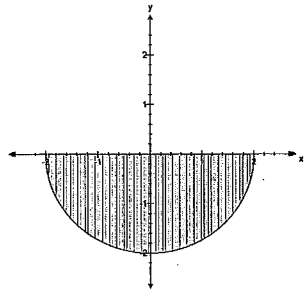
2011
Preliminary Final
EXAMINATION

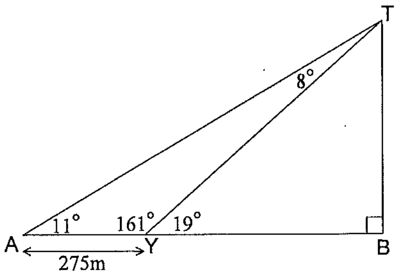
Mathematics

SOLUTIONS

Question 1	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a)	$5\pi\sqrt{\frac{2.75}{9.8}} = 8.32 = 8.3$ (2 *sign fig*)	1	For correct answer.
b)	$\frac{1}{2.7\times10^5} = 0.000003703 = 3.703\times10^{-6}$	1	For correct answer in SN
c)	$\frac{\lvert -2-6\rvert}{\lvert -2\rvert-6} = \frac{8}{-4} = -2$	1	For correct answer.
d)	$\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} + \frac{1}{3+\sqrt{3}}\times\frac{3-\sqrt{3}}{3-\sqrt{3}}$ $= \frac{\sqrt{3}}{3} + \frac{3-\sqrt{3}}{6}$ $= \frac{2\sqrt{3}+3-\sqrt{3}}{6}$ $= \frac{\sqrt{3}+3}{6}$	2	1 for rationalising. 1 for single fraction.
e)	$6(a^3-8) = 6(a-2)(a^2+2a+4)$	2	1 for common factor 1 for cubic factorisation
f) i.	$\frac{3x}{3} - \frac{2x-4}{3} = 0$ $3x-2x+4=0$ $x=-4$	1	For correct answer.
f) ii.	$2x^2 = 7x-5$ $2x^2-7x+5=0$ $(2x-5)(x-1)=0$ $x=\frac{5}{2},\ 1$	2	1 for quadratic equation 1 for solving
g)	$x-y=2$ ① $\times 2$ $3x+2y=1$ ② $2x-2y=4$ ③ $5x=5$ $x=1$ $1-y=2$ $y=-1 \quad (1,-1)$	2	1 for each value
		/12	

Question 2	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a)	All real x, except $x \neq 3$ *or* 1	1	
b)	iv. $x^2 + y^2 = 1$	1	
c)	$f(-3) + f\left(\frac{1}{2}\right) + 2f(4)$ $= -1 + \left(3 - \frac{1}{2}\right) + 2(3-4)$ $= -1 + 2\frac{1}{2} - 2$ $= -\frac{1}{2}$	2	1 for correct substitution 1 for answer
d)	 Range: $y \geq -1$	2	1 for correct diagram 1 for correct range
e)	$\lim\limits_{x \to 2} \frac{(2-x)(2+x)}{x-2}$ $= \lim\limits_{x \to 2} \frac{-(x-2)(2+x)}{x-2}$ $= \lim\limits_{x \to 2} -(x-2)$ $= -4$	2	1 for correct factorising 1 for answer
f)	$f(x) = 2x^3 - x$ $f(-x) = 2(-x)^3 - (-x)$ $f(-x) = -2x^3 + x$ $f(-x) = -(2x^3 - x)$ $f(-x) = -f(x)$ $\therefore$ *odd function*	2	1 for $f(-x)$ 1 for reason

Question 2	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
g)		2	1 for semi circle 1 for correct region
		/12	

Question 3	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a)	$\sin(180^\circ + \theta) = -\sin\theta = -0.251$	1	
b)	$\sec 585^\circ = \sec 225^\circ = -\sec 45^\circ = -\sqrt{2}$	1	
c)	$LHS = \dfrac{\dfrac{\cos\theta}{\sin\theta} \times \cos\theta}{\dfrac{\cos\theta}{\sin\theta} + \dfrac{\cos\theta\sin\theta}{\sin\theta}}$ $= \dfrac{\cos^2\theta}{\sin\theta} \times \dfrac{\sin\theta}{\cos\theta(1+\sin\theta)}$ $= \dfrac{\cos\theta}{1+\sin\theta}$ $= RHS$	3	1 for definitions 1 for simplifying fractions 1 for simplifying
d)			
i.	$\dfrac{YT}{\sin 11^\circ} = \dfrac{275}{\sin 8^\circ}$ $YT = \dfrac{275\sin 11^\circ}{\sin 8^\circ}$ $YT = 377\ m$ (*nearest m*)	2	1 for using The Sine rule correctly 1 for answer
ii.	$\sin 19^\circ = \dfrac{BT}{YT}$ $BT = YT \times \sin 19^\circ$ $BT = 122.7 = 123\ m$ (*nearest m*)	1	

Question 3	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
e) i.	$\angle MOL = 60^\circ$ (*revolution*) $\div 6$ $LM^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos 60^\circ$ $LM^2 = 2 - 2 \times \frac{1}{2}$ $LM = 1$ $OL = MO = 1$ (*radii*) $\therefore \Delta LMO$ *is equilateral*	2	1 for angle and cosine rule or alternate geometry solution 1 for equal sides
ii.	$A\Delta = \frac{1}{2} \times 1 \times 1 \times \sin 60^\circ$ $A\Delta = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\ cm^2$ $\therefore A\ hexagon = 6 \times \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}\ cm^2$	2	1 for area of triangle 1 for area of hexagon
		/12	

Question 4	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a)	$\angle XYP = 180^\circ - 132^\circ = 48^\circ$(*straight* $\angle$) $\angle MXY = 132^\circ$ (*corr angles* NP || *LM*) $\angle YXP = 48^\circ$ (*base* $\angle$ *isosceles* Δ *YXP*) $b^\circ = 132^\circ - 48^\circ = 84^\circ$	3	1 1 1
b) i	$\angle A$ *is common* $\angle AED = \angle ABC$ (*corr* $\angle$ *s*, BC || *DE*) $\angle AED = \angle ACB$ ($\angle$ *sum* Δ) $\Delta ABC \mid\mid\mid \Delta ADE$ (*equiangular*)	2	1 for part proof
ii	$\frac{AD}{8} = \frac{9}{12}$ (*corr sides in same ratio*) $AD = \frac{72}{12} = 6cm$ $BD = 8 - 6 = 2cm$	2	1 1 for answer
c) i	$\angle BAD = \angle CAE$ (*given*) AB = AC (*given*) $\angle ABD = \angle ACE$ (*base* $\angle$ *isoscles* ΔABC) $\therefore \Delta ABC \equiv \Delta ACE$ (*AAS*)	2	1 for part proof
ii	AB = AC (*given*) AD = AE (*corr sides of congruent* Δ *s above*) $\angle BAE = \angle BAD + \angle DAE$ $\angle CAD = \angle CAE + \angle DAE$ *since* $\angle BAD = \angle CAE$ (*given*) $\therefore \angle BAE = \angle CAD$ $\Delta ABE \equiv \Delta ACD$ (*SAS*)	3	1 1 1 or alternate methods
		/12	

Question 5	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a)	$2(2) + \mathbf{k}(-1) = 7$ $-\mathbf{k} = 3$ $\mathbf{k} = -3$	1	1 for k
b) i.	$AC: x = \frac{1+3}{2}$ $y = \frac{-1+5}{2}$ $BD: x = \frac{8-4}{2}$ $y = \frac{0+4}{2}$ $x = 2$ $y = 2$ $x = 2$ $y = 2$	2	1 for each midpoint
ii.	$m_{AC} = \frac{-1-5}{1-3}$ $m_{BD} = \frac{4-0}{-4-8}$ $m_{AC} = 3$ $m_{BD} = -\frac{1}{3}$ $m_{AC} \times m_{BD} = 3 \times -\frac{1}{3} = -1$ $\therefore AC \perp BD$	2	1 for gradients 1 for showing perpendicular
iii.	*ABCD* is a rhombus because diagonals *AC* and *BD* bisect each other at Right Angles.	1	1 for reason
iv.	$d_{AC} = \sqrt{(1-3)^2 + (-1-5)^2}$ $d_{BD} = \sqrt{(-4-8)^2 + (4-0)^2}$ $d_{AC} = \sqrt{4+36}$ $d_{BD} = \sqrt{144+16}$ $d_{AC} = \sqrt{40}$ $d_{BD} = \sqrt{160}$ $d_{AC} = 2\sqrt{10}$ $d_{BD} = 4\sqrt{10}$	2	1 for each distance
v.	$A = \frac{1}{2} \times AC \times BD$ $A = \frac{1}{2} \times 2\sqrt{10} \times 4\sqrt{10}$ $A = 40\ units^2$	1	1 for area
vi.	$m_{AD} = \frac{0+1}{8-1} = \frac{1}{7}$ *equation of AD* $y - 0 = \frac{1}{7}(x-8)$ $7y = x - 8$ $x - 7y - 8 = 0$	2	1 for gradient 1 for equation
vii.	$\tan\theta = -\frac{1}{3}$ $\theta = (180 - 18^\circ 26')$ $\theta = 161^\circ 34'$ $\theta = 162^\circ$ (*nearest degree*)	1	1 for angle
		/12	

Question 6	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a)	$12 + 4y - y^2 > 0$ $(6 - y)(2 + y) > 0$ *test* $y = 0$ *(true)* $-2 < y < 6$	2	1 for test 1 for correct solution
b)	$b^2 - 4ac \geq 0$ $2^2 - 4 \times 3 \times \mathbf{k} \geq 0$ $4 - 12\mathbf{k} \geq 0$ $-12\mathbf{k} \geq -4$ $\mathbf{k} \leq \frac{1}{3}$	1	
c)	$\alpha + 2\alpha = m + 1$ $\quad 2\alpha^2 = 2m + 2$ $3\alpha = m + 1$ $\quad \alpha^2 = m + 1$ $\alpha = \frac{m+1}{3}$ $\quad \left(\frac{m+1}{3}\right)^2 = m + 1$ $m^2 + 2m + 1 = 9m + 9$ $m^2 - 7m - 8 = 0$ $(m - 8)(m + 1) = 0$ $m = 8, -1$ *(not a solution: gvies 0, 0)* $\therefore \alpha = \frac{8+1}{3} = 3$ *roots are 3 and 6*	3	1 for definitions 1 for quadratic equation can find α first 1 for roots
d) i	$PO^2 = (3PA)^2$ $x^2 + y^2 = 9[(x - 8)^2 + y^2]$ $x^2 + y^2 = 9(x^2 - 16x + 64 + y^2)$ $x^2 + y^2 = 9x^2 - 144x + 576 + 9y^2$ $8x^2 + 8y^2 - 144x + 576 = 0$	2	1 for distances squared 1 for simplification
ii.	$x^2 + y^2 - 18x + 72 = 0$ $x^2 - 18x + (-9)^2 + y^2 = -72 + 81$ $(x - 9)^2 + y^2 = 9$ *in circle form* *centre:* $(9, 0)$ *radius:* 3	2	1 for completing the square 1 for centre and radius
e)	$a = 2$ *concave up*, *Vertex* $=(4\ -1)$ $(x - h)^2 = 4a(y - k)$ $(x - 4)^2 = 8(y + 1)$	2	1 for focus and vertex 1 for equation
		/12	

Question 7	Preliminary Final Examination - Mathematics		2011
Part	**Solution**	**Marks**	**Comment**
a) i.	$\frac{d}{dx}\left(\frac{2x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(2x^{\frac{5}{2}}\right)$ $= 2 \times \frac{5}{2}x^{\frac{3}{2}}$ $= 5\sqrt{x^3}$	1	For differentiation
ii.	$\frac{d}{dx}\left(3(2x-1)^{-\frac{1}{2}}\right) = 3 \times -\frac{1}{2}(2x-1)^{-\frac{3}{2}} \times 2$ $= -3(2x-1)^{-\frac{3}{2}}$ $= -\frac{3}{\sqrt{(2x-1)^3}}$	2	1 for differentiation 1 for simplifying
b)	$u = 2x, \quad v = (4x+3)^2$ $u' = 2, \quad v' = 2(4x+3) \times 4 = 8(4x+3)$ $\frac{dy}{dx} = \frac{(4x+3)^2 \times 2 - 2x \times 8(4x+3)}{(4x+3)^4}$ $\frac{dy}{dx} = \frac{(4x+3)[2(4x+3) - 16x]}{(4x+3)^4}$ $\frac{dy}{dx} = \frac{8x + 6 - 16x}{(4x+3)^3}$ $\frac{dy}{dx} = \frac{6 - 8x}{(4x+3)^3}$	3	1 for differentiation 1 correct rule 1 for simplifying
c)	$u = 5x^2, \quad v = (3x-1)^4$ $u' = 10x, \quad v' = 12(3x-1)^3$ $f'(x) = 5x^2 \times 12(3x-1)^3 + (3x-1)^4 \times 10x$ $f'(x) = 10x(3x-1)^3[6x + (3x-1)]$ $f'(x) = 10x(3x-1)^3(9x-1)$ $f'(2) = 10(2)(5)^3(17)$ $f'(2) = 42500$	3	1 for differentiation 1 for correct rule 1 for substitution
d)	$y' = 2x + 5$ *and* $y = \frac{x}{5}$ $\quad m_1 = \frac{1}{5}$ $\quad m_2 = -5$ $\therefore 2x + 5 = -5$ $2x = -10$ $x = -5 \quad y = 4$	3	1 for differentiation 1 for equation 1 for point
		/12	