OUR LADY OF THE SACRED HEART COLLEGE
KENSINGTON

REGNET CHRISTUS

OLSH COLLEGE

STUDENT – NAME / NUMBER......................................

MATHEMATICS TEACHER ....

2011

HSC Assessment

Extension 1 - Mathematics

Time allowed : 45 minutes

Directions to Candidates

- Question 1 **(15 Marks)**
- Question 2 **(15Marks)**
- Total marks 30
- Show all working
- Approved calculators may be used
- READ each question carefully

Question One: (15 marks)

a) Find $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$ **2 marks**

b) (i) Show that $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$ **2 marks**

(ii) Hence or otherwise solve $\sin x - \cos 2x = 0$ for $0 \le x \le 2\pi$ **3marks**

c) Find $\int \sin^2 6x \, dx$. **2 marks**

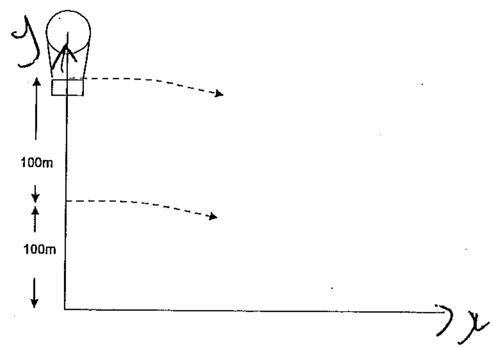
d) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. **2 marks**

(ii) Hence, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le \frac{\pi}{2}$. **1 mark**

e) Find the equation of the tangent to $y = \sin^{-1} 2x$ at the point where $x = 0.25$ **3 marks**

Question 2: (15 marks)

a) A balloon rises vertically from level ground. Two projectiles are fired horizontally in the same direction from the balloon at a velocity of 80ms^{-1}. The first is fired at a point 100 m from the ground and the second when it has risen a further 100 m from the ground. How far apart will the projectiles hit the ground? (Use $g = 10ms^{-2}$) **4 marks**

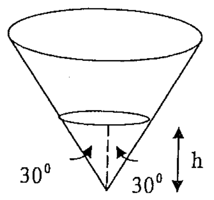


(b) The velocity of a particle moving along the x axis, in simple harmonic motion is given by:
$v^2 = 24 + 2x - x^2$.

(i) What are the endpoints of the motion? **1 mark**

(ii) Write an equation for the acceleration of the particle in terms of x. **2 marks**

(c) A hollow cone with a vertical angle of 60^0 is held with its axis vertical and vertex downwards.



Sand is being poured into the cone at a uniform rate of 15 cubic metres per second.

(i) Show that when the sand level has reached a height of h metres, the volume of sand in the cone, in cubic metres, is given by $v = \frac{1}{9}\pi h^3$. **2 marks**

(ii) Find the rate at which the surface of the sand level is rising when its depth is 4 metres. (leave your answer in terms of π) **2 marks**

(d) Let T be the temperature in a room at time t and let A be the temperature of its surroundings. Newton's Law of Cooling states that the rate of change of temperature T is proportional to (T-A).

(i) Verify that T=A+Be^{kt}, where B and k are constants, satisfies Newton's Law of Cooling. **1 mark**

(ii) The temperature of a substance in a room of constant temperature 6^0C is noted to be 29^0C and in 40 minutes to be 14^0C.

Find how long it takes the temperature of the substance to reach 9^0C. Give your answer to the nearest minute. **3 marks**

1

Q1-a) $\int \sin x \cos x \, dx$ let $u = \sin x$

$\frac{du}{dx} = -\cos x$ $\begin{pmatrix} S \\ C \\ -S \\ -C \end{pmatrix}$

$\cos x \, dx = -du$

$-\int u . du$

$= -\left[\frac{u^2}{2}\right] + c.$

$= -\frac{(\sin^2 x)}{2} + c.$ ✓ 2

b) $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$

LHS:

$\sin x - (1 - 2\sin^2 x)$ ✓

$= \sin x - 1 + 2\sin^2 x$

$= 2\sin^2 x + \sin x - 1$ = RHS. ✓ 2

ii) $2\sin^2 x + \sin x - 1 = 0$ $0 \le x \le 2\pi$

(let $\sin x = m$ $\quad x - 2$, $+1$

$2m^2 + m - 1$

$\frac{(2m-1)(2m+2)}{2} = 0$ $(-1, 2)$ ✓

$(2m-1)(m+1) = 0$

$\therefore m = \frac{1}{2}$ $\quad m = -1$ ✓

$\sin x = \frac{1}{2}$ $\quad \sin x = -1$

S | A ✓, T | C, 0

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad x = \frac{\pi}{2}$

S | A, T ✓ | C ✓ 3

$\therefore x = \frac{3\pi}{2},$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}.$ // ✓

2

c) $\int \sin^2 6x \, dx$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\sin^2 6x = \frac{1}{2}(1 - \cos 12x)$

$\frac{1}{2}\int (1 - \cos 12x) \, dx$ ✓

$\frac{1}{2}\left[x - \frac{1}{12}\sin 12x\right] + c.$ ✓ 2

$\int -\cos 12x \, dx$

S, C, $-$S, $-$C $\quad -\frac{1}{12}\sin 12x$

d) i. $\sqrt{3}\cos x - \sin x$ $\quad R\cos(x + \alpha)$

$R = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$

$\tan \alpha = \frac{1}{\sqrt{3}} \therefore \alpha = \frac{\pi}{6}$

$\sqrt{3}\cos x - \sin x \equiv 2\cos\left(x + \frac{\pi}{6}\right)$ ✓ 2

ii) $2\cos\left(x + \frac{\pi}{6}\right) = 1$

$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$ ✓ 1

$\frac{\pi}{6} \le x + \frac{\pi}{6} \le \frac{2\pi}{3}.$

$x + \frac{\pi}{6} = \frac{\pi}{3}$ ✓

$\frac{3\pi}{3}$ S | A ✓, T | C ✓ $\frac{6\pi}{3}$

$\therefore x + \frac{\pi}{6} = \frac{\pi}{3}, \cancel{\frac{5\pi}{3}}$

$\cancel{x = \frac{\pi}{3}}$ for $0 \le x \le \frac{\pi}{2}.$

$x = \frac{\pi}{6}.$

Be careful.

3

e) $y = \sin^{-1} 2x$ where $x = 0.25$ $x_1 = \frac{1}{4}$

at $x = 0.25$ $y = \sin^{-1} 2(0.25)$

$= \sin^{-1}(\frac{1}{2})$

$\boxed{y_1 = \frac{\pi}{6}}$ ✓

$y - y_1 = m(x - x_1)$

$y = \sin^{-1} 2x$

let $2x = u$

$u' = 2$

$y' = \frac{2}{\sqrt{1-(2x)^2}}$ ✓

$y' = \frac{2}{\sqrt{1-4x^2}}$

y' at $x = \frac{1}{4} = \frac{2}{\sqrt{1-0.25}}$

$= \frac{2}{\sqrt{\frac{3}{4}}}$

$y' = \frac{2}{\sqrt{\frac{3}{4}}} = 2 \div \sqrt{\frac{3}{4}}$

$2 \times \frac{2}{\sqrt{3}}$ $y' = \frac{4}{\sqrt{3}}$

$m = \frac{4}{\sqrt{3}}$

$m = \frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

$y - \frac{\pi}{6} = \frac{4\sqrt{3}}{3}(x - \frac{1}{4})$

$6y - \pi = 8\sqrt{3}(x - \frac{1}{4})$

$6y - \pi = 8\sqrt{3}x - 2\sqrt{3}$

$\therefore \boxed{6y - 8\sqrt{3}x + 2\sqrt{3} - \pi = 0}$ ✓

3

4

2. a) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = 80$ $\dot{y} = -10t$

$x = 80t$ $y = \frac{-10t^2}{2} + c$

$y = -5t^2 + c$

1st fire.

$y = -5t^2 + 100$

when $y = 0$ hits ground

$-5t^2 + 100 = 0$

$5t^2 = 100$

$t^2 = 20$

$t = \pm\sqrt{20}$

$t = \sqrt{20}$, since $t \geq 0$

x value for $t = \sqrt{20}$

$x = 80t$ ✓

$x = 80\sqrt{20}$ m.

$\sqrt{20} = 2\sqrt{5}$ ✓

$\therefore x_1 = 160\sqrt{5}$ m.

2nd fire

$y = -5t^2 + 200$ ✓

$-5t^2 + 200 = 0$

$5t^2 = 200$

$t^2 = 40$

~~$t = \pm 20$~~

~~since $t \geq 0$~~

~~then $t = 20$ s~~

x ~~$= 80t$~~

~~$x = 80(20)$~~

$t^2 = 40$ $t = \sqrt{40} = 2\sqrt{10}$

$x = 80t$

$x = 80(2\sqrt{10})$ ✓

$x_2 = 160\sqrt{10}$ m.

hence how far apart ✓

$= x_2 - x_1$

$= 160\sqrt{10} - 160\sqrt{5}$

$= 148.19$ m (2dp).

4

(5)

b) $V^2 = 24 + 2x - x^2$ SHM.

i) end points @ $v=0$.

$v=0$ $v=0$

$24 + 2x - x^2 = 0$

$x^2 - 2x - 24 = 0$ $\quad x - 24$ $\quad (-6, 4)$

$(x-6)(x+4)$ $\quad + - 2$

$x = 6, \quad x = -4.$

(1)

ii) find acceleration in terms of x.

since $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = \frac{1}{2}(24 + 2x - x^2)$

$\frac{1}{2}v^2 = 12 + x - \frac{x^2}{2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 12 + x - \frac{x^2}{2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 - \frac{2x}{2}$

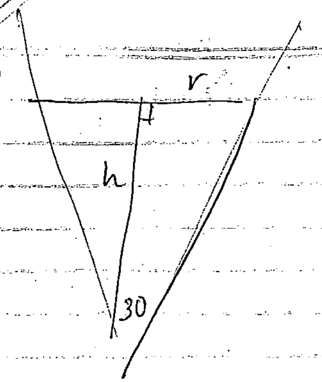
$= 1 - x = \ddot{x}$

$\ddot{x} = 1 - x$

(2)

(6)

c)



$V = \frac{1}{3}\pi r^2 h$

~~$\frac{dv}{dt} = 15\ cm$~~

$\boxed{\frac{dv}{dt} = 15 m^3/s.}$

$\tan 30 = \frac{o}{a}$

$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$ (1)

$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$= \frac{1}{3}\frac{\pi h^3}{3} = \frac{1}{9}\pi h^3$ as req.

$V = \frac{1}{9}\pi h^3$ (1)

$\frac{dV}{dh} = \frac{1}{3}\pi h^2$ at $h = 4$

$\boxed{\frac{dV}{dh} = \frac{16\pi}{3}}$

ii) $\frac{dh}{dt} = ?$ when $h = 4\ m$

$\frac{dh}{dV} = \frac{3}{16\pi}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$15 \times \frac{3}{16\pi}$

$\frac{dh}{dt} = \frac{45}{16\pi}\ m/s.$ (2)

Newtons law of cooling:

d) $\frac{dT}{dt} = k(T-A)$ where k constant.

i) $T = A + Be^{kt}$

$\frac{dT}{dt} = k.\,Be^{kt}$ since $Be^{kt} = T-A$

$= k(T-A)$

hence satisfys Newtons law of cooling.

$\frac{dT}{dt} = k(T-A)$

1

ii) cooling $\therefore$ k is neg.

$T = A + Be^{-kt}$

$T = 6 + Be^{-kt}$ $\quad T=29,\ t=0$

$T=14,\ t=40.$

at $t=0$.

$29 = 6 + B$

$\therefore B = 23$

$$\boxed{T = 6 + 23e^{-kt}}$$

at $t=40$

$14 = 6 + 23e^{-40k}$

$23e^{-40k} = 8.$

$e^{-40k} = \frac{8}{23}$

$$\boxed{k = \frac{\ln\left(\frac{8}{23}\right)}{-40}}$$

$T = 6 + 23e^{-kt}$ $\quad t = ?,\ T = 9.$

$9 = 6 + 23e^{-kt}$

$23e^{-kt} = 3$

$e^{-kt} = \frac{3}{23}$

$t = \frac{\ln\left(\frac{3}{23}\right)}{-k}$ where $k = \frac{\ln\left(\frac{8}{23}\right)}{-40}$

$k = 0{,}0264$

~~t = 77.15 mins~~

~~t = 77° 9'~~

~~t = 17.15 min~~

~~= 17 mins. (nearest min)~~

(nearest minute)

$t = 77.15$ min

~~t = 77 min (nearest minute)~~

~~because it takes 77.15 mins to reach 9°c then we~~

~~t = 77 min~~

$t = 77$ min (nearest min)

3