SYDNEY GIRLS HIGH SCHOOL

2011
HSC ASSESSMENT TASK 3

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 90 minutes
- Start each question on a new page.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.

Total marks – 75

- There are THREE (3) questions.
- All questions are of equal value.

Topics:

- Integration, Polynomials, Volumes (Sheets 1 to 5)

Name: ______________________

Class/Teacher: ______________________

Question 1: (25 marks)

a) Evaluate $\int_1^2 \frac{\ln x}{x}dx$ **3**

b) Find $\int \frac{1}{\sqrt{4x-x^2}}dx$ **3**

c) (i) Find A, B and C such that $\frac{3x^2+5x-1}{(x+3)(x^2+2)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+2}$ **3**

(ii) Hence, find $\int \frac{3x^2+5x-1}{(x+3)(x^2+2)}dx$. **2**

d) The roots of the equation $2x^3-9x^2+7=0$ are α, β and γ.

(i) Evaluate $\alpha^2+\beta^2+\gamma^2$. **3**

(ii) Hence, evaluate $\alpha^3+\beta^3+\gamma^3$. **3**

e) The polynomial $P(x)=4x^3+4x^2-39x+36$ has a double root. Solve $P(x)=0$. **4**

f) The base of a solid is the circle $x^2+y^2=25$. Each section of the solid perpendicular to the x axis is a right-angled isosceles triangle with the hypotenuse in the base. Find the volume of the solid. **4**

START QUESTION 2 ON A NEW PAGE.

Question 2: (25 marks)

a) Evaluate $\int_0^{\frac{\pi}{3}} \frac{dx}{1+\sin x}$. **3**

b) Evaluate $\int_0^{\frac{\pi}{6}} \sin^3 2x\, dx$. **3**

c) Use the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ to evaluate: **4**

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

d) If $ax^3+bx^2+d=0$ has a double root, show that $27a^2d+4b^3=0$. **3**

e) The equation $x^3+3x-5=0$ has roots α, β and γ. Find an equation with roots $\alpha\beta, \alpha\gamma$ and $\beta\gamma$. **3**

f) Factorise the polynomial $x^4+3x^3-9x+65$ into two quadratics with real coefficients, given that $-3-2i$ is a zero. **4**

g) The base of a solid is the region in the xy plane enclosed by the parabola $y=x^2$ and the line $y=x$. Each cross section perpendicular to the x axis is a semi-circle with its diameter in the base. Find the volume of the solid. **5**

START QUESTION 3 ON A NEW PAGE.

Question 3: (25 marks)

a) Find $\int e^x \sin x\, dx$. **4**

b) (i) If $I_n = \int \sec^n x\, dx$, show that $I_n = \frac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2)I_{n-2}\right]$. **4**

(ii) Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^4 x\, dx$. **3**

c) Solve the equation $2x^3-9x^2-27x+54=0$, given that the roots are in geometric progression. **3**

d) (i) For what values of k is $z-ki$ a factor of $P(z)=z^4+z^3+11z^2+9z+18$? **3**

(ii) Hence solve $P(x)=0$, $(k\neq 0)$. **4**

e) Find the volume of the torus generated when the circle $(x-1)^2+(y-4)^2=4$ is rotated about the x axis. **4**

END OF TEST

2011 Extension 2 Task 3 – Solutions

Solutions : Question 1

)	$I=\int_0^2 \frac{\ln x}{x}dx$ $u=\ln x,\ \frac{du}{dx}=\frac{1}{x}\Rightarrow du=\frac{dx}{x}$ $\therefore I=\int_{\ln 1}^{\ln 2} u\ du=\left[\frac{u^2}{2}\right]_0^{\ln 2}=\frac{(\ln 2)^2}{2}$ Note : $\frac{(\ln 2)^2}{2}\neq\frac{2(\ln 2)}{2}$	(d) (i)	$\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$ $=\left(-\frac{-9}{2}\right)^2-2\left(\frac{0}{2}\right)$ $\therefore\sum\alpha^2=\frac{81}{4}$
)	$I=\int\frac{dx}{\sqrt{4x-x^2}}=\int\frac{dx}{\sqrt{-(x^2-4x)}}$ $=\int\frac{dx}{\sqrt{4-(x^2-4x+4)}}$ $=\int\frac{dx}{\sqrt{4-(x-2)^2}}$ $\therefore I=\sin^{-1}\left(\frac{x-2}{2}\right)+C$	(d) (ii)	α is a root $\Rightarrow 2\alpha^3-9\alpha^2+7=0$ $\therefore 2\sum\alpha^3-9\sum\alpha^2+3\times 7=0$ $\sum\alpha^3=\frac{9\sum\alpha^2-3\times 7}{2}$ $=\frac{9\left(\frac{81}{4}\right)-21}{2}$ $\therefore\ \sum\alpha^3=\frac{645}{8}$
:))	$3x^2+5x-1=A(x^2+2)+(Bx+C)(x+3)$ let $x=-3\Rightarrow 27-15-1=11A\ \ \therefore\ \ A=1$ coeff. of $x^2\Rightarrow 3=A+B\ \ \therefore\ \ B=2$ constant $\Rightarrow -1=2A+3C$ $3C=-1-2A\ \ \therefore\ \ C=-1$ $\therefore\ \ A=1,B=2,C=-1$	(e)	Root are $\alpha,\alpha,\beta\ \ \Rightarrow P(\alpha)=P'(\alpha)=0$ $P'(\alpha)=12x^2+8x-39\Rightarrow 12x^2+8x-39=0$ $(2x-3)(6x+13)=0\Rightarrow x=\frac{3}{2}$ or $-\frac{13}{6}$ $P\left(\frac{3}{2}\right)=0\ \ \therefore\ \ \alpha=\frac{3}{2}$ $\sum\alpha=-1\Rightarrow\frac{3}{2}+\frac{3}{2}+\beta=-1\ \ \therefore\ \ \beta=-4$ i.e. The roots are $x=\frac{3}{2},\frac{3}{2},-4$
:) ii)	$I=\int\left(\frac{1}{x+3}+\frac{2x-1}{x^2+2}\right)dx$ $=\int\left(\frac{1}{x+3}+\frac{2x}{x^2+2}-\frac{1}{x^2+2}\right)dx$ $\therefore I=\ln(x+3)+\ln(x^2+2)$ $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$	(f)	$A=\frac{y\times 2y}{2}=y^2=25-x^2$ $V_{\text{slice}}=(25-x^2)\delta x$ $V=2\int_0^5(25-x^2)\ dx\ \ =2\left[25x-\frac{x^3}{3}\right]_0^5$ $\therefore V=\frac{500}{3}$ units^3

2 a) let $t=\tan\frac{x}{2}$

$\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$

$=\frac{1+t^2}{2}$

$\frac{2\,dt}{1+t^2}=dx$

$\int_0^{\frac{1}{\sqrt{3}}}\frac{2dt}{1+\frac{2t}{1+t^2}}\times\frac{1}{1+t^2}$

$=\int_0^{\frac{1}{\sqrt{3}}}\frac{2\,dt}{1+t^2+2t}$

$=2\int_0^{\frac{1}{\sqrt{3}}}(1+t)^{-2}\,dt$

$=-2\left[(1+t)^{-1}\right]_0^{\frac{1}{\sqrt{3}}}$

$=-2\left(\frac{1}{1+\frac{1}{\sqrt{3}}}-1\right)$

$=-2\left(\frac{\sqrt{3}}{\sqrt{3}+1}-1\right)$

$=-2\,\frac{\sqrt{3}-\sqrt{3}-1}{\sqrt{3}+1}$

$=\frac{2}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$

$=\sqrt{3}-1$

b) $\int_0^{\frac{\pi}{6}}\sin 2x\sin^2 2x\,dx$

$=\int_0^{\frac{\pi}{6}}\sin 2x(1-\cos^2 2x)\,dx$

$=\int_0^{\frac{\pi}{6}}\sin 2x\,dx-\int_0^{\frac{\pi}{6}}\sin 2x\cos^2 2x\,dx$

$=\left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{6}}+\left[\frac{\cos^3 2x}{6}\right]_0^{\frac{\pi}{6}}$

$=-\frac{\cos\frac{\pi}{3}}{2}+\frac{\cos 0}{2}+\frac{\cos^3\frac{\pi}{3}}{6}-\frac{\cos^3 0}{6}$

$=-\frac{1}{4}+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^3}{6}-\frac{1}{6}$

$=\frac{5}{48}$

c) $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx$

$=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$

$\therefore 2\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx$

$=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}\,dx$ let $u=\cos x$, $\frac{du}{dx}=-\sin x$, $-\frac{du}{\sin x}=dx$

$=\int_1^{-1}\frac{\pi\sin x}{1+u^2}\times-\frac{du}{\sin x}$

$=\pi\int_{-1}^{1}\frac{du}{1+u^2}$

$=\pi\left[\tan^{-1}u\right]_{-1}^{1}$

$=\pi\left(\frac{\pi}{4}--\frac{\pi}{4}\right)$

$=\frac{\pi^2}{2}$

$\therefore$ required ans is $\frac{\pi^2}{4}$

d) $3ax^2+2bx=0$

$x(3ax+2b)=0$

$3ax=-2b$

$x=-\frac{2b}{3a}$

$a\times\left(\frac{-2b}{3a}\right)^3+b\times\left(\frac{-2b}{3a}\right)^2+d=0$

$-\frac{8ab^3}{27a^3}+\frac{4b^3}{9a^2}+d=0$

$-8b^3+12b^3+27a^2d=0$

$27a^2d+4b^3=0$

) roots are

$\frac{\alpha\beta\gamma}{\gamma}, \frac{\alpha\beta\gamma}{\beta}, \frac{\alpha\beta\gamma}{\alpha}$

$\alpha\beta\gamma = -\frac{-5}{1}$

$= 5$

let $x = \frac{5}{\alpha}$

$\alpha = \frac{5}{x}$

$\left(\frac{5}{x}\right)^3 + 3 \cdot \frac{5}{x} - 5 = 0$

$\frac{125}{x^3} + \frac{15}{x} - 5 = 0$

$125 + 15x^2 - 5x^3 = 0$

$-3 + 2i$ is also a zero

one factor is

$(x+3+2i)(x+3-2i)$

$(x+3)^2 - (2i)^2$

$x^2 + 6x + 9 + 4$

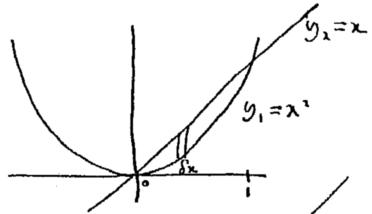
$x^2 + 6x + 9 + 4$

$x^2 + 6x + 13$

$$\begin{array}{r} x^2 - 3x + 5 \\ x^2+6x+13 \overline{) x^4 + 3x^3 \quad - 9x + 65} \\ x^4 + 6x^3 + 13x^2 \\ \hline -3x^3 - 13x^2 - 9x + 65 \\ -3x^3 - 18x^2 - 39x \\ \hline 5x^2 + 30x + 65 \\ 5x^2 + 30x + 65 \\ \hline 0 \end{array}$$

factors are $(x^2 + 6x + 13)(x^2 - 3x + 5)$

9)



$V_{slice} = \frac{1}{2}\pi r^2 h$

$= \frac{\pi}{2}\left(\frac{y_2 - y_1}{2}\right)^2 \delta x$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=0}^{1} \frac{\pi}{2}\left(\frac{y_2 - y_1}{2}\right)^2 \delta x$

$= \frac{\pi}{8}\int_0^1 (y_2 - y_1)^2\, dx$

$= \frac{\pi}{8}\int_0^1 (x - x^2)^2\, dx$

$= \frac{\pi}{8}\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$

$= \frac{\pi}{8}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$

$= \frac{\pi}{240}\ u^3$

Question 3:

a) $I = \int e^x \sin x\, dx$

$u = e^x \qquad v' = \sin x$

$u' = e^x \qquad v = -\cos x$

$I = -e^x \cos x + \int e^x \cos x\, dx$

$u = e^x \qquad v' = \cos x$

$u' = e^x \qquad v = \sin x$

$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$

$= e^x \sin x - I$

$\therefore I = -e^x \cos x + e^x \sin x - I$

$2I = e^x(\sin x - \cos x)$

$I = \frac{1}{2}e^x(\sin x - \cos x) + C$

bi) $I_n = \int \sec^n x\, dx$

$= \int \sec^{n-2} x . \sec^2 x\, dx$

$u = \sec^{n-2} x \qquad v' = \sec^2 x$

$u' = (n-2)\sec^{n-3} x . \sec x \tan x \qquad v = \tan x$

$= (n-2)\sec^{n-2} x \tan x$

$I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\, dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\, dx + (n-2)\int \sec^{n-2} x\, dx$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n = \frac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2) I_{n-2}\right]$$

ii) $$I_n = \frac{1}{n-1}\left(\sec^{n-2} x \tan x\right) + \frac{n-2}{n-1} I_{n-2}$$

$$I_4 = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^4 x \, dx$$

$$I_4 = \frac{1}{3}\left[\sec^2 x \tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \frac{2}{3} I_2$$

$$I_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$$

$$= \left[\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6}$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}$$

$$I_4 = \frac{1}{3}\left[\sec^2 x \tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \frac{2}{3} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{1}{3}\left\{\sec^2\frac{\pi}{3}\tan\frac{\pi}{3} - \sec^2\frac{\pi}{6}\tan\frac{\pi}{6}\right\} + \frac{4}{3\sqrt{3}}$$

$$= \frac{1}{3}\left\{4\sqrt{3} - \frac{4}{3} \cdot \frac{1}{\sqrt{3}}\right\} + \frac{4}{3\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{3} - \frac{4}{9\sqrt{3}} + \frac{4}{3\sqrt{3}}$$

$$= \frac{36 - 4 + 12}{9\sqrt{3}}$$

$$= \frac{44}{9\sqrt{3}}$$

c) $2x^3 - 9x^2 - 27x + 54 = 0$

Let roots be $\frac{\alpha}{r}, \alpha, \alpha r$

$$\frac{\alpha}{r} \times \alpha \times \alpha r = -27$$

$$\alpha^3 = -27$$

$$\alpha = -3$$

$$\frac{-3}{r} - 3 - 3r = \frac{9}{2}$$

$$\frac{-3}{r} - 3r = \frac{15}{2}$$

$-6-6r^2 = 15r$

$2r^2+5r+2=0$

$(2r+1)(r+2)=0$

$r = -\frac{1}{2}$ or $r=-2$

$\therefore$ roots are $\frac{3}{2}, -3, 6$

di) $P(z) = z^4 + z^3 + 11z^2 + 9z + 18$

If $z-ki$ is a factor, then $z=ki$ is a root.

$P(ki) = (ki)^4 + (ki)^3 + 11(ki)^2 + 9(ki) + 18$

$0 = k^4 i^4 + k^3 i^3 + 11k^2 i^2 + 9ki + 18$

$0+0i = k^4 - k^3 i - 11k^2 + 9ki + 18$

Equating real and imaginary parts:

$k^4 - 11k^2 + 18 = 0$ $\qquad$ $9k - k^3 = 0$

$(k^2-9)(k^2-2)=0$ $\qquad$ $k(9-k^2)=0$

$k=\pm 3$ or $k=\pm\sqrt{2}$ $\qquad$ $k=0$ (trivial case)

$k=\pm 3$

$\therefore k=\pm 3$

ii) $(z-3i)$ and $(z+3i)$ are factors.

$(z-3i)(z+3i) = z^2 - 9i^2$

$= z^2+9$

$z^4+z^3+11z^2+9z+18 = (z^2+9)(az^2+bz+c)$

Equating coefficients: $a=1$ $\qquad$ $b=1$

$9c=18$

$c=2$

$\therefore z^4+z^3+11z^2+9z+18 = (z^2+9)(z^2+z+2)$

when $P(x)=0$: $(x^2+9)(x^2+x+2)=0$

$x^2+9=0$ $\qquad$ $x^2+x+2=0$

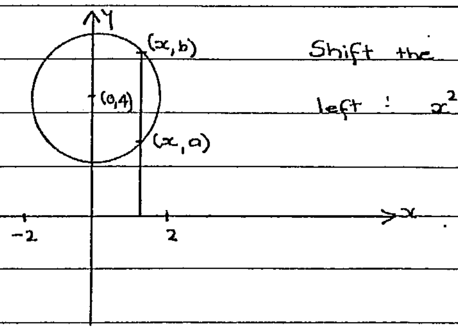
$x^2=-9$ $\qquad$ $= \frac{-1\pm\sqrt{1-4(2)}}{2}$

$=9i^2$ $\qquad$ $= \frac{-1\pm\sqrt{-7}}{2}$

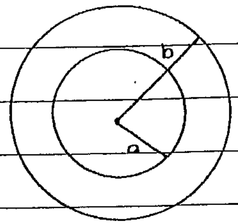
$x=\pm 3i$ $\qquad$ $= \frac{-1\pm i\sqrt{7}}{2}$

$\therefore x = \pm 3i, \frac{-1\pm i\sqrt{7}}{2}$

e)



Shift the circle 1 unit to the left : $x^2+(y-4)^2=4$



Area annulus $= \pi b^2 - \pi a^2$

$= \pi(b^2 - a^2)$

$= \pi(b-a)(b+a)$

$(y-4)^2 = 4 - x^2$

$y - 4 = \pm\sqrt{4-x^2}$

$y = 4 \pm \sqrt{4-x^2}$

$\therefore a = 4 - \sqrt{4-x^2}$ $\quad b = 4 + \sqrt{4-x^2}$

$b + a = 8$

$b - a = 2\sqrt{4-x^2}$

Area annulus $= \pi \times 8 \times 2\sqrt{4-x^2}$

$= 16\pi\sqrt{4-x^2}$

Volume torus $= 16\pi \int_{-2}^{2} \sqrt{4-x^2}\, dx$

$\int_{-2}^{2} \sqrt{4-x^2}\, dx$ = area of semi-circle with radius 2 units.

$= \frac{1}{2} \times \pi \times 2^2$

$= 2\pi$

Vol. torus $= 16\pi \times 2\pi$

$= 32\pi^2$ units^3