Sydney Girls High School

92%
average,
S.D ≈ 8

MATHEMATICS

ASSESSMENT TASK 1

December 2003

Topics: Chapters 10, 11, 12 and 16 from J & C:

Time Allowed: 90 minutes

Instructions:

- There are four (4) questions of equal value.
- Attempt all questions.
- Start each question on a new page.
- Show all necessary working.
- Marks may be deducted for careless or poor setting out.
- Board approved calculators may be used.

Total = 100 marks

QUESTION 1 (25 marks) — **Marks**

(a) On a number plane, mark the origin O, and A (5,-1), B (8, 3), and C (0, 9). Join A to B, B to C, and C to A. — 3

(b) Show that the gradient of the line BC is $-\frac{3}{4}$ — 3

(c) Show that the line AB has equation $4x-3y-23=0$. — 3

(d) Show that AB and BC are perpendicular. — 2

(e) Show that the length of AB is 5 units. — 3

(f) Find the coordinates of the point D such that $ABCD$ is a parallelogram. — 4

(g) If E is the point (8, -1) find the perpendicular distance of E from the line AB. — 3

(h) Find the point of intersection of the lines $4x-3y-23=0$ and $2x-4y+6=0$. — 4

QUESTION 2 (25 marks)

Marks

(a) Evaluate the following limits :

(i) $\lim_{x \to -2}\left(\frac{x^2+5x+6}{x+2}\right)$ 3

(ii) $\lim_{x \to \infty}\left(\frac{3x+5}{2x}\right)$ 3

(b)

(i) Sketch the curve $y = 2x^2 + 5x + 2$. 2

(ii) Find all values of x for which the curve is increasing. 1

(c) Find the primitive function of $6 - x^{-3}$. 2

(d) Differentiate $y = x^2 - 3x$ from first principles. 3

(e) Differentiate the following functions with respect to x :

(i) $y = 2x^3 + \frac{1}{x^4} - \sqrt{x}$ 2

(ii) $y = (2x^4 + 1)(3x + 12)$ 2

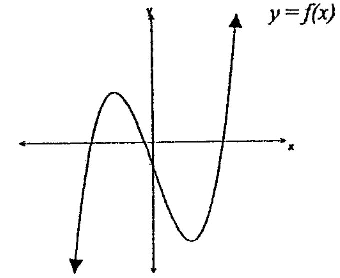
(iii) $y = (4x^2 + 7x)^{\frac{1}{3}}$ 2

(iv) $y = \frac{x^2+4}{2x-3}$ 3

(f) If $f(x) = 2x^3 + 3x^2 + 4$, for what values of x is $f'(x) = 7$? 2

QUESTION 3 (25 marks)

Marks



(a)

(i) Copy the diagram above by tracing it. 1

(ii) Sketch the gradient function $f'(x)$, for the function above. 2

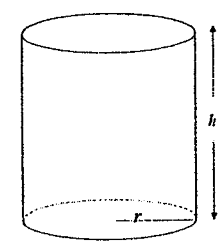
(b) If $A = \frac{5h+3}{7h-1}$ find $\frac{dA}{dh}$ when $h = 1$. 3

(c) Find the equation of the normal to the curve $y = 4x^3 - 7x^2 + 3$ at the point where $x = 2$. 3

(d) At what point on the curve $y = 2x^2 - 6x + 4$ is the tangent parallel to the line $y = 6x + 4$? 4

(e) Find the stationary points on the curve $y = x^3 + 3x^2 - 9x + 4$ and determine their nature. 4

(f) A can of 'Sparkle' soft drink is in the shape of a closed cylinder with height h cm and radius r cm, as shown below.



(i) The volume of the can is 500 cm^3. Show that the surface area, S cm^2, of the can is given by $S = 2\pi r^2 + \frac{1000}{r}$. 4

(ii) If the area of metal used to make the can is to be minimised, find the exact radius of the can. 4

(25 marks) **Marks**

(a) Consider the curve $y = x^3 - 3x^2 + 1$.

(i) Find any stationary points on the curve. 3

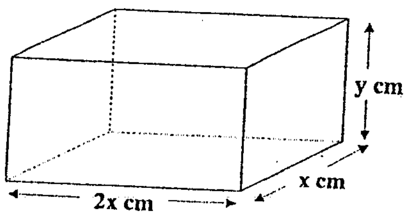
(ii) Determine their nature. 3

(iii) Find any points of inflexion on the curve. 3

(iv) Hence sketch the curve in the domain $-2 \le x \le 3$ 3

(v) Find the minimum point of the curve in this domain. 1

(b) An open rectangular box has four sides and a base, but no lid, as shown below:



(i) Write down the formulae for the area A cm^2 of the outer surface of the box, and the volume V cm^3 contained by the box. 4

(ii) Given that $A = 150$, show that the volume is given by $V(x) = 50x - \frac{2}{3}x^3$ 4

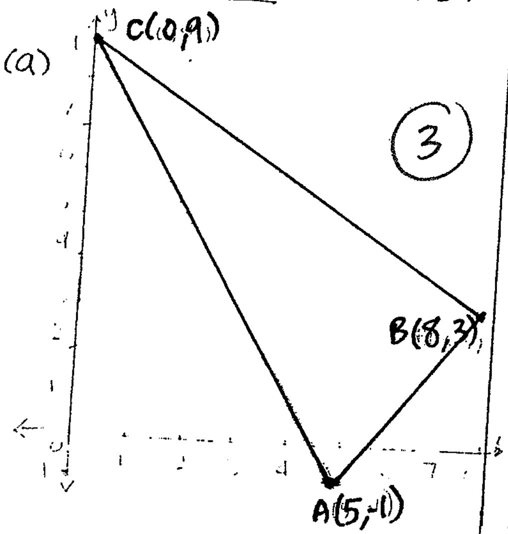
(iii) Find the value of x for which $V(x)$ is maximum, and verify that the maximum value of V is $\frac{500}{3}$. 4

End of exam

Solutions to Yr 11 2/3 Unit Exam Task 1 2003

QUESTION 1 (25 marks)

(a)



3

(b) $m_{BC} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{9-3}{0-8}$

$= -\dfrac{6}{8}$

$= -\dfrac{3}{4}$

3

(c) $m_{AB} = \dfrac{3+1}{8-5}$

$= \dfrac{4}{3}$

using point-gradient formula

$y + 1 = \dfrac{4}{3}(x-5)$

$3y + 3 = 4(x-5)$

$0 = 4x - 3y - 23$

3

(d) $AB \perp BC$ if $m_{AB} m_{BC} = -1$

$m_{AB} \times m_{BC} = \dfrac{4}{3} \times \left(\dfrac{-3}{4}\right)$

$= -\dfrac{12}{12}$

$= -1$

2

(e) $AB = \sqrt{(8-5)^2 + (3+1)^2}$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$= 5$

3

(f) If ABCD is a parallelogram then the midpoint of AC = midpt. of DB

let D have co-ords x_1, y_1

midpoint of $AC = \left(\dfrac{5+0}{2}, \dfrac{9-1}{2}\right)$

$= \left(\dfrac{5}{2}, 4\right)$

4

midpoint of $BD = \left(\dfrac{8+x_1}{2}, \dfrac{3+y_1}{2}\right)$

$\dfrac{8+x_1}{2} = \dfrac{5}{2}$ and $\dfrac{3+y_1}{2} = 4$

$x_1 + 8 = 5$ $\quad$ $y_1 + 3 = 8$

$x_1 = -3$ $\quad$ $y_1 = 5$

$\therefore (x_1, y_1) = (-3, 5) = D$

* P.T.O for alternative point

(g) $4x - 3y - 23 = 0$ $\quad$ $\overset{x_1\ \ y_1}{(8,-1)}$

$p.d. = \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

$= \left|\dfrac{4(8) + 3(-1) - 23}{\sqrt{16+9}}\right|$

$= \dfrac{12}{5}$

3

Question 1 (cont)

(h) $4x - 3y - 23 = 0 \quad \ldots (1)$

$2x - 4y + 6 = 0 \quad \ldots (2)$

(2)×2 $\quad 4x - 8y + 12 = 0 \quad \ldots (3)$

$-\ 4x - 3y - 23 = 0$

$-5y + 35 = 0$

$35 = 5y$

$7 = y$

$4x - 3(7) - 23 = 0$

$4x - 44 = 0$

$4x = 44$

$x = 11$

$\therefore (x,y) = (11,7)$

4

QUESTION 2 (25 marks)

(a)(i) $\lim\limits_{x \to -2} \dfrac{x^2+5x+6}{x+2}$

$= \lim\limits_{x \to -2} \dfrac{(x+3)\cancel{(x+2)}}{\cancel{x+2}}$

$= \lim\limits_{x \to -2} (x+3)$

$= -2 + 3 = +1$

3

(ii) $\lim\limits_{x \to \infty} \dfrac{3x+5}{2x}$

$= \lim\limits_{x \to \infty} \dfrac{\frac{3x}{x} + \frac{5}{x}}{\frac{2x}{x}}$

$= \lim\limits_{x \to \infty} \dfrac{3 + \frac{5}{x}}{2}$

$= \dfrac{3}{2}$ as $\dfrac{5}{x} \to 0$

3

(b) $y = 2x^2 + 5x + 2$

$= 2x^2 + 4x + x + 2$

$= 2x(x+2) + (x+2)$

$= (2x+1)(x+2)$

(i) x-intercepts, let $y = 0$

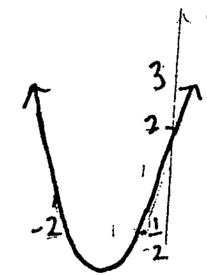
$0 = (2x+1)(x+2)$

$-\frac{1}{2}, -2 = x$

y-intercept = 2

$a > 0$

$\therefore$ concave up



2

(ii) values for which x is increasing.

axis of symmetry, $x = -\dfrac{5}{4}$

$= -1\frac{1}{4}$

$\therefore$ increasing for $x > -1\frac{1}{4}$

1

(c) primitive function of $6 - x^{-3}$

$\int 6 - x^{-3}\,dx = 6x + \dfrac{x^{-2}}{2} + C$

$= 6x + \dfrac{2}{x^2} + C$

2

or using the same method, $D = (3, 13)$.

(d) $y = x^2 - 3x$

By definition

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) - x^2 + 3x}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h}$$

$$= \lim_{h\to 0} \frac{2xh - 3h + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x-3+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x - 3 + h \quad 3$$

$$= 2x - 3$$

(e) Differentiate with respect to x:

(i) $y = 2x^3 + \frac{1}{x^4} - \sqrt{x}$

$= 2x^3 + x^{-4} - x^{\frac{1}{2}}$

2 $\frac{dy}{dx} = 6x^2 - 4x^{-5} - \frac{1}{2}x^{-\frac{1}{2}}$

$= 6x^2 - \frac{4}{x^5} - \frac{1}{2\sqrt{x}}$

(ii) $y = (2x^4+1)(3x+12)$

let $u = 2x^4+1$ and $v = 3x+12$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

2 $= (2x^4+1)3 + (3x+12)8x^3$

$= 6x^4 + 3 + 24x^4 + 96x^3$

$= 30x^4 + 96x^3 + 3$

(iii) $y = (4x^2+7x)^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}(4x^2+7x)^{-\frac{2}{3}} \cdot (8x+7)$

$= \frac{8x+7}{3(4x^2+7x)^{\frac{2}{3}}}$ 2

$= \frac{8x+7}{3\sqrt[3]{(4x^2+7x)^2}}$

(iv) $y = \frac{x^2+4}{2x-3}$

let $u = x^2+4$ & $v = 2x-3$ 3

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2x-3)(2x) - (x^2+4)(2)}{(2x-3)^2}$

$= \frac{4x^2 - 6x - 2x^2 - 8}{(2x-3)^2}$

$= \frac{2x^2 - 6x - 8}{(2x-3)^2}$ 2

(f) $f(x) = 2x^3 + 3x^2 + 4$

$f'(x) = 6x^2 + 6x$

$12 = 6x^2 + 6x$

$0 = 6x^2 + 6x - 12$ $\quad p = -72$

$0 = 6x^2 + 12x - 6x - 12$ $\quad s = 6$

$0 = 6x(x+2) - 6(x+2)$ $\quad \boxed{F = 12, -6}$

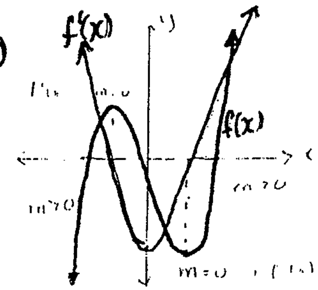
$0 = (6x-6)(x+2)$

$0 = 6(x-1)(x+2)$

$\therefore x = 1, -2$

QUESTION 3 (25 marks)

(a) (i)



3

(ii) see above

(b) $A = \frac{5h+3}{7h-1}$ $\quad$ let $u = 5h+3$ & $v = 7h-1$

$\frac{dA}{dh} = \frac{v\frac{du}{dh} - u\frac{dv}{dh}}{v^2}$

$= \frac{(7h-1)5 - (5h+3)7}{(7h-1)^2}$

$= \frac{35h - 5 - 35h - 21}{(7h-1)^2}$ 3

$= \frac{-26}{(7h-1)^2}$

if $h = 1$ $\quad \frac{dA}{dh} = \frac{-26}{(7-1)^2}$

$= \frac{-26}{36}$

$= -\frac{13}{18}$

(c) $y = 4x^3 - 7x^2 + 3$ 3

$\frac{dy}{dx} = 12x^2 - 14x$

if $x = 2$ $\quad \frac{dy}{dx} = 12(2)^2 - 14(2)$

$= 20$

$m_1 m_2 = -1$

$m_2 = -\frac{1}{20}$

if $x = 2$ $\quad y = 4(2)^3 - 7(2)^2 + 3$

$= 32 - 28 + 3$

$= 7$

$y - y_1 = m_2(x - x_1)$ $\quad$ where $x_1 = 2$, $y_1 = 7$

$y - 7 = -\frac{1}{20}(x - 2)$

$20y - 140 = -x + 2$

$x + 20y - 142 = 0$

(d) $y = 2x^2 - 6x + 4$

$\frac{dy}{dx} = 4x - 6$

$\therefore m_1 = 6$

for what value of x does $\frac{dy}{dx} = 6$

$6 = 4x - 6$ $\quad$ 4

$12 = 4x$

$3 = x$

Hence for $x = 3$

$y = 4$

(e) $y = x^3 + 3x^2 - 9x + 4$

$\frac{dy}{dx} = 3x^2 + 6x - 9$

$\frac{d^2y}{dx^2} = 6x + 6$

for stationary points $\frac{dy}{dx} = 0$

$3x^2 + 6x - 9 = 0$ $\quad$ 4

$3(x^2 + 2x - 3) = 0$

$3(x+3)(x-1) = 0$

$x = -3, \quad x = 1$

if $x = -3$ $\quad y = (-3)^3 + 3(-3)^2 - 9(-3) + 4$

$= 31 \quad \therefore \max(-3, 31)$

if $x = 1$ $\quad y = 1 + 3 - 9 + 4 = -1 \quad \min(1, -1)$

(f)(i)

$S = 2\pi r^2 + 2\pi r h \quad \ldots ①$

and $V = \pi r^2 h$, $V = 500\,cm^3$

$\therefore 500 = \pi r^2 h$

$\frac{500}{\pi r^2} = h$

4

substitute h into ①

$S = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{1000}{r}$

(ii) $\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2}$

for max / min pts $\frac{ds}{dr} = 0$

$4\pi r - \frac{1000}{r^2} = 0$

$4\pi r = \frac{1000}{r^2}$

$4\pi r^3 = 1000$

$r^3 = \frac{1000}{4\pi}$

$r = \frac{10}{\sqrt[3]{4\pi}}$

4

Question 4

(a)(i) $y = x^3 - 3x^2 + 1$

$\frac{dy}{dx} = 3x^2 - 6x$

3

for stat pts. $\frac{dy}{dx} = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

if $x = 0$ $y = 1$ if $x = 2$ $y = -3$

(ii) $\frac{d^2y}{dx^2} = 6x - 6$

at $x = 0$ $\frac{d^2y}{dx^2} < 0$ At (0,1) we have a max pt.

$\therefore$ max

3

at $x = 2$ $\frac{d^2y}{dx^2} > 0$ At (2,-3) we have a min pt.

$\therefore$ min

(iii) for points of inflexion

$\frac{d^2y}{dx^2} = 0$ $6x - 6 = 0$

$6x = 6$

$x = 1$

3

if $x = 1$ $y = (1)^3 - 3(1)^2 + 1$

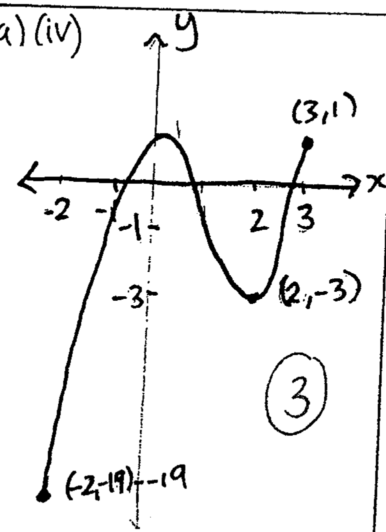
$= -1$

Check if concavity changes.

x	-1	1	2
$\frac{d^2y}{dx^2}$	-12	0	6

Since concavity changes (1,-1) is a point of inflexion.

Q4(a)(iv)



3

(v) minimum pt is (-2,-19)

1

(b)(i) Area of base $= 2x \times x$

$= 2x^2$

Area of 2 sides $= (x \times y) \times 2$

$= 2xy$

Area of front face = Area of back face

Area of back face $= 2x \times y$

$= 2xy$

Total area $= 2x^2 + 2xy + 2xy + 2xy$

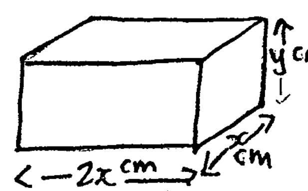
$\therefore A = 2x^2 + 6xy$

4

$V = l \times b \times h$

$= 2x \times x \times y$

$\therefore V = 2x^2 y$



(ii) If $A = 150$

$150 = 2x^2 + 6xy$

$150 - 2x^2 = 6xy$

$\frac{150 - 2x^2}{6x} = y$

Now sub. y into $V = 2x^2 y$

$\therefore V(x) = 2x^2\left(\frac{150 - 2x^2}{6x}\right)$

$= \frac{300x^2}{6x} - \frac{4x^4}{6x}$

$= 50x - \frac{2}{3}x^3$

4

(iii) $V(x) = 50x - \frac{2}{3}x^3$

$V'(x) = 50 - 2x^2$

$V''(x) = -4x$

For stat. pts $V'(x) = 0$

$50 - 2x^2 = 0$

$50 = 2x^2$

$25 = x^2$

$5 = x$

(x must be positive)

4

$V''(5) = -4(5)$

$= -20$

< 0

$\therefore$ max. turning pt at $x = 5$

$V(5) = 50(5) - \frac{2}{3}(5)^3$

$= 250 - \frac{2}{3}(125)$

$= \frac{750}{3} - \frac{250}{3}$

$= \frac{500}{3}$

$\therefore$ max value of V is $\frac{500}{3}$.